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# A General Composite Algorithms for Solving General Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

**Rabian Wangkeeree**[1] **and Thanatporn Bantaojai**

Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok 65000, Thailand
e-mail : rabianw@nu.ac.th,
princess_13322@hotmail.com

**Abstract :** In this paper, we introduce a general composite algorithm for finding a common element of the set of solutions of an general equilibrium problem and the fixed point set a nonexpansive mapping in the framework of Hilbert spaces. Strong convergence of such iterative scheme is obtained which solving some variational inequalities for a strongly monotone and strictly pseudo-contractive mapping. Our results extend the corresponding recent results of Yao and Liou [Y. Yao, Y.C. Liou, Composite Algorithms for Minimization over the Solutions of Equilibrium Problems and Fixed Point Problems, Abstract and Applied Analysis, Volume 2010 (2010), Article ID 763506, 19 pages].

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## 1 Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Recall that a mapping $A : C \to H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C$. It is clear that any $\alpha$-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz

[1]Corresponding author email: rabianw@nu.ac.th (R. Wangkeeree)

continuous. Let $f : C \to H$ be a $\rho$-contraction; that is, there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of $S$ by $Fix(S)$.

Let $A : C \to H$ be a nonlinear mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction. Consider a general equilibrium problem:

$$\text{Find } z \in C \text{ such that } \phi(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of all solutions of the general equilibrium problem (1.1) is denoted by $EP$, i.e.,

$$EP = \{z \in C : \phi(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}.$$

If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$\phi(z, y) \geq 0, \forall y \in C. \tag{1.2}$$

If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.3}$$

We note that the problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, e.g., [1–4].

In 2005, Combettes and Hirstoaga [5] introduced an iterative algorithm of finding the best approximation to the initial data and proved a strong convergence theorem. In 2007, by using the viscosity approximation method, Takahashi and Takahashi [6] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed point points of a nonexpansive mapping. Subsequently, algorithms constructed for solving the equilibrium problems and fixed point problems have further developed by some authors. In particular, Ceng and Yao [7] introduced an iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of common fixed points of finitely many nonexpansive mappings. Mainge and Moudafi [8] introduced an iterative algorithm for equilibrium problems and fixed point problems. Yao et. al. [9] considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of an infinite nonexpansive mappings. Noor et. al. [10] introduced an iterative method for solving fixed point problems and variational inequality problems. Wangkeeree [11] introduced a new iterative scheme for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality. Wangkeeree and Kamraksa [12] introduced an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature. Some works related to the equilibrium problem, fixed point problems and the variational inequality problem, please see [1–54] and the references therein.

However, we note that all constructed algorithms in [6–13] do not work to find the minimum-norm solution of the corresponding fixed point problems and the equilibrium problems. Very recently, Yao and Liou [14] purposed some algorithms for finding the minimum-norm solution of the fixed point problems and the equilibrium problems. They first suggested two new composite algorithms (one implicit and one explicit) for solving the above minimization problem. To be more precisely, let $C$ be a nonempty closed convex subset of $H$, $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying certain conditions and $S : C \to C$ be a nonexpansive mapping such that $\Omega := F(S) \cap EP \neq \emptyset$. Let $f$ be a contraction on a Hilbert space $H$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \mu_n P_C[\alpha_n f(x_n) + (1-\alpha_n)Sx_n] + (1-\mu_n)u_n, n \geq 0, \end{cases} \tag{1.4}$$

where $A$ is an $\alpha$-inverse strongly monotone mapping. They proved that if $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in [0,1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

(ii) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ and $\lim_{n\to\infty} \frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$,

then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $x^* \in \Omega$ which is the unique solution of variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega.$$

In particular, if we take $f = 0$ in (1.4), then the sequence $\{x_n\}$ generated by

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \mu_n P_C[(1-\alpha_n)Sx_n] + (1-\mu_n)u_n, n \geq 0, \end{cases} \tag{1.5}$$

converges strongly to a solution of the minimization problem which is the problem of finding $x^*$ such that

$$x^* = \arg\min_{x\in\Omega} \|x\|^2 \tag{1.6}$$

where $\Omega$ stands for the intersection set of the solution set of the general equilibrium problem and the fixed points set of a nonexpansive mapping.

On the other hand, iterative approximation methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [15–17] and the references therein. Let $B$ be a strongly positive bounded linear operator on $H$: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2 \ \text{ for all } x \in H. \tag{1.7}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$ :

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.8}$$

where $b$ is a given point in $H$. In 2003, Xu [16] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n u,\ n \geq 0, \tag{1.9}$$

converges strongly to the unique solution of the minimization problem (1.8) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [18] introduced the following iterative iterative process for nonexpansive mappings (see [16] for further developments in both Hilbert and Banach spaces). Let $f$ be a contraction on $H$. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n),\ n \geq 0, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved [16, 18] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.10) strongly converges to the unique solution $x^*$ in $C$ of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, x \in H. \tag{1.11}$$

Recently, Marino and Xu [19] mixed the iterative method (1.9) and the viscosity approximation method (1.10) introduced by Moudafi [18] and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n\gamma f(x_n), n \geq 0, \tag{1.12}$$

where $B$ is a strongly positive bounded linear operator on $H$. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the certain conditions, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution $x^*$ in $H$ of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, x \in H \tag{1.13}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where $h$ is a potential function for $\gamma f(i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

Recall that a mapping $F : H \to H$ is called *$\delta$-strongly monotone* if there exists a positive constant $\delta$ such that

$$\langle Fx - Fy, x - y\rangle \geq \delta\|x - y\|^2, \forall x, y \in H. \tag{1.14}$$

Recall also that a mapping $F$ is called *$\lambda$-strictly pseudo-contractive* if there exists a positive constant $\lambda$ such that

$$\langle Fx - Fy, x - y\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Fx - Fy)\|^2, \forall x, y \in H. \tag{1.15}$$

It is easy to see that (1.15) can be rewritten as

$$\langle (I - F)x - (I - F)y, x - y\rangle \geq \lambda\|(I - F)x - (I - F)y\|^2. \tag{1.16}$$

**Remark 1.1.** *If $F$ is a strongly positive bounded linear operator on $H$ with coefficient $\bar{\gamma}$, then $F$ is $\bar{\gamma}$-strongly monotone and $\frac{1}{2}$-strictly pseudo-contractive. In fact, since $F$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, we have*

$$\langle Fx - Fy, x - y\rangle = \langle F(x - y), x - y\rangle \geq \overline{\gamma}\|x - y\|^2.$$

*Therefore, $F$ is $\overline{\gamma}$-strongly monotone. On the other hand,*

$$\begin{aligned}
&\|(I - F)x - (I - F)y\|^2 \\
&\qquad = \langle (x - y) - (Fx - Fy), (x - y) - (Fx - Fy)\rangle \\
&\qquad = \langle x - y, x - y\rangle - 2\langle Fx - Fy, x - y\rangle + \langle Fx - Fy, Fx - Fy\rangle \\
&\qquad = \|x - y\|^2 - 2\langle Fx - Fy, x - y\rangle + \|Fx - Fy\|^2 \\
&\qquad \leq \|x - y\|^2 - 2\langle Fx - Fy, x - y\rangle + \|F\|^2\|x - y\|^2.
\end{aligned} \tag{1.17}$$

*Since $F$ is strongly positive if and only if $\frac{1}{\|F\|}F$ is strongly positive, we may assume, with out loss of generality, that $\|F\| = 1$. From* (1.17)*, we have*

$$\begin{aligned}
\langle Fx - Fy, x - y\rangle &\leq \|x - y\|^2 - \frac{1}{2}\|(I - F)x - (I - F)y\|^2 \\
&= \|x - y\|^2 - \frac{1}{2}\|(x - y) - (Fx - Fy)\|^2.
\end{aligned}$$

*Hence $F$ is $\frac{1}{2}$-strictly pseudo-contractive.*

In this paper, motivated by the above results, we introduce a general iterative scheme below in a real Hilbert space $H$, with the initial guess $x_0 \in C$ chosen arbitrary,

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ x_{n+1} = \mu_n P_C[\alpha_n\gamma f(x_n) + (1 - \alpha_n F)Sx_n] + (1 - \mu_n)u_n, n \geq 0. \end{cases} \tag{1.18}$$

where $C$ is a nonempty closed and convex subset of $H$, $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in [0,1], $\phi : C \times C \to \mathbb{R}$ is a bifunction satisfying certain conditions,

$S : C \to C$ is a nonexpansive mapping such that $\Omega := Fix(S) \cap EP \neq \emptyset$, $f : C \to H$ is a contraction with coefficient $0 < \rho < 1$, $F$ is $\delta$-strongly monotone and $\lambda$-strictly pseudo-contractive with $\delta + \lambda > 1$, $\gamma$ is a positive real number such that $\gamma < \frac{1}{\rho}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$ and $A$ is an $\alpha$-inverse strongly monotone mapping. We prove that the proposed algorithm converges strongly to $x^* \in \Omega$ which is the unique solution of the following variational inequality

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \qquad x \in \Omega.$$

In particular,

(I) if $F$ is a strongly positive bounded linear operator on $H$, then $x^*$ is the unique solution of the variational inequality (1.13);

(II) if $F = I$, the identity mapping on $H$ and $\gamma = 1$, then $x^*$ is the unique solution of the variational inequality (1.11);

(III) if $F = I$, the identity mapping on $H$ and $f = 0$, then $x^*$ is the unique solution of minimization problem (1.6).

The results presented in this paper extend and improve the main results in Yao and Liou [14], Marino and Xu [19] and many others.

## 2 Preliminaries

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

$P_C$ is called the *metric projection* of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H, y \in C$. For more details see [20].

Throughout this paper, we assume that a bifunction $\phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;

(A2) $\phi$ is monotone, i,e., $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y)$;

(A4) for each $x \in C$, the mapping $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

**Lemma 2.1** ([5]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

*Further, if $T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:*

*(i) $T_r$ is single-valued and $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

*(ii) $EP$ is closed and convex and $EP = Fix(T_r)$.*

**Lemma 2.2** ([21]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let the mapping $A : C \to H$ be $\alpha$-inverse strongly monotone and $r > 0$ be a constant. Then, we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in C.$$

*In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.*

**Lemma 2.3** ([22]). *Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in $C$ such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

**Lemma 2.4** ([16]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n \gamma_n,$$

*where $\{\gamma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that*

*(1)* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

*(2)* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

The following lemma can be found in [23, Lemma 2.7]. For the sake of the completeness, we include its proof in a Hilbert space'version.

**Lemma 2.5.** *Let $H$ be a real Hilbert space and $F : H \to H$ a mapping.*

*(i)* *If* $F$ *is* $\delta$*-strongly monotone and* $\lambda$*-strictly pseudo-contractive with* $\delta+\lambda>1$*, then* $I-F$ *is contractive with constant* $\sqrt{(1-\delta)/\lambda}$*.*

*(ii)* *If* $F$ *is* $\delta$*-strongly monotone and* $\lambda$*-strictly pseudo-contractive with* $\delta+\lambda>1$*, then for any fixed number* $\tau\in(0,1)$*,* $I-\tau F$ *is contractive with constant* $1-\tau(1-\sqrt{(1-\delta)/\lambda})$*.*

*Proof.* (i) For any $x,y\in H$, we have

$$\lambda\|(I-F)x-(I-F)y\|^2\le\|x-y\|^2-\langle Fx-Fy,x-y\rangle\le(1-\delta)\|x-y\|^2,\forall x,y\in H.$$

Thus

$$\|(I-F)x-(I-F)y\|\le\sqrt{\frac{1-\delta}{\lambda}}\|x-y\|,\ \text{for all } x,y\in H.$$

Since $\delta+\lambda>1$, we have $\frac{1-\delta}{\lambda}\in(0,1)$. Hence $I-F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.

(ii) Since $I-F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$, we have for any $\tau\in(0,1)$,

$$\begin{aligned}\|x-y-\tau(Fx-Fy)\| &= \|(1-\tau)(x-y)+\tau[(I-F)x-(I-F)y]\|\\ &\le (1-\tau)\|x-y\|+\tau\|(I-F)x-(I-F)y\|\\ &\le (1-\tau)\|x-y\|+\tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\|\\ &= \left(1-\tau\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|,\ \text{for all } x,y\in H.\end{aligned}$$

Hence $I-\tau F$ is contractive with constant $1-\tau(1-\sqrt{(1-\delta)/\lambda})$. □

# 3 Main Results

Now, we give a lemma in order to prove our main results.

**Lemma 3.1.** *Let* $C$ *be a nonempty closed and convex subset of a real Hilbert space* $H$*. Let* $S:C\to C$ *be a nonexpansive mapping and* $A:C\to H$ *be an* $\alpha$*-inverse strongly monotone mapping. Let* $\phi:C\times C\to\mathbb{R}$ *be a bifunction which satisfies conditions (A1)-(A4) such that* $\Omega:=EP\cap Fix(S)$ *is nonempty. Let* $F:C\to H$ *be* $\delta$*-strongly monotone and* $\lambda$*-strictly pseudo-contractive with* $\delta+\lambda>1$*,* $f:C\to H$ *a* $\rho$*-contraction,* $\gamma$ *a positive real number such that* $\gamma<(1-\sqrt{(1-\delta)/\lambda})/\rho$ *and* $r$ *a constant such that* $r\in(0,2\alpha)$*. For given* $x_0\in C$ *arbitrarily, let the sequence* $\{x_n\}$ *be generated iteratively by* (1.18)*. Suppose that* $\{\alpha_n\}$ *and* $\{\mu_n\}$ *are two sequences in* $[0,1]$*. Then, the sequence* $\{x_n\}$ *is bounded. Furthermore, if the sequences* $\{\alpha_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*

*(C1)* $\lim_{n\to\infty}\alpha_n=0$ *and* $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$*;*

*(C2)* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ *and* $\lim_{n\to\infty} \frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$,

*then* $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

*Proof.* First, we rewrite the sequence $\{x_n\}$ by the following :

$$x_{n+1} = \mu_n P_C[\alpha_n \gamma f(x_n) + (1-\alpha_n F)Sx_n] + (1-\mu_n)T_r(x_n - rAx_n), n \geq 0, \quad (3.1)$$

where the mapping $T_r$ is defined in Lemma 2.1. Pick $z \in \Omega$. Let

$$u_n = T_r(x_n - rAx_n) \text{ and } y_n = \alpha_n \gamma f(x_n) + (1-\alpha_n F)Sx_n$$

for all $n \geq 0$. The nonexpansivity of $T_r$ and $I - rA$ implies that

$$\begin{aligned} \|u_n - z\| &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\| \\ &\leq \|x_n - z\| \text{ for all } z \in \Omega. \end{aligned} \quad (3.2)$$

Applying Lemma 2.5, we can calculate the following

$$\begin{aligned} \|x_{n+1} - z\| &= \|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\| \\ &\leq \mu\|P_C[y_n] - z\| + (1-\mu_n)\|u_n - z\| \\ &\leq \mu_n\|y_n - z\| + (1-\mu_n)\|x_n - z\| \\ &= \mu_n\|\alpha_n\gamma f(x_n) + (I-\alpha_n F)Sx_n - z\| + (1-\mu_n)\|x_n - z\| \\ &= \mu_n\|\alpha_n\gamma f(x_n) + (I-\alpha_n F)Sx_n - (I-\alpha_n F)z - \alpha_n F(z)\| \\ &\quad + (1-\mu_n)\|x_n - z\| \\ &\leq \mu_n\alpha_n\|\gamma f(x_n) - F(z)\| + \mu_n\|(I-\alpha_n F)Sx_n - (I-\alpha_n F)z\| \\ &\quad + (1-\mu_n)\|x_n - z\| \\ &\leq \mu_n\alpha_n\|\gamma f(x_n) - \gamma f(z)\| + \mu_n\alpha_n\|\gamma f(z) - F(z)\| \\ &\quad + \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\| + (1-\mu_n)\|x_n - z\| \\ &\leq \mu_n\alpha_n\gamma\rho\|x_n - z\| + \mu_n\alpha_n\|\gamma f(z) - F(z)\| \\ &\quad + \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\| + (1-\mu_n)\|x_n - z\| \\ &= \left(1 - \mu_n\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\rho\right)\right)\|x_n - z\| \\ &\quad + \mu_n\alpha_n\|\gamma f(z) - F(z)\| \\ &= \left(1 - \mu_n\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\rho\right)\right)\|x_n - z\| \\ &\quad + \frac{\mu_n\alpha_n(1 - \sqrt{(1-\delta)/\lambda} - \gamma\rho)}{(1 - \sqrt{(1-\delta)/\lambda} - \gamma\rho)}\|\gamma f(z) - F(z)\|. \end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{\|\gamma f(z) - F(z)\|}{(1 - \sqrt{(1-\delta)/\lambda} - \gamma\rho)}\right\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{f(x_n)\}$, $\{Sx_n\}$ and $\{y_n\}$ are all bounded.

Next, assume that the control conditions (C1) and (C2) hold. From (3.1), we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\mu_{n+1}P_C[y_{n+1}]+(1-\mu_{n+1})u_{n+1}-\mu_nP_C[y_n]-(1-\mu_n)u_n\| \\
&= \|\mu_{n+1}(P_C[y_{n+1}]-P_C[y_n])+(\mu_{n+1}-\mu_n)P_C[y_n] \\
&\quad +(1-\mu_{n+1})(u_{n+1}-u_n)+(\mu_n-\mu_{n+1})u_n\| \\
&\leq \mu_{n+1}\|y_{n+1}-y_n\|+(1-\mu_{n+1})\|u_{n+1}-u_n\| \\
&\quad +|\mu_{n+1}-\mu_n|(\|P_C[y_n]\|+\|u_n\|).
\end{aligned}$$

Now, we estimate $\|y_{n+1}-y_n\|$ and $\|u_{n+1}-u_n\|$. We have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}F)Sx_{n+1}-\alpha_n\gamma f(x_n)-(I-\alpha_nF)Sx_n\| \\
&= \|(\alpha_{n+1}\gamma f(x_{n+1})-\alpha_{n+1}\gamma f(x_n))+(\alpha_{n+1}\gamma f(x_n)-\alpha_n\gamma f(x_n)) \\
&\quad +[(I-\alpha_{n+1}F)Sx_{n+1}-(I-\alpha_{n+1}F)Sx_n] \\
&\quad +[(I-\alpha_{n+1}F)Sx_n-(I-\alpha_nF)Sx_n]\| \\
&\leq \alpha_{n+1}\gamma\rho\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\| \\
&\quad +\left(1-\alpha_{n+1}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n+1}-x_n\|+|\alpha_n-\alpha_{n+1}|\|F(Sx_n)\| \\
&= \left(1-\alpha_{n+1}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n+1}-x_n\|+\alpha_{n+1}\gamma\rho\|x_{n+1}-x_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|F(Sx_n)\|),
\end{aligned}$$

and

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_r(x_{n+1}-rAx_{n+1})-T_r(x_n-rAx_n)\| \\
&\leq \|(x_{n+1}-rAx_{n+1})-(x_n-rAx_n)\| \\
&\leq \|x_{n+1}-x_n\|.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \mu_{n+1}\left(1-\alpha_{n+1}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n+1}-x_n\| \\
&\quad +\mu_{n+1}\alpha_{n+1}\gamma\rho\|x_{n+1}-x_n\| \\
&\quad +\mu_{n+1}|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|F(Sx_n)\|) \\
&\quad +(1-\mu_{n+1})\|x_{n+1}-x_n\|+|\mu_{n+1}-\mu_n|(\|P_C[y_n]\|+\|u_n\|) \\
&\leq \left(1-\alpha_{n+1}\mu_{n+1}\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\rho\right)\right)\|x_{n+1}-x_n\| \\
&\quad +(|\alpha_{n+1}-\alpha_n|+|\mu_{n+1}-\mu_n|)M,
\end{aligned}$$

where $M>0$ is a constant satisfying

$$\sup_n\{\mu_{n+1}(\gamma\|f(x_n)\|+\|F(Sx_n)\|),\|P_C[y_n]\|+\|u_n\|\}\leq M.$$

This together with (C1), (C2) and Lemma 2.4 implies that

$$\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0. \tag{3.3}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\|^2 \\
&\leq \mu_n\|P_C[y_n] - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\|y_n - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n\|\alpha_n\gamma f(x_n) + (I - \alpha_n F)Sx_n - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n\|\alpha_n\gamma f(x_n) - \alpha_n F(z) + (I - \alpha_n F)Sx_n - (I - \alpha_n F)z\|^2 \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\|(I - \alpha_n F)Sx_n - (I - \alpha_n F)z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\mu_n\alpha_n\langle (I - \alpha_n F)Sx_n - (I - \alpha_n F)z, \gamma f(x_n) - F(z)\rangle \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)\|u_n - z\|^2. \qquad (3.4)
\end{aligned}$$

From Lemma 2.2, we get

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\
&\leq \|(x_n - rAx_n) - (z - rAz)\|^2 \\
&\leq \|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2. \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.4), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)[\|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2]
\end{aligned}$$

$$
\begin{aligned}
= \quad & \left(1-\alpha_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-z\|^2+\mu_n\alpha_n^2\|\gamma f(x_n)-F(z)\|^2 \\
& +2\alpha_n\mu_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n)-F(z)\|\|x_n-z\| \\
& +(1-\mu_n)r(r-2\alpha)\|Ax_n-Az\|^2. \qquad (3.6)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(1-\mu_n)&r(2\alpha-r)\|Ax_n-Az\|^2 \\
&\le \left(1-\alpha_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-z\|^2-\|x_{n+1}-z\|^2 \\
&\quad +\mu_n\alpha_n^2\|\gamma f(x_n)-F(z)\|^2 \\
&\quad +2\alpha_n\mu_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n)-F(z)\|\|x_n-z\| \\
&\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\mu_n\alpha_n^2\|\gamma f(x_n)-F(z)\|^2 \\
&\quad +2\alpha_n\mu_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n)-F(z)\|\|x_n-z\| \\
&\le (\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|+\mu_n\alpha_n^2\|\gamma f(x_n)-F(z)\|^2 \\
&\quad +2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n)-F(z)\|\|x_n-z\|.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\mu_n)r(2\alpha-r)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we derive

$$\lim_{n\to\infty}\|Ax_n-Az\|=0.$$

From Lemma 2.1, we obtain

$$
\begin{aligned}
\|u_n-z\|^2 &= \|T_r(x_n-rAx_n)-T_r(z-rAz)\|^2 \\
&\le \langle (x_n-rAx_n)-(z-rAz),u_n-z\rangle \\
&= \frac{1}{2}\Big(\|(x_n-rAx_n)-(z-rAz)\|^2+\|u_n-z\|^2 \\
&\qquad -\|(x_n-z)-r(Ax_n-Az)-(u_n-z)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n-z\|^2+\|u_n-z\|^2 \\
&\qquad -\|(x_n-u_n)-r(Ax_n-Az)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2 \\
&\qquad +2r\langle x_n-u_n,Ax_n-Az\rangle-r^2\|Ax_n-Az\|^2\Big).
\end{aligned}
$$

Thus, we deduce

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|. \quad (3.7)$$

By (3.4) and (3.7), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)[\|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|] \\
&\leq \left(1 - \alpha_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)[-\|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|].
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\mu_n)\|x_n - u_n\|^2 \\
&\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)[2r\|x_n - u_n\|\|Ax_n - Az\|] \\
&\leq (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + 2r(1-\mu_n)\|x_n - u_n\|\|Ax_n - Az\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\mu_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Az\| \to 0$, we derive that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \quad (3.8)$$

Note that $x_{n+1} - x_n = \mu_n(P_C[y_n] - x_n) + (1-\mu_n)(u_n - x_n)$. Using (3.3), (3.8) and the condition (C2), we have

$$\|P_C[y_n] - x_n\| \to 0.$$

Therefore,

$$\begin{aligned}
\|Sx_n - x_n\| &\leq \|Sx_n - P_C[y_n]\| + \|P_C[y_n] - x_n\| \\
&\leq \|Sx_n - y_n\| + \|P_C[y_n] - x_n\| \\
&\leq \alpha_n\|\gamma f(x_n) - F(Sx_n)\| + \|P_C[y_n] - x_n\| \to 0. \quad (3.9)
\end{aligned}$$

This completes the proof. □

Now we show the strong convergence of the sequence $\{x_n\}$ generated by (1.18).

**Theorem 3.2.** *Let $C, H, S, A, \phi, F, \Omega, f, r$ and $\{x_n\}$ be as in Lemma 3.1. Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions :*

*(D1)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

*(D2)* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ *and* $\lim_{n\to\infty} \frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$.

*Then the sequence $\{x_n\}$ converges strongly to $x^*$ of the following variational inequality*

$$\langle (F-\gamma f)x^*, x-x^*\rangle \geq 0, \qquad x \in \Omega \tag{3.10}$$

*or equivalently $x^* = P_\Omega(I - F + \gamma f)x^*$, where $P_\Omega$ is the metric projection of $H$ onto $\Omega$.*

*Proof.* Let $\Phi = P_\Omega$. Then $\Phi(I - F - \gamma f)$ is a contraction on $C$. In fact, from Lemma 2.5 (i), we have

$$\begin{aligned}
&\|\Phi(I-F-\gamma f)x - \Phi(I-F-\gamma f)y\| \\
&\leq \|(I-F-\gamma f)x - (I-F-\gamma f)y\| \\
&\leq \|(I-F)x - (I-F)y\| + \gamma\|f(x) - f(y)\| \\
&\leq \sqrt{\frac{1-\delta}{\lambda}}\|x-y\| + \alpha\gamma\|x-y\| \\
&= \left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right)\|x-y\|, \text{ for all } x, y \in C.
\end{aligned} \tag{3.11}$$

Therefore, $\Phi(I - F - \gamma f)$ is a contraction on $C$ with coefficient $\left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right) \in (0, 1)$. Thus, by Banach contraction principal, $P_\Omega(I - F - \gamma f)$ has a uninique fixed point $x^*$. That is $P_\Omega(I - F - \gamma f)x^* = x^*$ which mean that $x^*$ is the unique solution in $\Omega$ of the variational inequality (3.10). Next, we show that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), Sx_n - x^*\rangle \leq 0. \tag{3.12}$$

Indeed, we can choose a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), Sx_n - x^*\rangle = \limsup_{k\to\infty}\langle \gamma f(x^*) - F(x^*), Sx_{n_k} - x^*\rangle. \tag{3.13}$$

Without loss of generality, we may further assume that $Sx_{n_k} \to \tilde{x}$ weakly. This together with Lemma 3.1, we have $x_{n_k} \to \tilde{x}$ weakly. Applying Lemma 3.1 and Lemma 2.3, we obtain $\tilde{x} \in Fix(S)$. Next we show $\tilde{x} \in EP$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$ we have

$$\phi(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0.$$

From the monotonicity of $F$, we have

$$\frac{1}{r}\langle y-u_n, u_n-(x_n-rAx_n)\rangle \geq \phi(y,u_n), \forall y \in C.$$

Hence,

$$\langle y-u_n, \frac{u_{n_i}-x_{n_i}}{r}+Ax_{n_i}\rangle \geq \phi(y,u_{n_i}), \forall y \in C. \tag{3.14}$$

Put $z_t = ty+(1-t)\tilde{x}$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.14) we have

$$\begin{aligned}
\langle z_t-u_{n_i}, Az_t\rangle &\geq \langle z_t-u_{n_i}, Az_t\rangle - \langle z_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r}+Ax_{n_i}\rangle \\
&\quad + \phi(z_t,u_{n_i}) \\
&= \langle z_t-u_{n_i}, Az_t-Au_{n_i}\rangle + \langle z_t-u_{n_i}, Au_{n_i}-Ax_{n_i}\rangle \\
&\quad + \langle z_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r}\rangle + \phi(z_t,u_{n_i}).
\end{aligned} \tag{3.15}$$

Note that $\|Au_{n_i}-Ax_{n_i}\| \leq \frac{1}{\alpha}\|u_{n_i}-x_{n_i}\| \to 0$. Further, from monotonicity of $A$, we have $\langle z_t-u_{n_i}, Az_t-Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.15), we have

$$\langle z_t-\tilde{x}, Az_t\rangle \geq \phi(z_t,\tilde{x}). \tag{3.16}$$

From (A1), (A4) and (3.16), we also have

$$\begin{aligned}
0 &= \phi(z_t,z_t) \leq t\phi(z_t,y)+(1-t)\phi(z_t,\tilde{x}) \\
&\leq t\phi(z_t,y)+(1-t)\langle z_t-\tilde{x}, Az_t\rangle \\
&= t\phi(z_t,y)+(1-t)t\langle y-\tilde{x}, Az_t\rangle
\end{aligned}$$

and hence

$$0 \leq \phi(z_t,y)+(1-t)\langle Az_t, y-\tilde{x}\rangle. \tag{3.17}$$

Letting $t \to 0$ in (3.17), we have, for each $y \in C$,

$$0 \leq \phi(\tilde{x},y)+\langle y-\tilde{x}, A\tilde{x}\rangle. \tag{3.18}$$

This implies that $\tilde{x} \in EP$. Therefore, $\tilde{x} \in \Omega$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(x^*)-F(x^*), Sx_n-x^*\rangle = \langle \gamma f(x^*)-F(x^*), \tilde{x}-x^*\rangle \leq 0.$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. From Lemma 2.5 and (1.18), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\mu_n(P_C[y_n] - x^*) + (1-\mu_n)(u_n - x^*)\|^2 \\
&\leq \mu_n\|P_C[y_n] - x^*\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu_n\|y_n - x^*\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu\|\alpha_n\gamma f(x_n) - \alpha_n F(x^*) + (I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*\|^2 \\
&\quad + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu_n\|(I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*\|^2 + \alpha_n^2\|\gamma f(x_n) - F(x^*)\|^2 \\
&\quad + 2\alpha_n\mu_n\langle (I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*, \gamma f(x_n) - \gamma f(x^*)\rangle \\
&\quad + 2\alpha_n\mu_n\langle (I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu_n\|(I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*\|^2 + \alpha_n^2\|\gamma f(x_n) - F(x^*)\|^2 \\
&\quad + 2\alpha_n\mu_n\|(I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*\|\|\gamma f(x_n) - \gamma f(x^*)\| \\
&\quad + 2\alpha_n\mu_n\langle (I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2\|x_n - x^*\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(x^*)\|^2 \\
&\quad + 2\alpha_n\mu_n\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\mu_n\langle (I-\alpha_n F)Sx_n - (I-\alpha_n F)x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + (1-\mu_n)\|u_n - x^*\|^2 \\
&\leq \mu_n\left(1-2\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right) + 2\alpha_n\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right) \\
&\quad \times\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle Sx_n - x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + 2\alpha_n^2\mu_n\langle F(x^*) - F(Sx_n), \gamma f(x^*) - F(x^*)\rangle \\
&\quad + \mu_n\alpha_n^2[\|x_n - x^*\|^2 + \|\gamma f(x_n) - F(x^*)\|^2] + (1-\mu_n)\|x_n - x^*\|^2 \\
&= \left(1-2\alpha_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right) + 2\alpha_n\gamma\rho\mu_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right) \\
&\quad \times\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle Sx_n - x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + 2\alpha_n^2\mu_n\langle F(x^*) - F(Sx_n), \gamma f(x^*) - F(x^*)\rangle \\
&\quad + \mu_n\alpha_n^2[\|x_n - x^*\|^2 + \|\gamma f(x_n) - F(x^*)\|^2] \\
&= \left(1-2\alpha_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}} - \gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right)\right) \\
&\quad \times\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle Sx_n - x^*, \gamma f(x^*) - F(x^*)\rangle \\
&\quad + 2\alpha_n^2\mu_n\langle F(x^*) - F(Sx_n), \gamma f(x^*) - F(x^*)\rangle \\
&\quad + \mu_n\alpha_n^2[\|x_n - x^*\|^2 + \|\gamma f(x_n) - F(x^*)\|^2] \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \delta_n\gamma_n,
\end{aligned}$$

where $\gamma_n = 2\alpha_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right)$ and
$\delta_n = \frac{\langle Sx_n-x^*,\gamma f(x^*)-F(x^*)\rangle}{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right)} + \frac{\alpha_n\langle F(x^*)-F(Sx_n),\gamma f(x^*)-F(x^*)\rangle}{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right)}$
$+\frac{\alpha_n[\|x_n-x^*\|^2+\|\gamma f(x_n)-F(x^*)\|^2]}{2\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\rho\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\right)}$. It clear that $\sum_{n=0}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\delta_n\leq 0$. Hence, all conditions of Lemma 2.4 are satisfied. Therefore, $x_n\to x^*$. This completes the proof. □

The following example shows that there exist the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfying the conditions (D1) and (D2) of Theorem 3.2.

**Example 3.1.** *For each $n\geq 0$, let $\alpha_n=\frac{1}{n+1}$ and $\mu_n=\frac{1}{2}+\frac{1}{n+1}$. Then, it is easily to obtain $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$, $0<\frac{1}{2}=\liminf_{n\to\infty}\mu_n\leq\limsup_{n\to\infty}\mu_n=\frac{1}{2}<1$ and $\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}}=0$. Hence conditions (D1) and (D2) of Theorem 3.2 are satisfied.*

**Corollary 3.3.** *Let $C,H,S,A,\phi,\Omega,f,r$ be as in Lemma 3.1. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases}\phi(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall y\in C,\\ x_{n+1}=\mu_nP_C[\alpha_n\gamma f(x_n)+(1-\alpha_nB)Sx_n]+(1-\mu_n)u_n, n\geq 0.\end{cases}$$

*where $B$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma}>\frac{1}{2}$ and $0<\gamma<(1-\sqrt{2-2\overline{\gamma}})/\rho$. Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions :*

*(D1)* $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ *and* $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$;

*(D2)* $0<\liminf_{n\to\infty}\mu_n\leq\limsup_{n\to\infty}\mu_n<1$ *and* $\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}}=0$.

*Then the sequence $\{x_n\}$ converges strongly to $x^*$ of the following variational inequality*

$$\langle(B-\gamma f)x^*,x-x^*\rangle\geq 0,\qquad x\in\Omega$$

*or equivalently $\tilde{x}=P_\Omega(I-B+\gamma f)\tilde{x}$, where $P_\Omega$ is the metric projection of $H$ onto $\Omega$.*

Setting $\gamma=1$ and $F=I$, the identity mapping on $C$ in Theorem 3.2, we obtain the following result.

**Corollary 3.4** ([14, Theorem 3.7])**.** *Let $C,H,S,A,\phi,\Omega,f,r$ be as in Lemma 3.1. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases}\phi(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall y\in C,\\ x_{n+1}=\mu_nP_C[\alpha_nf(x_n)+(1-\alpha_n)Sx_n]+(1-\mu_n)u_n, n\geq 0.\end{cases}$$

*Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions :*

*(D1)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$;

*(D2)* $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ *and* $\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$.

*Then the sequence* $\{x_n\}$ *converges strongly to* $x^*$ *of the following variational inequality*

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \qquad x \in \Omega.$$

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