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Existence of Solutions for New Systems of Generalized Mixed Vector Variational-like Inequalities in Reflexive Banach Spaces

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Abstract : In this paper, we introduce and study a new type of generalized mixed vector variational-like inequalities and a new type of system of generalized mixed vector variational-like inequalities in reflexive Banach spaces. Firstly, we introduce the new concept of pseudo-monotonicity for vector multi-valued mappings, and prove an existence theorem of solution for generalized mixed vector variational-like inequalities by using the Fan-KKM theorem. Secondly, we introduce a new type of system of generalized mixed vector variational-like inequalities and by using the Kakutani-Fan-Glicksberg fixed point theorem, some new existence results of solution for system of generalized mixed vector variational-like inequalities are obtained under some suitable conditions.

Keywords : Set-valued mapping; Monotone mapping; Pseudomonotone mapping; Semi-monotone mapping; Semi-pseudomonotone mapping; Generalized mixed vector variational-like inequality; Fan-KKM theorem; Kakutani-Fan-Glicksberg fixed point theorem.
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1 Introduction

Vector variational inequality (VVI) was first introduced by Giannessi [1] in finite dimensional Euclidean space in 1980. Later on, Chen and Cheng [2] proposed

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the vector variational inequality in infinite-dimensional spaces and applied it to the vector optimization problem. Since then, a lot of applications have been found. It has shown to be a powerful tool in the mathematical investigation of optimization topics. For the past years, vector variational inequalities and their generalizations have been studied and applied in various directions. (see [3–10]). It is worth noting that vector variational-like inequalities are important generalization of vector variational inequalities related to the class of η-connected sets which is much more general than the class of convex sets (see [11, 12]).

It is well known that one of the most frequently used hypotheses in the theory of the variational inequality problems is the monotonicity of a nonlinear mapping. There are many kinds of generalizations of the monotonicity in the literature of recent years, such as pseudo-monotonicity, quasi-monotonicity, semi-monotonicity relaxed $\eta - \alpha$-semi-monotonicity, etc. (see [3–5]).

On the other hand, some critical and attractive problems related to variational inequalities and complementarity problems were considered in recent papers. In 2003, Huang and Fang [13] introduced systems of order complementarity problems and established some existence results by fixed point theory. In [14] Kassy and Kolumbn introduced systems of variational inequalities and proved the existence theorem by using the Ky Fan lemma. Later on, Kassay et al. [15], introduced and studied Minty and Stampacchia variational inequality systems by the Kakutani-Fan-Glicksberg fixed point theorem. Recently, in [10], Zhao and Xia introduced and studied systems of vector variational-like inequalities by the same fixed point theorem.

Motivated and inspired by the research going on in this direction, in this paper, we introduce and study a new type of generalized mixed vector variational-like inequalities and a new type of system of generalized mixed vector variational-like inequalities in reflexive Banach spaces. Firstly, we introduce the new concept of pseudo-monotonicity for vector multi-valued mappings, and prove an existence theorem of solution for generalized mixed vector variational-like inequality problem by using the Fan-KKM theorem. Secondly, we introduce a new type of system of generalized mixed vector variational-like inequality problem and prove some existence theorems of solution for system of generalized mixed vector variational-like inequality problem by using the Kakutani-Fan-Glicksberg fixed point theorem.

2 Preliminaries

Let X and Y be two real Banach spaces, $L(X,Y)$ be the family of all linear bounded operators from X to Y, and K be a nonempty closed and convex subset of X. Recall that a subset C of Y is said to be a closed convex cone if C is closed and $C + C \subset C$, $\lambda C \subset C$ for $\lambda > 0$. In addition, if $C \neq Y$, then C is called a proper closed convex cone. A closed convex cone is pointed if $C \cap (-C) = \{0\}$. A mapping $C : K \to 2^Y$ is said to be a cone mapping if $C(x)$ is a proper closed convex pointed cone and int $C(x) \neq \emptyset$ for each $x \in K$.

Let $S, T : K \times K \to 2^{L(X,Y)}$, $F : K \to 2^{L(X,Y)}$ be three vector set-valued mappings and let $g : K \times K \to Y$ and $\eta : K \times K \to K$ are two bi-mappings. In this paper, we consider the following two kinds of generalized mixed vector variational-like inequalities:

(i) *generalized mixed vector variational-like inequality problem* (for short, the (GMVVLIP)): Find $x_0 \in K$ such that for each $y \in K$, there exists $\zeta \in F(x_0)$ such that

$$\langle \zeta, \eta(y, x_0) \rangle + g(y, x_0) \notin \text{-int } C(x_0) \tag{2.1}$$

(ii) *system of generalized mixed vector variational-like inequality problem* (for short, the (SGMVVLIP)): Find $(x_0, y_0) \in K \times K$ such that for each $z \in K$, there exist $\xi \in S(x_0, y_0)$ and $\zeta \in T(x_0, y_0)$ satisfying

$$\begin{cases} \langle \xi, \eta(z, x_0) \rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta, \eta(z, y_0) \rangle + g(z, y_0) \notin \text{-int } C(y_0) \end{cases} \tag{2.2}$$

For our main results, we need the following definitions and lemmas.

Let $C : K \to 2^Y$ be a set-valued maping such that for each $x \in K$, $C(x)$ is a closed convex pointed cone with $\text{int}C(x) \neq \emptyset$. The following notations will be used in the sequel:

$$C_- = \bigcap_{x \in K} C(x)$$

Definition 2.1 ([7, 16])**.** Let X, Y be Banach spaces, K be nonempty subset of X. Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping.

(i) T is *monotone* on K if for any $x, y \in K$, it holds that

$$\langle \xi - \eta, y - x \rangle \in C_-, \quad \forall\, \xi \in T(x),\ \eta \in T(y).$$

(ii) T is *C_x-pseudomonotone on* K if for every pair of points $x \in K$, $y \in K$ and for all $\xi \in T(x)$, $\zeta \in T(y)$, we have

$$\langle \xi, y - x \rangle \notin \text{-int } C(x) \quad \text{implies} \quad \langle \zeta, y - x \rangle \notin \text{-int } C(x).$$

(iii) T is *generalized C-pseudomonotone on* K if for every pair of points $x \in K$, $y \in K$, there exists $\xi \in T(x)$ such that

$$\langle \xi, y - x \rangle \notin \text{-int } C(x)$$

implies that there exists $\zeta \in T(y)$ such that

$$\langle \zeta, y - x \rangle \notin \text{-int } C(x).$$

Definition 2.2 ([4])**.** A vector set-valued mapping $T : K \times K \to 2^{L(X,Y)}$ is said to be a vector set-valued semi-monotone mapping on K if it satisfies the following conditions:

(1) for each $u \in K$, the mapping $T(u, \cdot) : K \to 2^{L(X,Y)}$ is a vector set-valued monotone mapping in the sense of Definition 2.1;

(2) for each $v \in K$, the mapping $T(\cdot, v) : K \to 2^{L(X,Y)}$ is lower semi-continuous on K, where K is equipped with the weak topology, and $L(X, Y)$ is equipped with the uniform convergence topology of operators.

For more details see, for instances, [4].

Now, we introduce some new definitions that we will use in our results.

Definition 2.3. Let X, Y be Banach spaces, K be nonempty subset of X. Let $\eta : K \times K \to K$ and $g : K \times K \to Y$ be two bi-mappings. Let $T : K \to 2^{L(X,Y)}$ be a vector set-valued mapping.

(i) T is *η-pseudomonotone with respect to g on K* if for every pair of points $x \in K$, $y \in K$ and for all $\xi \in T(x)$, $\zeta \in T(y)$, we have

$$\langle \xi, \eta(y, x)\rangle + g(y, x) \notin \text{-int } C(x) \quad \text{implies} \quad \langle \zeta, \eta(y, x)\rangle + g(y, x) \notin \text{-int } C(x).$$

(ii) T is *weakly η-pseudomonotone with respect to g on K* if for every pair of points $x \in K$, $y \in K$, there exists $\xi \in T(x)$ such that

$$\langle \xi, \eta(y, x)\rangle + g(y, x) \notin \text{-int } C(x)$$

implies that there exists $\zeta \in T(y)$ such that

$$\langle \zeta, \eta(y, x)\rangle + g(y, x) \notin \text{-int } C(x).$$

Remark 2.4.

(i) If T is η-pseudomonotone with respect to g on K, then T is weakly η-pseudomonotone with respect to g on K.

(ii) If $g(y, x) = 0$ for all $x,\ y \in K$ then the concept of weakly η-pseudomonotone with respect to g reduces to η-pseudomonotone introduced in [17].

(iii) If $\eta(y, x) = y - x$ and $g(y, x) = 0$ for all $x,\ y \in K$ then the concept of η-pseudomonotone with respect to g reduces to C_x-pseudomonotone introduced in [16] and the concept of weakly η-pseudomonotone with respect to g reduces to generalized C-pseudomonotone introduced in [7].

Definition 2.5. A vector set-valued mapping $T : K \times K \to 2^{L(X,Y)}$ is said to be semi-η-pseudomonotone with respect to g on K if it satisfies the following conditions:

(1) for each $u \in K$, the mapping $T(u, \cdot) : K \to 2^{L(X,Y)}$ is η-pseudomonotone with respect to g on K in the sense of Definition 2.3;

(2) for each $v \in K$, the mapping $T(\cdot, v) : K \to 2^{L(X,Y)}$ is lower semi-continuous on K, where K is equipped with the weak topology, and $L(X, Y)$ is equipped with the uniform convergence topology of operators.

Definition 2.6 ([17]). Let $\eta : K \times K \to K$ be a bi-mapping. Then a mapping $T : K \to 2^{(X,Y)}$ is said to be V-hemicontinuous on K if for every $x,\ y \in K$, $\alpha > 0$ and $t_\alpha \in T(x+\alpha y)$, there exists $t_0 \in T(x)$ such that for any $z \in X$, $\langle t_\alpha, z\rangle \to \langle t_0, z\rangle$ as $\alpha \to 0^+$.

Lemma 2.7. *Let X, Y be Banach spaces and K be a nonempty convex subset of X and let $C : K \to 2^Y$ be a cone mapping. Let $\eta : K\times K \to K$ and $g : K\times K \to Y$ be two continuous and affine mappings such that $\eta(x, x) = 0 = g(x, x)$, $\forall x \in K$. Let $T : K \to 2^{L(X,Y)}$ be a vector set-valued mapping and we consider the following problms:*

(I) $x_0 \in K$ such that for each $y \in K$, there exists $\xi \in T(x_0)$ such that

$$\langle \xi, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0);$$

(II) $x_0 \in K$ such that for each $y \in K$, there exists $\xi \in T(y)$, such that

$$\langle \xi, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0);$$

(III) $x_0 \in K$ such that for each $y \in K$ and for each $\xi \in T(y)$, it holds that

$$\langle \xi, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0).$$

Then,

(i) Problem (III) implies Problem (II);

(ii) Problem (II) implies Problem (I) if T is V-hemicontinuous;

(iii) Problem (I) implies Problem (III) if T is η-pseudomonotone with respect to g on K and implies Problem (II) if T is weakly η-pseudomonotone with respect to g on K.

Proof. (i) It is clear from the definition.

(ii) Suppose that T is V-hemicontinuous on K and let $x_0 \in K$ be a solution of Problem (II). Then for each $y \in K$, there exists $\xi \in T(y)$, such that

$$\langle \xi, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0)$$

Now, for each $y \in K$ and $\alpha \in (0, 1)$, we let $x_\alpha = \alpha y + (1 - \alpha)x_0$. By the convexity of K, we have $x_\alpha \in K$, $\forall\alpha \in (0, 1)$. It implies that for any $\alpha \in (0, 1)$, there exists $\xi_\alpha \in T(\alpha y + (1 - \alpha)x_0)$, such that

$$\langle \xi_\alpha, \eta(x_\alpha, x_0)\rangle + g(x_\alpha, x_0) \notin -int\ C(x_0),$$

since η and g are affine and $\eta(x, x) = 0 = g(x, x)$, we obtain that

$$\alpha(\langle \xi_\alpha, \eta(y, x_0)\rangle + g(y, x_0)) \notin -int\ C(x_0).$$

Since $-int\ C(x_0)$ is a convex cone, we get

$$\langle \xi_\alpha, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0).$$

By V- hemicontinuity of T, there exists $\xi_0 \in T(x_0)$ such that

$$\langle \xi_0, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0).$$

Consequently, for each $x_0 \in K$, there exists $\xi \in T(x_0)$ such that

$$\langle \xi_0, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0).$$

(iii) The result follow from the definition of η-pseudomonotone with respect to g and weakly η-pseudomonotone with respect to g, respectively. □

Definition 2.8 (KKM mapping [18]). Let K be a nonempty subset of a topological vector space E. A multivalued mapping $G : K \to 2^E$ is said to be a *KKM mapping* if for any finite subset $\{y_1, y_2, ..., y_n\}$ of K, we have

$$co\{y_1, y_2, ..., y_n\} \subset \cup_{i=1}^n G(y_i)$$

where $co\{y_1, y_2, ..., y_n\}$ denotes the convex hull of $\{y_1, y_2, ..., y_n\}$.

Lemma 2.9 (Fan-KKM Theorem [18]). *Let K be a nonempty convex subset of a Hausdorff topological vector space E and let $G : K \to 2^E$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is a compact subset of K, then $\cap_{y\in K} G(y) \neq \emptyset$.*

Lemma 2.10 (Kakutani-Fan-Glicksberg [19]). *Suppose that X is a Hausdorff locally convex space and K is a nonempty convex compact subset of X. If $T : K \to 2^K$ is an upper semi-continuous mapping with nonempty convex closed values, then T has a fixed point in K, i.e., there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.*

Lemma 2.11 ([4]). *Let X, Y be two Banach spaces, $K \subset X$. Suppose that the set-valued mapping $T : K \to 2^Y$ is upper semi-continuous at x_0 with $T(x_0)$ compact. If $x_n \in K,\ n = 1, 2, ...$ with $x_n \to x_0$, and $y_n \in T(x_n)$, then there exists $y_0 \in T(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

Definition 2.12. A multivalued mapping $T : K \to 2^Y$ is called *concave* if for any $x, y \in K,\ \alpha \in (0, 1)$,

$$\alpha T(x) + (1 - \alpha)T(y) \subseteq T(\alpha x + (1 - \alpha)y).$$

Definition 2.13. Let $W : X \to 2^Y$. The graph of W, denoted by $\mathcal{G}(W)$, is

$$\mathcal{G}(W) = \{(x, y) \in X \times Y \mid x \in X,\ y \in W(x)\}.$$

3 Existence results for generalized mixed variational-like inequalities

In this section, we will prove the existence theorem of solutions for generalized mixed variational-like inequalities with weakly η-pseudomonotone with respect to g on K by using the Fan-KKM theorem.

Theorem 3.1. *Let X be a reflexive Banach space and Y a Banach space. Let K be a nonempty closed bounded convex subset of X. Let $C : K \to 2^Y$ be a multivalued mapping such that for every $x \in K$, $C(x)$ is proper closed convex cone with $intC(x) \neq \emptyset$, and $W : K \to 2^Y$ be defined by $W(x) = Y \backslash \{-int\ C(x)\}$ such that the graph $\mathcal{G}(W)$ of W is weakly closed in $X \times Y$ and W is concave. Let $\eta : K \times K \to K$ and $g : K \times K \to Y$ be two continuous and affine mappings satisfy $\eta(x, x) = 0 = g(x, x)$, $\forall x \in K$. Suppose that $T : K \to 2^{L(X,Y)}$ is nonempty compact valued, weakly η-pseudomonotone with respect to g and V-hemicontinuous on K. Then (GMVVLIP) is solvable.*

Proof. Let $G, H : K \to 2^K$ be two multivalued mapping defined by

$$G(y) = \{x \in K : \exists \xi \in T(x) \text{ such that } \langle \xi, \eta(y, x)\rangle + g(y, x) \notin -int\ C(x)\},$$

$$H(y) = \{x \in K : \exists \zeta \in T(y) \text{ such that } \langle \zeta, \eta(y, x)\rangle + g(y, x) \notin -int\ C(x)\},$$

for all $y \in K$. Since $y \in G(y)$ and $y \in H(y)$, we get that $G(y)$ and $H(y)$ are nonempty. Next, we divided the proof of the theorem in to the following five steps.

Step 1: We prove that G is a KKM mapping on K. Assume that G is not KKM mapping. Then there exists a finite set $\{x_1, x_2, ..., x_n\} \subset K$ and $t_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that $x = \sum_{i=1}^n t_i x_i \notin \cup_{i=1}^n G(x_i)$. It follows from the definition of G that for all $\xi \in T(x)$,

$$\langle \xi, \eta(x_i, x)\rangle + g(x_i, x) \in -int\ C(x),\ \ i = 1, 2, ..., n.$$

Since $-int\ C(x)$ is a convex cone and $t_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$, we have

$$\sum_{i=1}^n t_i[\langle \xi, \eta(x_i, x)\rangle + g(x_i, x)] \in -int\ C(x).$$

Since η and g are affine, we get that

$$\begin{aligned}
\theta &= \langle \xi, \eta(x, x)\rangle + g(x, x) \\
&= \langle \xi, \eta(\sum_{i=1}^n t_i x_i, x)\rangle + g(\sum_{i=1}^n t_i x_i, x) \\
&= \sum_{i=1}^n t_i \langle \xi, \eta(x_i, x)\rangle + \sum_{i=1}^n t_i g(x_i, x) \\
&= \sum_{i=1}^n t_i[\langle \xi, \eta(x_i, x)\rangle + g(x_i, x)] \in -int\ C(x),
\end{aligned}$$

where θ denotes the zero vector in Y. Thus $\theta \in -int\ C(x)$. This is a contradiction with $C(x)$ is proper. Hence G is a KKM mapping on K.

Step 2: We prove that $G(y) \subset H(y)$ for all $y \in K$ and H is a KKM mapping on K. Since T is weakly η-pseudomonotone with respect to g, we derive that $G(y) \subset H(y)$ for all $y \in K$. Thus H is also a KKM mapping since G is a KKM mapping.

Step 3: We prove that for each $y \in K$, $H(y)$ is closed. For any $y \in K$, let $\{x_n\}$ be a sequence in $H(y)$ such that $x_n \to x^* \in K$. Since $x_n \in H(y)$, for all $n \in \mathbb{N}$, there exists $\zeta_n \in T(y)$ such that

$$\langle \zeta_n, \eta(y, x_n)\rangle + g(y, x_n) \notin -int\ C(x_n),$$

or

$$\langle \zeta_n, \eta(y, x_n)\rangle + g(y, x_n) \in W(x_n).$$

Since $T(y)$ is compact, without lost of generality, we assume that there exists $\zeta_0 \in T(y)$ such that $\zeta_n \to \zeta_0$. Since W is concave, we have that the graph $\mathcal{G}(W)$ of W is convex. Thus we obtain that graph $\mathcal{G}(W)$ of W is closed in $X \times Y$ since it is convex and weakly closed. Now, since $\eta(\cdot,\cdot)$, $\langle \cdot,\cdot\rangle$ and g are continuous, W has a closed graph in $X \times Y$ and $\zeta_n \to \zeta_0$, $x_n \to x^*$, we have

$$\langle \zeta_n, \eta(y, x_n)\rangle + g(y, x_n) \to \langle \zeta_0, \eta(y, x^*)\rangle + g(y, x^*) \in W(x^*).$$

Consequently, we have

$$\langle \zeta_0, \eta(y, x^*)\rangle + g(y, x^*) \notin -int\ C(x^*).$$

Hence $x^* \in H(y)$ and therefore $H(y)$ is closed.

Step 4: For any $y \in K$, we prove that $H(y)$ is convex. Let $x_1,\ x_2 \in H(y)$ and $\alpha_1,\ \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$. Then there exist $\zeta \in T(y)$ such that

$$\langle \zeta, \eta(y, x_1)\rangle + g(y, x_1) \notin -int\ C(x_1) \tag{3.1}$$

and

$$\langle \zeta, \eta(y, x_2)\rangle + g(y, x_2) \notin -int\ C(x_2). \tag{3.2}$$

Multiplying (3.1) and (3.2) by α_1 and α_2 respectively and combining, we get that

$$\alpha_1[\langle \zeta, \eta(y, x_1)\rangle + g(y, x_1)] + \alpha_2[\langle \zeta, \eta(y, x_2)\rangle + g(y, x_2)] \in \alpha_1 W(x_1) + \alpha_2 W(x_2).$$

Since η and g are affine and W is concave, we have

$$\begin{aligned}\langle \zeta, \eta(y, \alpha_1 x_1 + \alpha_2 x_2)\rangle &+ g(y, \alpha_1 x_1 + \alpha_2 x_2)\\ &= \alpha_1[\langle \zeta, \eta(y, x_1)\rangle + g(y, x_1)] + \alpha_2[\langle \zeta, \eta(y, x_2)\rangle + g(y, x_2)]\\ &\in \alpha_1 W(x_1) + \alpha_2 W(x_2) \subseteq W(\alpha_1 x_1 + \alpha_2 x_2).\end{aligned}$$

That is

$$\langle \zeta, \eta(y, \alpha_1 x_1 + \alpha_2 x_2)\rangle + g(y, \alpha_1 x_1 + \alpha_2 x_2) \notin -intC(\alpha_1 x_1 + \alpha_2 x_2).$$

Hence $\alpha_1 x_1 + \alpha_2 x_2 \in H(y)$ and so $H(y)$ is convex.

Step 5: We prove that the generalized mixed vector variational-like inequality (GMVVLIP) is solvable. Firstly, we prove that $\bigcap_{y\in K} H(y) \neq \emptyset$. Now, we equip X with the weak topology. Then K is weakly compact since X is a reflexive Banach space and K is a closed bounded convex subset of X. Also, since $H(y)$ is closed convex subset of a reflexive Banach space, we get that $H(y)$ is weakly closed. Since K is weakly compact and $H(y) \subseteq K$, it follows directly that $H(y)$ is weakly compact. Then by KKM-Fan Theorem Lemma 2.9, we have

$$\bigcap_{y\in K} H(y) \neq \emptyset.$$

Next, we claim that $\bigcap_{y\in K} G(y) = \bigcap_{y\in K} H(y)$. From step 2, we get that $\bigcap_{y\in K} G(y) \subseteq \bigcap_{y\in K} H(y)$ and from Lemma 2.7, we have $\bigcap_{y\in K} G(y) \supseteq \bigcap_{y\in K} H(y)$, so we obtain that $\bigcap_{y\in K} G(y) = \bigcap_{y\in K} H(y)$. Thus $\bigcap_{y\in K} G(y) \neq \emptyset$. Therefore, there exists $x_0 \in K$ such that for each $y \in K$, there exists $\xi \in T(x_0)$ such that

$$\langle \xi, \eta(y, x_0)\rangle + g(y, x_0) \notin -int\ C(x_0).$$

This complete the proof. □

Now, if we setting $g(y,x) = 0$ for all $y,\ x \in K$ in Theorem 3.1 then we get the following Corollary.

Corollary 3.2 ([17])**.** *Let X be a reflexive Banach space and Y a Banach space. Let K be a nonempty closed bounded convex subset of X. Let $C : K \to 2^Y$ be a multivalued mapping such that for every $x \in K$, $C(x)$ is proper closed pointed convex cone with $intC(x) \neq \emptyset$, and $W : K \to 2^Y$ be defined by $W(x) = Y \backslash \{-int\ C(x)\}$ such that W is upper semicontinuous concave. Let $\eta : K \times K \to K$ be continuous and affine mappings such that $\eta(x,x) = 0,\ \ \forall x \in K$. Suppose that $T : K \to 2^{L(X,Y)}$ is nonempty compact valued, η-pseudomonotone and V-hemicontinuous on K. Then, there exists $x_0 \in K$ such that for each $y \in K$, there exists $\xi \in T(x_0)$ such that*

$$\langle \xi, \eta(y, x_0)\rangle \notin -int\ C(x_0).$$

4 Existence results for systems of generalized mixed vector variational-like inequalities

In this section, by using the Kakutani-Fan-Glicksberg fixed point theorem, we prove some existence theorems of solutions for systems of generalized mixed vector variational-like inequality problems (SGMVVLIP). We first consider a (SG-MVVLIP) defined on a bounded closed convex subset of a real reflexive Banach space and finally, we also consider a (SGMVVLIP) defined on an unbounded closed convex set.

Theorem 4.1. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty bounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping, and $W : K \to 2^Y$ be defined by $W(x) = Y \backslash \{-int\ C(x)\}$ such that the graph $\mathcal{G}(W)$ of W is weakly closed in $X \times Y$ and W is concave. Let $\eta : K \times K \to K$ and $g : K \times K \to Y$ be two continuous and affine mappings satisfy $\eta(x,x) = 0 = g(x,x)$,$\forall x \in K$. Let $S,\ T : K \times K \to 2^{L(X,Y)}$ with nonempty convex compact values and satisfies the following conditions:*

- *(i) For each $z \in K$, $S(\cdot,z) : K \to 2^{L(X,Y)}$ and $T(z,\cdot) : K \to 2^{L(X,Y)}$ are η-pseudomonotone with respect to g on K;*
- *(ii) for each $z \in K$, $S(z,\cdot) : K \to 2^{L(X,Y)}$ and $T(\cdot,z) : K \to 2^{L(X,Y)}$ are lower semicontinuous on K, where K equipped with the weak topology and $L(X,Y)$ is equipped with the uniform convergence topology operator;*
- *(iii) for each $z \in K$, the mappings $S(\cdot,z) : K \to 2^{L(X,Y)}$ and $T(z,\cdot) : K \to 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X.*

Then the (SGMVVLIP) has a solution in K.

Proof. Let M be a finite dimentional subspace of X such that $K_M = K \cap M \neq \emptyset$. For any $(x,y) \in K \times K$, we consider the following problem:
$(P)_M$ Find $(x_0,y_0) \in K_M \times K_M$ such that for all $z \in K_M$ there exist $\xi_0 \in S(x_0,x)$ and $\zeta_0 \in T(y,y_0)$ satisfying

$$\begin{cases} \langle \xi_0, \eta(z,x_0)\rangle + g(z,x_0) \notin \text{-int } C(x_0) \\ \langle \zeta_0, \eta(z,y_0)\rangle + g(z,y_0) \notin \text{-int } C(y_0). \end{cases} \tag{4.1}$$

By our assumptions, condition (i) and (iii), we know that $S(\cdot,x)$, $T(y,\cdot)$, η and g are satisfy the condition of Theorem 3.1. It follows from Theorem 3.1 that the problem $(P)_M$ is solvable.

Define a multi-valued mapping $F : K_M \times K_M \to 2^{K_M \times K_M}$ by

$$F(x,y) = \{(x_0,y_0) \in K_M \times K_M \ :\ (x_0,y_0) \text{ solve problem } (P)_M\},$$

$\forall (x,y) \in K_M \times K_M$. Next, we will show that this mapping has at least one fixed point in K_M.

Step 1: It is clear that $F(x,y)$ is nonempty and bounded for each $(x,y) \in K_M \times K_M$.

Step 2: We show that $F(x,y)$ is convex for each $(x,y) \in K_M \times K_M$. Let $(x_1,y_1),(x_2,y_2) \in F(x,y)$. Then we note that for each $z \in K_M$ there exist $\xi_i \in S(x_i,x),\ i = 1,2$ and $\zeta_j \in T(y,y_j),\ j = 1,2$ satisfying

$$\begin{cases} \langle \xi_i, \eta(z,x_i)\rangle + g(z,x_i) \notin \text{-int } C(x_i), \quad i = 1,2 \\ \langle \zeta_j, \eta(z,y_j)\rangle + g(z,y_j) \notin \text{-int } C(y_j), \quad j = 1,2. \end{cases}$$

By lemma 2.7, for each $z \in K_M$, $\xi \in S(z,x)$ and for each $\zeta \in T(y,z)$, we have

$$\begin{cases} \langle \xi, \eta(z,x_i)\rangle + g(z,x_i) \notin \text{-int } C(x_i), & i = 1,2 \\ \langle \zeta, \eta(z,y_j)\rangle + g(z,y_j) \notin \text{-int } C(y_j), & j = 1,2. \end{cases}$$

Since η and g are affine and W is concave, we get that for each $\lambda \in [0,1]$, we have

$$\begin{aligned} &\langle \xi, \eta(z,(\lambda x_1 + (1-\lambda)x_2))\rangle + g(z,(\lambda x_1 + (1-\lambda)x_2)) \\ &\qquad = \lambda[\langle \xi, \eta(z,x_1)\rangle + g(z,x_1)] + (1-\lambda)[\langle \xi, \eta(z,x_2)\rangle + g(z,x_2)] \\ &\qquad \in \lambda W(x_1) + (1-\lambda)W(x_2) \\ &\qquad \subset W(\lambda x_1 + (1-\lambda)x_2) \\ &\qquad = Y \setminus -\mathit{int}\ C(\lambda x_1 + (1-\lambda)x_2) \end{aligned}$$

and

$$\begin{aligned} &\langle \zeta, \eta(z,(\lambda y_+ (1-\lambda)y_2))\rangle + g(z,(\lambda y_1 + (1-\lambda)y_2)) \\ &\qquad = \lambda[\langle \zeta, \eta(z,y_1)\rangle + g(z,y_1)] + (1-\lambda)[\langle \zeta, \eta(z,y_2)\rangle + g(z,y_2)] \\ &\qquad \in \lambda W(y_1) + (1-\lambda)W(y_2) \\ &\qquad \subset W(\lambda y_1 + (1-\lambda)y_2) \\ &\qquad = Y \setminus -\mathit{int}\ C(\lambda y_1 + (1-\lambda)y_2). \end{aligned}$$

By using Lemma 2.7 again, we get that for $z \in K_M$, there exist $\bar{\xi} \in S(\lambda x_1 + (1-\lambda)x_2, x)$ and $\bar{\zeta} \in T(y, \lambda y_1 + (1-\lambda)y_2)$ such that

$$\begin{cases} \langle \bar{\xi}, \eta(z,(\lambda x_1 + (1-\lambda)x_2))\rangle + g(z,(\lambda x_1 + (1-\lambda)x_2)) \notin \text{-int } C(\lambda x_1 + (1-\lambda)x_2) \\ \langle \bar{\zeta}, \eta(z,(\lambda y_1 + (1-\lambda)y_2))\rangle + g(z,(\lambda y_1 + (1-\lambda)y_2)) \notin \text{-int } C(\lambda y_1 + (1-\lambda)y_2). \end{cases}$$

This mean that $\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) \in F(x,y)$. Consequently $F(x,y)$ is convex.

Step 3: We show that $F(x,y)$ is closed for each $(x,y) \in K_M \times K_M$. Let $\{(x_n,y_n)\}$ be a sequence in $F(x,y)$ such that $(x_n,y_n) \to (x_0,y_0)$. Then it follows from the definition of $F(x,y)$ that for each $z \in K_M$ there exist $\xi_n \in S(x_n,x)$ and $\zeta_n \in T(y,y_n)$ such that

$$\begin{cases} \langle \xi_n, \eta(z,x_n)\rangle + g(z,x_n) \notin \text{-int } C(x_n) \\ \langle \xi_n, \eta(z,y_n)\rangle + g(z,y_n) \notin \text{-int } C(y_n) \end{cases}$$

for all $n \in \mathbb{N}$. According to Lemma 2.11, there exist $\xi_0 \in S(x_0,x)$, $\zeta_0 \in T(y,y_0)$ and subsequences $\{\xi_{n_k}\}$ of $\{\xi_n\}$, $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ such that $\xi_{n_k} \to \xi_0$ and $\zeta_{n_j} \to \zeta_0$. Thus letting $k \to \infty$ and $j \to \infty$, we get that $\langle \xi_0, \eta(z,x_0)\rangle + g(z,x_0) \notin$ -int $C(x_0)$ and $\langle \zeta_0, \eta(z,y_0)\rangle + g(z,y_0) \notin$ -int $C(y_0)$ since $\eta(\cdot,\cdot)$, $\langle \cdot,\cdot\rangle$ and g are continuous, W has a closed graph in $X \times Y$, $\xi_{n_k} \to \xi_0$, $\zeta_{n_j} \to \zeta_0$, $x_n \to x_0$ and $y_n \to y_0$. Hence $(x_0,y_0) \in F(x,y)$ and $F(x,y)$ is closed.

Step 4: We show that the mapping $F : K_M \times K_M :\to 2^{K_M \times K_M}$ is upper semi-continuous. Since $K_M \times K_M$ is compact, we only need to show that the mapping $F : K_M \times K_M :\to 2^{K_M \times K_M}$ is closed. Suppose that $(x_n, y_n) \in K_M \times K_M$ for all $n = 1, 2, 3, ...$ with $(x_n, y_n) \to (x_0, y_0)$ and $(u_n, v_n) \in F(x_n, y_n)$ with $(u_n, v_n) \to (u_0, v_0)$. We will show that $(u_0, v_0) \in F(x_0, y_0)$. By the definition of $F(x, y)$, we have, for each $z \in K_M$ there exist $\xi_n \in S(u_n, x_n)$ and $\zeta_n \in T(y_n, v_n)$ such that

$$\begin{cases} \langle \xi_n, \eta(z, u_n)\rangle + g(z, u_n) \notin \text{-int } C(u_n) \\ \langle \xi_n, \eta(z, v_n)\rangle + g(z, v_n) \notin \text{-int } C(v_n) \end{cases}$$

for all $n = 1, 2, 3, ...$ Thus for all $\varphi_n \in S(z, x_n)$ and $\phi_n \in T(y_n, z)$, we have

$$\begin{cases} \langle \varphi_n, \eta(z, u_n)\rangle + g(z, u_n) \notin \text{-int } C(u_n) \\ \langle \phi_n, \eta(z, v_n)\rangle + g(z, v_n) \notin \text{-int } C(v_n) \end{cases}$$

for all $n = 1, 2, 3, ...$ Since $S(z, \cdot)$ and $T(\cdot, z)$ are lower semi-continuous, for each $\varphi \in S(z, x_0)$ and $\phi \in T(y_0, z)$, there exist $\varphi_n \in S(z, x_n)$ and $\phi_n \in T(y_n, z)$ such that $\varphi_n \to \varphi$ and $\phi_n \to \phi$. Now, letting $n \to \infty$, since W is closed and $\eta(\cdot, \cdot)$, $\langle \cdot, \cdot \rangle$ and $g(\cdot, \cdot)$ are continuous, we get that

$$\begin{cases} \langle \varphi, \eta(z, u_0)\rangle + g(z, u_0) \notin \text{-int } C(u_0) \\ \langle \phi, \eta(z, v_0)\rangle + g(z, v_0) \notin \text{-int } C(v_0). \end{cases}$$

By Lemma 2.7, there exist $\xi_0 \in S(u_o, x_0)$ and $\zeta_0 \in T(y_0, v_0)$ such that

$$\begin{cases} \langle \xi_0, \eta(z, u_0)\rangle + g(z, u_0) \notin \text{-int } C(u_0) \\ \langle \zeta_0, \eta(z, v_0)\rangle + g(z, v_0) \notin \text{-int } C(v_0). \end{cases}$$

Thus $(u_0, v_0) \in F(x_0, y_0)$. Therefore F is upper semi-continuous. By the Kakutani-Fan-Glicksberg fixed point theorem, there exist $(x_0, y_0) \in K_M \times K_M$ such that $(x_0, y_0) \in F(x_0, y_0)$. That is for each $z \in K_M$, there exist $\xi \in S(x_0, y_0)$ and $\zeta \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0). \end{cases}$$

Now, we generalize this result to the whole space. Let

$$\Gamma = \{N : N \text{ is a finite dimennsional subspace of } X \text{ with } K_N = K \cap N \neq \emptyset\}$$

and A_N be the solution set of the following problem: Find $(x^*, y^*) \in K \times K$ such that for each $z \in K_N$ there exist $\xi \in S(x^*, y^*)$ and $\zeta \in T(x^*, y^*)$ such that

$$\begin{cases} \langle \xi, \eta(z, x^*)\rangle + g(z, x^*) \notin \text{-int } C(x^*) \\ \langle \zeta, \eta(z, y^*)\rangle + g(z, y^*) \notin \text{-int } C(y^*). \end{cases}$$

From the previous discussion, we know that A_N is nonempty and bounded for all $N \in \Gamma$. Let $\overline{A}_N^w$ denote the weak closure of A_N. Obviously, we have

$$A_{\bigcup_{i=1}^n N_i} \subset \bigcap_{i=1}^n A_{N_i} \subset \bigcap_{i=1}^n \overline{A}_{N_i}^w.$$

Since X is reflexive, we have $\overline{A}_N^w$ is weakly compact for all $N \in \Gamma$. Thus $\{\overline{A}_N^w : N \in \Gamma\}$ has the finite intersection property. It implies that $\bigcap_{N\in\Gamma} \overline{A}_N^w \neq \phi$. Let $(x_0, y_0) \in \bigcap_{N\in\Gamma} \overline{A}_N^w$. Then for each $z \in K_N$, there exist $\xi \in S(x_0, y_0)$ and $\zeta \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0). \end{cases}$$

Next, for any given $z \in K$, choose $N \in \Gamma$ such that $z, x_0, y_0 \in K_N$. Since $(x_0, y_0) \in \overline{A}_N^w$, there exists $(x_n, y_n) \in A_N$ such that (x_n, y_n) converse weakly to (x_0, y_0). Therefore for each $z \in K_N$ and all $\xi_n \in S(z, y_n)$, $\zeta \in T(x_n, z)$, we have

$$\begin{cases} \langle \xi_n, \eta(z, x_n)\rangle + g(z, x_n) \notin \text{-int } C(x_n) \\ \langle \zeta_n, \eta(z, y_n)\rangle + g(z, y_n) \notin \text{-int } C(y_n). \end{cases}$$

Since $S(z, \cdot)$ and $T(\cdot, z)$ are lower semi-continuous, for each $\xi \in S(z, y_0)$ and $\zeta \in T(x_0, z)$ there exist $\xi_n \in S(z, y_n)$ and $\zeta_n \in T(x_n, z)$ such that $\xi_n \to \xi$ and $\zeta_n \to \zeta$. Letting $n \to \infty$ and as W is weakly closed and η and g are continuous, we have

$$\begin{cases} \langle \xi, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0). \end{cases}$$

By Lemma 2.7, there exist $\xi_0 \in S(y_0, x_0)$ and $\zeta_0 \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta_0, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0). \end{cases}$$

This complete the proof. □

Next, we consider the system of generalized mixed vector variational-like inequality problem in which K is an unbounded. We have the following result.

Theorem 4.2. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty unbounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping, and $W : K \to 2^Y$ be defined by $W(x) = Y\backslash\{-int\ C(x)\}$ such that the graph $\mathcal{G}(W)$ is weakly closed in $X \times Y$ and W is concave. Let $\eta : K\times K \to K$ and $g : K\times K \to Y$ be two continuous and affine mappings satisfy $\eta(x, x) = 0 = g(x, x)$, $\forall x \in K$. Let S, $T : K \times K \to 2^{L(X,Y)}$ with nonempty convex compact values and satisfies the following conditions:*

(i) For each $z \in K$, $S(\cdot, z) : K \to 2^{L(X,Y)}$ and $T(z, \cdot) : K \to 2^{L(X,Y)}$ are η-pseudomonotone with respect to g on K;

(ii) for each $z \in K$, $S(z, \cdot) : K \to 2^{L(X,Y)}$ and $T(\cdot, z) : K \to 2^{L(X,Y)}$ are lower semicontinuous on K, where K equipped with the weak topology and $L(X, Y)$ is equipped with the uniform convergence topology operator;

(iii) for each $z \in K$, the mappings $S(\cdot, z) : K \to 2^{L(X,Y)}$ and $T(z, \cdot) : K \to 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X;

(iv) there exists $u_0 \in K$ such that if $(x_n, y_n) \in K \times K$ with $(x_n, y_n) \to \infty$ as $n \to \infty$, then for each n large enough it holds that $\exists \xi_n \in S(u_0, y_n)$ and $\zeta_n \in T(x_n, u_0)$ satisfying

$$\begin{cases} \langle \xi_n, \eta(u_0, x_n)\rangle + g(u_0, x_n) \in \text{-}int\ C(x_n) \\ \langle \zeta_n, \eta(u_0, y_n)\rangle + g(u_0, y_n) \in \text{-}int\ C(y_n) \end{cases}$$

Then the (SGMVVLIP) has a solution in K.

Proof. For each $n \in \mathbb{N}$, let $K_n = K \cap B(\theta, n)$, where $B(\theta, n)$ is the closed ball with center at θ and radius n. Hence, from Theorem 4.2, we get that there exists $(x_n, y_n) \in K_n \times K_n$ such that for each $z \in K_n$ there exists $\xi_n \in S(x_n, y_n)$ and $\zeta_n \in T(x_n, y_n)$ satisfying

$$\begin{cases} \langle \xi_n, \eta(z, x_n)\rangle + g(z, x_n) \notin \text{-int}\ C(x_n) \\ \langle \zeta_n, \eta(z, y_n)\rangle + g(z, y_n) \notin \text{-int}\ C(y_n). \end{cases}$$

By Lemma 2.7, for all $\varphi_n \in S(z, y_n)$ and $\phi_n \in T(x_n, z)$, we have

$$\begin{cases} \langle \varphi_n, \eta(z, x_n)\rangle + g(z, x_n) \notin \text{-int}\ C(x_n) \\ \langle \phi_n, \eta(z, y_n)\rangle + g(z, y_n) \notin \text{-int}\ C(y_n). \end{cases}$$

By condition (4), we know that $\{(x_n, y_n)\}$ is bounded. If not, without loss of generality, we assume that $(x_n, y_n) \to \infty$. Thus for $z = u_0$, $\varphi_n \in S(u_0, y_n)$ and $\phi_n \in T(x_n, u_0)$, we have

$$\begin{cases} \langle \varphi_n, \eta(u_0, x_n)\rangle + g(u_0, x_n) \notin \text{-int}\ C(x_n) \\ \langle \phi_n, \eta(u_0, y_n)\rangle + g(u_0, y_n) \notin \text{-int}\ C(y_n). \end{cases}$$

This is a contradiction according to condition (4). Thus $\{(x_n, y_n)\}$ is bounded. Without loss of generality, we assume that $(x_n, y_n) \to^w (x_0, y_0)$. We shall show that (x_0, y_0) is the solution of the (SGMVVLIP). Consider, for each $z \in K$ and each $\xi \in S(z, y_0)$ and $\zeta_n \in T(x_0, z)$, it follow from the lower semi-continuity of $S(z, \cdot)$ and $T(\cdot, z)$ that there exist $\xi_n \in S(z, y_n)$ and $\zeta \in T(x_n, z)$ such that $\xi_n \to \xi$ and $\zeta_n \to \zeta$ satisfying

$$\begin{cases} \langle \xi_n, \eta(z, x_n)\rangle + g(z, x_n) \notin \text{-int}\ C(x_n) \\ \langle \zeta_n, \eta(z, y_n)\rangle + g(z, y_n) \notin \text{-int}\ C(y_n). \end{cases}$$

Now, letting $n \to \infty$, by the continuity of $\eta(\cdot,\cdot)$, $\langle\cdot,\cdot\rangle$ and $g(\cdot,\cdot)$, we can show that

$$\langle \xi_n, \eta(z, x_n)\rangle + g(z, x_n) \to^{w} \langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0)$$

and

$$\langle \zeta_n, \eta(z, y_n)\rangle + g(z, y_n) \to^{w} \langle \zeta_0, \eta(z, y_0)\rangle + g(z, y_0).$$

Since W is weakly closed, we obtain that

$$\langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0)$$

and

$$\langle \zeta_0, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0).$$

Using Lemma 2.7 again, we have that for each $z \in K$, there exist $\xi_0 \in S(x_0, y_0)$ and $\zeta_0 \in T(x_0, y_0)$ satisfying

$$\begin{cases} \langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0) \\ \langle \zeta_0, \eta(z, y_0)\rangle + g(z, y_0) \notin \text{-int } C(y_0). \end{cases}$$

This completes the proof. □

Next, we consider the generalized mixed vector variational-like inequalities with set-valued semi-η-pseudomonotone with respect to g on K.

Corollary 4.3. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty bounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping, and $W : K \to 2^Y$ be defined by $W(x) = Y \backslash \{-int\ C(x)\}$ such that the graph $\mathcal{G}(W)$ of W is weakly closed in $X \times Y$ and W is concave. Let $\eta : K \times K \to K$ and $g : K \times K \to Y$ be two continuous and affine mappings satisfy $\eta(x, x) = 0 = g(x, x)$, $\forall x \in K$. Let $T : K \times K \to 2^{L(X,Y)}$ with nonempty convex compact values and satisfies the following conditions:*

- *(i) T is a set valued semi-η-pseudomonotone with respect to g on K;*
- *(iii) for each $z \in K$, the mappings $T(z, \cdot) : K \to 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X.*

Then there exists $x_0 \in K$ such that for each $z \in K$, there exists $\xi \in T(x_0, x_0)$ satisfying

$$\langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-}int\ C(x_0).$$

Proof. Define a set-valued mapping $S : K \times K \to 2^{L(X,Y)}$ by $S(u, v) = T(v, v)$ for all $u, v \in K$. We observe that $S(\cdot, z)$ is η-pseudomonotone with respect to g and $S(z, \cdot)$ is lower semicontinuous for all $z \in K$. Moreover, $S(\cdot, z)$ continuous on each finite dimensional subspace of X. Hence, by Theorem 4.1, there exists $x_0 \in K$ such that for each $z \in K$, there exists $\xi \in T(x_0, x_0)$ satisfying

$$\langle \xi_0, \eta(z, x_0)\rangle + g(z, x_0) \notin \text{-int } C(x_0).$$

□

If we set $\eta(y,x) = y - x$ and $g(y,x) = 0$ for all $y, x \in K$, then the concept of η-pseudomonotone with respect to g reduces to C_x-pseudomonotone introduced in [16] and we also known from Proposition 2.1 of [16] that every set-valued monotone is C_x-pseudomonotone. Therefore, the following result follow directly from Corollary 4.3.

Corollary 4.4 ([4]). *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty bounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping, and $W : K \to 2^Y$ be defined by $W(x) = Y \backslash \{-int\ C(x)\}$ such that the graph $\mathcal{G}(W)$ of W is weakly closed in $X \times Y$ and W is concave. Let $T : K \times K \to 2^{L(X,Y)}$ with nonempty convex compact values and satisfies the following conditions:*

(i) T is a set valued semi-monotone mapping on K;

(iii) for each $z \in K$, the mappings $T(z, \cdot) : K \to 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X.

Then there exists $x_0 \in K$ such that for each $z \in K$, there exists $\xi \in T(x_0, x_0)$ satisfying

$$\langle \xi_0, z - x_0 \rangle \notin \text{-}int\ C(x_0).$$

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References

[1] F. Giannessi, Theory of alternative, quadratic programs and complementarity problems, in: R.W. Cottle, F. Giannessi, J.L. Lions (Eds.), Variational Inequalities and Complementarity Problems, Wiley, 1980, pp. 151–186.

[2] G.Y.Chen, G.M.Cheng, Vector variational inequality and vector optimization, in Lecture Notes in Economics and Mathematical Systems, vol. 258, Springer Verlag, NewYork, 1987, pp. 408–416.

[3] Y.Q. Chen, On the semi-monotone operator theory and applications, J. Math. Anal. Appl. 231 (1999) 177–192.

[4] Z. Fang, A generalized vector variational inequality problem with a set-valued semi-monotone mapping, Nonlinear Anal. 69 (2008) 1824–1829.

[5] Y.P. Fang, N.J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, J. Optim. Theory Appl. 118 (2003) 327–338.

[6] F. Giannessi, Vector Variational Inequalities and Vector Equilibrium, Kluwer Academic, 1999.

[7] K.L. Lin, D.P. Yang, J.C. Yao, Generalized vector variational inequalities, J. Optim. Theory Appl. 92 (1997) 117–126.

[8] X.Q. Yang, Vector variational inequality and its dual, Nonlinear Anal. 21 (1993) 869–877.

[9] F. Zheng, Vecter variational inequalities with semi-monotone operators, Journal of Global Obtimization 35 (2005) 633–642.

[10] Y. Zhao, Z. Xia, Existence results for systems of vector variational-like inequalities, Nonlinear Analysis: Real World Applications 8 (2007) 1370–1378.

[11] A.H. Siddiqi, Q.H. Ansari, R. Ahmad, On vector variational-like inequalities, Indian J. Pure Appl. Math. 26 (1995) 1135–1141.

[12] X.Q. Yang, G.Y. Chen, A class of non convex functions and variational inequalities, J. Math. Anal. Appl. 169 (1992) 359–373.

[13] N.J. Huang, Y.P. Fang, Fixed point theorems and a new system of multivalued generalized order complementarity problems, Positivity 7 (2003) 257–265.

[14] G. Kassay, J. Kolumbn, System of multi-valued variational inequalities, Publ. Math. Debrecen 56 (2000) 185–195.

[15] G. Kassay, J. Kolumbn, Z. Ples, Factorization of Minty and Stampacchia variational inequality system, Eur. J. Oper. Res. 143 (2002) 377–389.

[16] I.V. Konnov, J.C. Yao, On the generalized variational inequality problem, J. Math. Anal. Appl. 206 (1997) 42–58.

[17] Q.H. ansari, On generalized vector variational-like inequalities, Ann. Sci. Math. Quebec 19 (1995) 131–137.

[18] K. Fan, A generalization of Tychonoff's fixed-point theorem, Mathematische Annalen. 142 (1961) 305–310 .

[19] Zeidle, Nonlinear Functional Analysis and its Applications, vol. 4, Springer-Verlag, New York, Berlin, 1988.

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