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Generalized Mixed Equilibrium Problems for Maximal Monotone Operators and Two Relatively Quasi-Nonexpansive Mappings[1]

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Abstract : In this paper, we prove the strong convergence theorems of modified hybrid projection methods for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of solution of the variational inequality operators of an inverse strongly monotone, the zero point of a maximal monotone operator and the set of fixed point of two relatively quasi-nonexpansive mappings in a Banach space. Our results modify and improve the recently ones announced by many authors.

Keywords : Strong convergence; Hybrid projection methods; Generalized mixed equilibrium problem; Variational inequality operators; Inverse-strongly monotone; Maximal monotone operator; Relatively quasi-nonexanasive mappings.

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1 Introduction

Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction, $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function, and $B : C \longrightarrow E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem*, is to find $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions to (1.1) is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}. \tag{1.2}$$

If $B = 0$, the problem (1.1) reduce into the *mixed equilibrium problem for* Θ, denoted by $MEP(\Theta, \varphi)$, is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $\Theta \equiv 0$, the problem (1.1) reduce into the *mixed variational inequality* of Browder type, denoted by $MVI(C, B, \varphi)$, is to find $x \in C$ such that

$$\langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $B = 0$ and $\varphi = 0$ the problem (1.1) reduce into the *equilibrium problem for* Θ, denoted by $EP(\Theta)$, is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The above formulation (1.5) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\Theta)$. In other words, the $EP(\Theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\Theta)$; see, for example [1–4] and references therein. Some solution methods have been proposed to solve the $EP(\Theta)$; see, for example, [3–7] and references therein. In 2005, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Theta)$ is nonempty and they also proved a strong convergence theorem.

Let E be a Banach space with norm $\|\cdot\|$, C be a nonempty closed convex subset of E and let E^* denote the dual of E. Let B be a *monotone* operator of C into E^*. The *variational inequality problem* is to find a point $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \text{for all} \quad y \in C. \tag{1.6}$$

The set of solutions of the variational inequality problem is denoted by $VI(C, B)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Bu$ and so on. An operator B of C into E^* is said to be *inverse-strongly monotone*, if there exists a positive real number α such that

$$\langle x-y, Bx-By\rangle \geq \alpha\|Bx-By\|^2 \tag{1.7}$$

for all $x, y \in C$. In such a case, B is said to be *α-inverse-strongly monotone*. If an operator B of C into E^* is α-inverse-strongly monotone, then B is *Lipschitz continuous*, that is $\|Bx-By\| \leq \frac{1}{\alpha}\|x-y\|$ for all $x, y \in C$.

In Hilbert space H, Iiduka et al. [8] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Bx_n), \tag{1.8}$$

where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive real numbers, converges weakly to some element of $VI(C,B)$.

In 2008, Iiduka and Takahashi [9] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator B in a Banach space: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \tag{1.9}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to some element of $VI(C,B)$.

Consider the problem of finding:

$$v \in E \quad \text{such that} \quad 0 \in A(v), \tag{1.10}$$

where A is an operator from E into E^*. Such $v \in E$ is called a *zero point* of A. When A is a maximal monotone operator, a well-know methods for solving (1.10) in a Hilbert space H is the *proximal point algorithm*: $x_1 = x \in H$ and,

$$x_{n+1} = J_{r_n}x_n, \quad n = 1, 2, 3, \ldots, \tag{1.11}$$

where $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_n A)^{-1}$, then Rockafellar [10] proved that the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}(0)$.

In 2008, Li and Song [11] proved a strong convergence theorem in a Banach space, by the following algorithm: $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}x_n)), \\ x_{n+1} &= J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)J(y_n)), \end{aligned} \tag{1.12}$$

with the coefficient sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\longrightarrow\infty}\beta_n = 0$, and $\lim_{n\longrightarrow\infty} r_n = \infty$, where J is the duality mapping from E into E^* and $J_r = (I + rT)^{-1}J$. Then they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_C x$, where Π_C is the generalized projection from E onto C.

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty (see [12]). A mapping S is said to be *quasi-nonexpansive* if $F(S) \neq \emptyset$ and $\|Sx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(S)$. It is easy to see that if S is nonexpansive with $F(S) \neq \emptyset$, then it is quasi-nonexpansive. We write $x_n \longrightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. Let E be a real Banach space with norm $\|\cdot\|$ and let J be the n*ormalized duality mapping* from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Let C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of S [13] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widetilde{F(S)}$. A mapping S from C into itself is said to be *relatively nonexpansive* [14–16] if $\widetilde{F(S)} = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [17, 18]. S is said to be ϕ-nonexpansive, if $\phi(Sx, Sy) \leq \phi(x, y)$ for $x, y \in C$. S is said to be *relatively quasi-nonexpansive* if $F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(S)$. Recall that an operator S in a Banach space is call *closed*, if $x_n \longrightarrow x$ and $Sx_n \longrightarrow y$, then $Sx = y$.

In 2008, Takahashi and Zembayashi [19] introduced the following shrinking projection method of closed relatively nonexpansive mappings as follow:

$$\begin{cases} x_0 = x \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) JS(x_n)), \\ u_n \in C \ \text{ such that } \ \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases} \tag{1.13}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\longrightarrow\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP(\Theta)} x$. Qin and Su [20] proved the following iteration for relatively nonexpansive mappings T in a Banach space E:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{1.14}$$

the sequence $\{x_n\}$ generated by (1.14) converges strongly to $\Pi_{F(T)}x_0$. In 2009, Cholamjiak [21], proved the following iteration

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n), \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.15}$$

where J is the duality mapping on E. Assume that α_n, β_n and γ_n are sequence in $[0,1]$. Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$, where $F := F(T) \cap F(S) \cap EP(\Theta) \cap VI(A,C)$. Moreover, Saewan et al. [22] proved the strong convergence for two relatively quasi-nonexpansive mappings in a Banach space under certain appropriate conditions. In 2009, Ceng et al. [23] proved the following strong convergence theorem for finding a common element of the set of solutions for an equilibrium and the set of a zero point for a maximal monotone operator T in a Banach space E,

$$\begin{cases} y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}(x_n))), \\ H_n = \{z \in C : \phi(z, T_{r_n}y_n) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0. \end{cases} \tag{1.16}$$

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0 \cap EP(\Theta)}x_0$, where $\Pi_{T^{-1}0 \cap EP(\Theta)}$ is the generalized projection of E onto $A^{-1}0 \cap EP(\Theta)$.

Recently, Inoue et al. [24] proved the strong convergence for finding a common fixed point set of relatively nonexpansive mappings and the zero point set of maximal monotone operators in Banach spaces E: $x_0 = x \in C$ and

$$\begin{cases} y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)JTJ_{r_n}(x_n))), \\ C_n = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x. \end{cases} \tag{1.17}$$

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection of E onto $F(T) \cap A^{-1}0$.

In this paper, motivated and inspired by Li and Song [11], Iiduka and Takahashi [9], Takahashi and Zembayashi [19], Ceng et al. [23] and Inoue et al. [24], we introduce the new hybrid algorithm defined by: $x_1 = x \in C$ and

$$\begin{cases} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)JT(J_{r_n}w_n)), \\ y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)JSz_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x, \quad \forall n \geq 1. \end{cases} \tag{1.18}$$

Under appropriate conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (1.18) converges strongly to the point $\Pi_{F(T)\cap F(S)\cap VI(C,B)\cap A^{-1}(0)\cap\Omega}x$. The results presented in this paper extend and improve the corresponding ones announced by Li and Song [11], Inoue et al. [24] and many authors in the literature.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}. \tag{2.1}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [25, 26] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no Banach space is p-uniformly convex for $1 < p < 2$. It is well known that a Hilbert space is *2*-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{2.2}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [27]):

(1) if E is smooth, then J is single-valued;

(2) if E is strictly convex, then J is one-to-one and $\langle x-y, x^*-y^*\rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$;

(3) if E is reflexive, then J is surjective;

(4) if E is uniformly convex, then it is reflexive;

(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [28] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.1 ([29, 30]). *If E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$ we have*

$$\|x-y\| \leq \frac{2}{c^2}\|Jx-Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *2*-uniformly convex constant of E; see [25].

Lemma 2.2 ([29, 31]). *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x-y, Jx-Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x-y\|^p,$$

where J_p is the generalized duality mapping of E and $\dfrac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.3 (Xu [30]). *Let E be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \longrightarrow [0,\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda y)\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.3}$$

for all $x, y \in \{z \in E : \|z\| \leq r\}$ and $\lambda \in [0,1]$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E. \tag{2.4}$$

Following Alber [32], the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x) \tag{2.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function ϕ that (see [32])

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.6}$$

If E is a Hilbert space, then $\phi(x,y) = \|x - y\|^2$.

If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.6), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [27, 33] for more details.

Lemma 2.4 (Kamimura and Takahashi [34]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5 (Mutsushita and Takahashi [35]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.6 (Alber [32]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 (Alber [32]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x), \quad \forall y \in C.$$

Let E be a strictly convex, smooth and reflexive Banach space, let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V : E \times E^* \longrightarrow \mathbb{R}$ as follows (see [36]):

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.7}$$

for all $x \in Ex \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x,x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x,y)$.

Lemma 2.8 (Kohsaka and Takahashi [36, Lemma 3.2]). *Let E be a strictly convex, smooth and reflexive Banach space, and let V be as in (2.7). Then*

$$V(x,x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.8}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E. Because $\phi(x,y)$ is strictly convex and coercive in the first variable, we know that the minimization problem $\inf_{y\in C}\phi(x,y)$ has a unique solution. The operator $\Pi_C x := \arg\min_{y\in C}\phi(x,y)$ is said to be the generalized projection of x on C.

A set-valued mapping $A : E \longrightarrow E^*$ with domain $D(A) = \{x \in E : A(x) \neq \emptyset\}$ and range $R(A) = \{x^* \in E^* : x^* \in A(x), x \in D(A)\}$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x^* \in A(x), y^* \in A(y)$. We denote the set $\{x \in E : 0 \in Ax\}$ by $A^{-1}0$. A is *maximal* monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is knows that A is a maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.

Define the *resolvent* of A by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(A)$. Also, $A^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of A by $A_r = (J - JJ_r)/r$. We know that $A_r x \in A(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 2.9 (Kohsaka and Takahashi [36, Lemma 3.1]). *Let E be a smooth, strictly convex and reflexive Banach space, let $A \subset E\times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, let $r > 0$ and let $J_r = (J + rA)^{-1}J$. Then*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y)$$

for all $x \in A^{-1}0$ and $y \in E$.

Let B be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = B(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \quad \forall y \in C\}. \tag{2.9}$$

Theorem 2.10 (Rockafellar [10]). *Let C be a nonempty, closed convex subset of a Banach space E and B a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Bv + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.10}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, B)$.

Lemma 2.11 (Tan and Xu [37]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \textit{for all} \;\; n \geq 0.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.12 (Blum and Oettli [1]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in C.$$

Lemma 2.13 (Takahashi and Zembayashi [19]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \longrightarrow C$ as follows:*

$$T_r x = \{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\} \tag{2.11}$$

for all $x \in E$. Then, the followings hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = EP(\Theta)$;

(4) *$EP(\Theta)$ is closed and convex.*

Lemma 2.14 (Takahashi and Zembayashi [19]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.15 (Zhang [38]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \longrightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and Θ be a* *bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \longrightarrow C$ as follows:

$$K_r(x) = \{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C\} \tag{2.12}$$

for all $x \in C$. Then, the followings hold:

(i) K_r *is single-valued;*

(ii) K_r *is firmly nonexpansive, i.e., for all* $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;

(iii) $F(K_r) = \Omega$;

(iv) Ω *is closed and convex.*

(v) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ $\forall p \in F(K_r)$, $z \in E$.

Remark 2.16 (Zhang [38])**.** It follows from Lemma 2.13 that the mapping $K_r : C \longrightarrow C$ defined by (2.12) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

3 Strong Convergence Theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of mixed equilibrium problems, the set of solution of the variational inequality operators, the zero point of a maximal monotone operators and the set of fixed piint of two relatively quasi-nonexpansive mappings in a Banach space by using the shrinking hybrid projection method.

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, let $A : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(A) \subset C$. Let $J_r = (J + rA)^{-1}J$ for $r > 0$ and let W be an α-inverse-strongly monotone operator of C into E^*. Let T and S are closed relatively quasi-nonexpansive from C into itself such that $F := F(T) \cap F(S) \cap VI(C, W) \cap A^{-1}(0) \cap \Omega \neq \emptyset$ and $\|Wy\| \leq \|Wy - Wu\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated* *by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$,*

$$\begin{cases} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n W x_n), \\ z_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n) JT(J_{r_n} w_n)), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n) JSz_n), \\ u_n \in C \quad \text{such that} \quad \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1-\beta_n)\phi(z, z_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfying $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If T and S are uniformly continuous, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Let $H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)$, $y \in C$ and $K_{r_n} = \{u \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0.\ \forall y \in C\}$. We first show that $\{x_n\}$ is bounded. Put $v_n = J^{-1}(Jx_n - \lambda_n W x_n)$, let $p \in F := F(T) \cap F(S) \cap VI(C, W) \cap A^{-1}(0) \cap \Omega$ and $u_n = K_{r_n} y_n$. By (3.1) and Lemma 2.8, the convexity of the function V in the second variable, we have

$$\begin{aligned} \phi(p, w_n) &= \phi(p, \Pi_C v_n) \\ &\leq \phi(p, v_n) = \phi(p, J^{-1}(Jx_n - \lambda_n W x_n)) \\ &\leq V(p, Jx_n - \lambda_n W x_n + \lambda_n W x_n) - 2\langle J^{-1}(Jx_n - \lambda_n A x_n) - p, \lambda_n W x_n\rangle \\ &= V(p, Jx_n) - 2\lambda_n \langle v_n - p, W x_n\rangle \\ &= \phi(p, x_n) - 2\lambda_n\langle x_n - p, W x_n\rangle + 2\langle v_n - x_n, -\lambda_n W x_n\rangle. \end{aligned} \tag{3.2}$$

Since $p \in VI(C, W)$ and W is α-inverse-strongly monotone, we have

$$\begin{aligned} -2\lambda_n\langle x_n - p, W x_n\rangle &= -2\lambda_n\langle x_n - p, W x_n - Wp\rangle - 2\lambda_n\langle x_n - p, Wp\rangle \\ &\leq -2\alpha\lambda_n \|W x_n - Wp\|^2, \end{aligned} \tag{3.3}$$

and by Lemma 2.1, we obtain

$$\begin{aligned} 2\langle v_n - x_n, -\lambda_n W x_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n W x_n) - x_n, -\lambda_n W x_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n W x_n) - x_n\|\|\lambda_n W x_n\| \\ &\leq \frac{4}{c^2}\|Jx_n - \lambda_n W x_n - Jx_n\|\|\lambda_n W x_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|W x_n\|^2 \\ &\leq \frac{4}{c^2}\lambda_n^2\|W x_n - Wp\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we get

$$\begin{aligned}
\phi(p,w_n) &\leq \phi(p,x_n) - 2\alpha\lambda_n\|Wx_n - Wp\|^2 + \frac{4}{c^2}\lambda_n^2\|Wx_n - Wp\|^2 \\
&\leq \phi(p,x_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\|Wx_n - Wp\|^2 \\
&\leq \phi(p,x_n). \qquad (3.5)
\end{aligned}$$

By Lemma 2.8, Lemma 2.9 and (3.5), we have

$$\begin{aligned}
\phi(p,z_n) &= \phi(p, J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)JT(J_{r_n}w_n))) \\
&= V(p, \alpha_n J(x_n) + (1-\alpha_n)JT(J_{r_n}w_n)) \\
&\leq \alpha_n V(p, J(x_n)) + (1-\alpha_n)V(p, JT(J_{r_n}w_n)) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, TJ_{r_n}w_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, J_{r_n}w_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)(\phi(p,w_n) - \phi(J_{r_n}w_n, w_n)) \qquad (3.6) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,w_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,x_n) \\
&= \phi(p,x_n),
\end{aligned}$$

it follows that

$$\begin{aligned}
\phi(p,y_n) &= \phi(p, J^{-1}(\beta_n J(x_n) + (1-\beta_n)JS(z_n))) \\
&= V(p, \beta_n J(x_n) + (1-\beta_n)JS(z_n)) \\
&\leq \beta_n V(p, J(x_n)) + (1-\beta_n)V(p, JS(z_n)) \\
&= \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p, Sz_n) \qquad (3.7) \\
&\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,z_n) \\
&\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,x_n) \\
&\leq \phi(p,x_n).
\end{aligned}$$

From (3.1) and (3.7), we obtain

$$\phi(p,u_n) = \phi(p, K_{r_n}y_n) \leq \phi(p,y_n) \leq \phi(p,x_n). \qquad (3.8)$$

So, we have $p \in C_{n+1}$. This implies that $F \subset C_n$ for all $n \in \mathbb{N}$, $\{x_n\}$ is well defined.

From Lemma 2.6 and $x_n = \Pi_{C_n}x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n \qquad (3.9)$$

and

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0, \quad \forall p \in F. \qquad (3.10)$$

From Lemma 2.7, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p,x_0) - \phi(p,x_n) \leq \phi(p,x_0)$$

for all $p \in F \subset C_n$ and $n \geq 1$. Then, the sequence $\{\phi(x_n, x_0)\}$ is bounded. Thus $\{x_n\}$ is bounded and $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{J_{r_n} w_n\}$ are also bounded. Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \in \mathbb{N}.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. Hence the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.11}$$

Letting $m, n \longrightarrow \infty$ in (3.11), we get $\phi(x_m, x_n) \longrightarrow 0$. It follows from Lemma 2.4, that $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$. That is, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we can assume that $x_n \longrightarrow u \in C$, as $n \longrightarrow \infty$. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0)$$

for all $n \in \mathbb{N}$, we also have $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$ and by definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).$$

Noticing the $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$, we obtain

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

From again Lemma 2.4, that

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \longrightarrow \infty} \|Jx_{n+1} - Jx_n\| = \lim_{n \longrightarrow \infty} \|Jx_{n+1} - Ju_n\| = 0. \tag{3.13}$$

So, by the triangle inequality, we get

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0. \tag{3.15}$$

On the other hand, we observe that

$$\begin{aligned} \phi(p, x_n) - \phi(p, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2 \langle p, Jx_n - Ju_n \rangle \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|. \end{aligned}$$

It follows that

$$\phi(p,x_n)-\phi(p,u_n)\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty. \tag{3.16}$$

From (3.1), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
\phi(p,u_n) &\leq \phi(p,y_n)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)\phi(p,z_n)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\alpha_n\phi(p,x_n)+(1-\alpha_n)(\phi(p,w_n)-\phi(J_{r_n}w_n,w_n))]\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\alpha_n\phi(p,x_n)+(1-\alpha_n)(\phi(p,x_n)-\phi(J_{r_n}w_n,w_n))]\\
&\leq \phi(p,x_n)-(1-\alpha_n)(1-\beta_n)\phi(J_{r_n}w_n,w_n)
\end{aligned}$$

and then

$$(1-\alpha_n)(1-\beta_n)\phi(J_{r_n}w_n,w_n)\leq\phi(p,x_n)-\phi(p,u_n). \tag{3.17}$$

From conditions $\limsup_{n\longrightarrow\infty}\alpha_n<1$, $\limsup_{n\longrightarrow\infty}\beta_n<1$ and (3.16), we obtain

$$\lim_{n\longrightarrow\infty}\phi(J_{r_n}w_n,w_n)=0.$$

By again Lemma 2.4, we have

$$\lim_{n\longrightarrow\infty}\|J_{r_n}w_n-w_n\|=0. \tag{3.18}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty}\|J(J_{r_n}w_n)-J(w_n)\|=0. \tag{3.19}$$

Now, we claim that $u\in F$. First we show that $u\in F(T)\cap F(S)$.
From the definition of C_n, we have

$$\beta_n\phi(z,x_n)+(1-\beta_n)\phi(z,z_n)\leq\phi(z,x_n)\Leftrightarrow\phi(z,z_n)\leq\phi(z,x_n),\quad \forall z\in C_{n+1}.$$

Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}$, we obtain

$$\phi(x_{n+1},z_n)\leq\phi(x_{n+1},x_n).$$

From $\lim_{n\longrightarrow\infty}\phi(x_{n+1},x_n)=0$, we get

$$\lim_{n\longrightarrow\infty}\phi(x_{n+1},z_n)=0. \tag{3.20}$$

From again Lemma 2.4, that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-z_n\|=0. \tag{3.21}$$

By (3.12) and (3.21), we get

$$\lim_{n\longrightarrow\infty}\|x_n-z_n\|=0. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty}\|Jx_{n+1}-Jz_n\|=\lim_{n\longrightarrow\infty}\|Jx_n-Jz_n\|=0. \tag{3.23}$$

From (3.1) again

$$\begin{aligned}\|Jx_{n+1}-Jz_n\| &= \|Jx_{n+1}-\alpha_nJx_n-(1-\alpha_n)JTJ_{r_n}w_n\|\\ &= \|\alpha_n(Jx_{n+1}-Jx_n)+(1-\alpha_n)(Jx_{n+1}-JTJ_{r_n}w_n)\|\\ &= \|(1-\alpha_n)(Jx_{n+1}-JTJ_{r_n}w_n)-\alpha_n(Jx_n-Jx_{n+1})\|\\ &\geq (1-\alpha_n)\|Jx_{n+1}-JTJ_{r_n}w_n\|-\alpha_n\|Jx_n-Jx_{n+1}\|.\end{aligned}$$

It follows that

$$\|Jx_{n+1}-JTJ_{r_n}w_n\|\leq\frac{1}{1-\alpha_n}(\|Jx_{n+1}-Jz_n\|+\alpha_n\|Jx_n-Jx_{n+1}\|).$$

From conditions $\limsup_{n\longrightarrow\infty}\alpha_n<1$, (3.13) and (3.23), we have

$$\lim_{n\longrightarrow\infty}\|Jx_{n+1}-JTJ_{r_n}w_n\|=0. \tag{3.24}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-TJ_{r_n}w_n\|=0. \tag{3.25}$$

$\lim_{n\longrightarrow\infty}\phi(J_{r_n}x_n,w_n)=0$.
Apply (3.5) and (3.6), we observe that

$$\begin{aligned}\phi(p,u_n) &\leq \phi(p,y_n)\\ &\leq \beta_n\phi(p,x_n)+(1-\beta_n)\phi(p,z_n)\\ &\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\alpha_n\phi(p,x_n)+(1-\alpha_n)\phi(p,w_n)]\\ &\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\alpha_n\phi(p,x_n)+(1-\alpha_n)[(\phi(p,x_n)\\ &\quad -2\lambda_n\left(\alpha-\tfrac{2}{c^2}\lambda_n\right)\|Wx_n-Wp\|^2]]\\ &\leq \phi(p,x_n)-(1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha-\tfrac{2}{c^2}\lambda_n\right)\|Wx_n-Wp\|^2\end{aligned}$$

and hence

$$2\lambda_n\left(\alpha-\frac{2}{c^2}\lambda_n\right)\|Wx_n-Wp\|^2\leq\frac{1}{(1-\alpha_n)(1-\beta_n)}(\phi(p,x_n)-\phi(p,u_n))$$

for all $n\in\mathbb{N}$. Since $0<a\leq\lambda_n\leq b<\frac{c^2\alpha}{2}$, $\limsup_{n\longrightarrow\infty}\alpha_n<1$, $\limsup_{n\longrightarrow\infty}\beta_n<1$ and (3.16), we have

$$\lim_{n\longrightarrow\infty}\|Wx_n-Wp\|=0. \tag{3.26}$$

From Lemma 2.7, Lemma 2.8 and (3.4), we get

$$\begin{aligned}\phi(x_n,w_n)=\phi(x_n,\Pi_Cv_n) &\leq \phi(x_n,v_n)\\ &= \phi(x_n,J^{-1}(Jx_n-\lambda_nWx_n))\\ &= V(x_n,Jx_n-\lambda_nWx_n)\\ &\leq V(x_n,(Jx_n-\lambda_nWx_n)+\lambda_nWx_n)\\ &\quad -2\langle J^{-1}(Jx_n-\lambda_nWx_n)-x_n,\lambda_nWx_n\rangle\\ &= \phi(x_n,x_n)+2\langle v_n-x_n,-\lambda_nWx_n\rangle\\ &= 2\langle v_n-x_n,-\lambda_nWx_n\rangle\\ &\leq \frac{4\lambda_n^2}{c^2}\|Wx_n-Wp\|^2.\end{aligned}$$

From Lemma 2.4 and (3.26), we have

$$\lim_{n\longrightarrow\infty} \|x_n - w_n\| = 0. \tag{3.27}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty} \|Jx_n - Jw_n\| = 0. \tag{3.28}$$

From (3.18) and (3.27), we obtain

$$\lim_{n\longrightarrow\infty} \|J_{r_n}w_n - x_n\| = 0. \tag{3.29}$$

So, by the triangle inequality, we get

$$\|J_{r_n}w_n - TJ_{r_n}w_n\| \leq \|J_{r_n}w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - TJ_{r_n}w_n\|.$$

Again by (3.12), (3.25) and (3.29), we also have

$$\lim_{n\longrightarrow\infty} \|J_{r_n}w_n - TJ_{r_n}w_n\| = 0. \tag{3.30}$$

From (3.29), (3.30) and T is uniformly continuous, we get

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0.$$

Since T is closed and $x_n \longrightarrow u$, we have $u \in F(T)$.
Applying (3.7), (3.8) and Lemma 2.14, we get

$$\begin{array}{rcl} \phi(u_n, y_n) &=& \phi(K_{r_n}y_n, y_n) \\ &\leq& \phi(p, y_n) - \phi(p, K_{r_n}y_n) \\ &\leq& \phi(p, x_n) - \phi(p, u_n) \\ &=& \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 - (\|p\|^2 - 2\langle p, Ju_n\rangle + \|u_n\|^2) \\ &=& \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ &\leq& \|x_n - u_n\|(\|x_n + u_n\|) + 2\|p\|\|Jx_n - Ju_n\|. \end{array}$$

From (3.14) and (3.15) and Lemma 2.4, we get

$$\lim_{n\longrightarrow\infty} \|u_n - y_n\| = 0. \tag{3.31}$$

From (3.12) and (3.31), we have

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - y_n\| = 0. \tag{3.32}$$

By (3.1), we get

$$\begin{array}{rcl} \|Jx_{n+1} - Jy_n\| &=& \|Jx_{n+1} - \beta_nJx_n - (1-\beta_n)JSz_n\| \\ &=& \|\beta_n(Jx_{n+1} - Jx_n) + (1-\beta_n)(Jx_{n+1} - JSz_n)\| \\ &=& \|(1-\beta_n)(Jx_{n+1} - JSz_n) - \beta_n(Jx_n - Jx_{n+1})\| \\ &\geq& (1-\beta_n)\|Jx_{n+1} - JSz_n\| - \beta_n\|Jx_n - Jx_{n+1}\|. \end{array}$$

It follows that

$$\|Jx_{n+1} - JSz_n\| \leq \frac{1}{1-\beta_n}(\|Jx_{n+1} - Jy_n\| - \beta_n\|Jx_n - Jx_{n+1}\|)$$

By condition $\limsup_{n\longrightarrow\infty} \beta_n < 1$, (3.13) and (3.32), we have

$$\lim_{n\longrightarrow\infty} \|Jx_{n+1} - JSz_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - Sz_n\| = 0. \tag{3.33}$$

By the triangle inequality, we get

$$\|z_n - Sz_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - Sz_n\|.$$

By (3.21) and (3.33), we have

$$\lim_{n\longrightarrow\infty} \|z_n - Sz_n\| = 0.$$

From (3.22), it follow that

$$\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0.$$

Thus by the closedness of S and $x_n \longrightarrow u$, we get $u \in F(S)$. Hence $u \in F(T) \cap F(S)$.

Next, we show that $u \in A^{-1}0$. Indeed, since $\liminf_{n\longrightarrow\infty} r_n > 0$, it follows from (3.19) that

$$\lim_{n\longrightarrow\infty} \|A_{r_n} w_n\| = \lim_{n\longrightarrow\infty} \frac{1}{r_n}\|Jw_n - J(J_{r_n} w_n)\| = 0. \tag{3.34}$$

If $(z, z^*) \in A$, then it holds from the monotonicity of A that

$$\langle z - J_{r_{n_i}} w_{n_i}, z^* - A_{r_{n_i}} w_{n_i}\rangle \geq 0$$

for all $i \in \mathbb{N}$. Letting $i \longrightarrow \infty$, we get $\langle z - u, z^*\rangle \geq 0$. Then, the maximality of A implies $u \in A^{-1}0$.

Next, we show that $u \in VI(C, W)$. Let $Y \subset E \times E^*$ be an operator as follows:

$$Yv = \begin{cases} Wv + N_C(v), & v \in C; \\ \emptyset, & \textit{otherwise}. \end{cases}$$

By Theorem 2.10, Y is maximal monotone and $Y^{-1}0 = VI(C, W)$. Let $(v, w) \in G(Y)$. Since $w \in Yv = Wv + N_C(v)$, we get $w - Wv \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Wv\rangle \geq 0. \tag{3.35}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n W x_n)$. Then by Lemma 2.6, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n W x_n)\rangle \geq 0,$$

thus

$$\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Wx_n\rangle \leq 0. \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$\begin{aligned}
\langle v - w_n, w\rangle &\geq \langle v - w_n, Wv\rangle \\
&\geq \langle v - w_n, Wv\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Wx_n\rangle \\
&= \langle v - w_n, Wv - Wx_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&= \langle v - w_n, Wv - Ww_n\rangle + \langle v - w_n, Ww_n - Wx_n\rangle \\
&\quad + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&\geq -\|v - w_n\|\frac{\|w_n - x_n\|}{\alpha} - \|v - w_n\|\frac{\|Jx_n - Jw_n\|}{a} \\
&\geq -M(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a}),
\end{aligned}$$

where $M = \sup_{n\geq 1}\{\|v - w_n\|\}$. From (3.27) and (3.28), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of Y, we have $u \in Y^{-1}0$ and hence $u \in VI(C, W)$.

Next, we show that $u \in \Omega$. From (3.31) and J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\longrightarrow\infty} \|Ju_n - Jy_n\| = 0. \tag{3.37}$$

From the assumption $r_n \geq a$, we get

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.$$

Noticing that $u_n = K_{r_n} y_n$, we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

Hence,

$$H(u_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jy_{n_i}\rangle \geq 0, \quad \forall y \in C.$$

From the (A2), we note that

$$\|y - u_{n_i}\|\frac{\|Ju_{n_i} - Jy_{n_i}\|}{r_{n_i}} \geq \frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jy_{n_i}\rangle \geq -H(u_{n_i}, y) \geq H(y, u_{n_i}),$$

$\forall y \in C$. Taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n \longrightarrow u$, we have $H(y, u) \leq 0, \quad \forall y \in C$. For $0 < t < 1$ and $y \in C$, define

$y_t = ty + (1-t)u$. Noticing that $y, u \in C$, we obtains $y_t \in C$, which yields that $H(y_t, u) \leq 0$. It follows from (A1) that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, u) \leq tH(y_t, y).$$

That is, $H(y_t, y) \geq 0$.
Let $t \downarrow 0$, from (A3), we obtain $H(u, y) \geq 0, \quad \forall y \in C$. This implies that $u \in \Omega$. Hence $u \in F := F(T) \cap F(S) \cap VI(C, B) \cap A^{-1}(0) \cap \Omega$.

Finally, we show that $u = \Pi_F x_0$. Indeed from $x_n = \Pi_{C_n} x_0$ and Lemma 2.6, we have

$$\langle Jx_0 - Jx_n, x_n - z\rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - p\rangle \geq 0, \quad \forall p \in F. \tag{3.38}$$

Taking limit $n \longrightarrow \infty$, we obtain

$$\langle Jx_0 - Ju, u - p\rangle \geq 0, \quad \forall p \in F.$$

By again Lemma 2.6, we can conclude that $u = \Pi_F x_0$. This completes the proof. □

Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, let $A : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(A) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let W be an α-inverse-strongly monotone operator of C into E^*. Let T be closed relatively quasi-nonexpansive from C into itself such that $F := F(T) \cap VI(C, W) \cap A^{-1}(0) \cap \Omega \neq \emptyset$ and $\|Wy\| \leq \|Wy - Wu\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$,*

$$\begin{cases} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n W x_n), \\ z_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n) JT(J_{r_n} w_n)), \\ u_n \in C \quad \text{such that} \quad \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.39}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If T is uniformly continuous, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. In Theorem 3.1, if $S = I$ and $\beta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ then (3.1) reduced to (3.39). □

Corollary 3.3. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, let W be an α-inverse-strongly monotone operator of C into E^*. Let T and S are closed relatively quasi-nonexpansive from C into itself such that $F := F(T) \cap F(S) \cap VI(C, W) \cap \Omega \neq \emptyset$ and $\|Wy\| \leq \|Wy - Wu\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$,*

$$\begin{cases} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n W x_n), \\ z_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n) JT(w_n)), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n) JS(z_n)), \\ u_n \in C \ \ \text{such that} \ \ \Theta(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1-\beta_n)\phi(z, z_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.40}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfying $\limsup_{n \longrightarrow \infty} \alpha_n < 1$, $\limsup_{n \longrightarrow \infty} \beta_n < 1$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2 \alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If T and S are uniformly continuous, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. In Theorem 3.1, set $A = \partial i_C$ where i_C is the indicator function; that is

$$i_C = \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise}. \end{cases}$$

Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$, in fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.5 that

$$\begin{aligned} z = J_r x \ &\Leftrightarrow \ Jz + r\partial i_C(z) \ni Jx \\ &\Leftrightarrow \ Jx - Jz \in r\partial i_C(z) \\ &\Leftrightarrow \ i_C(y) \geq \langle y - z, \tfrac{Jx - Jz}{r} \rangle + i_C(z), \ \ \forall y \in E \\ &\Leftrightarrow \ 0 \geq \langle y - z, Jx - Jz \rangle, \ \ \forall y \in C \\ &\Leftrightarrow \ z = \arg\min_{y \in C} \phi(y, x) \\ &\Leftrightarrow \ z = \Pi_C x. \end{aligned}$$

So, we obtain the desired result by using Theorem 3.1. □

4 Application to Complementarity Problems

Let K be a nonempty, closed convex cone in E, W an operator of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in K\}. \tag{4.1}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Wu \in K^*, \ \ \langle u, Wu \rangle = 0. \tag{4.2}$$

The set of solutions of the complementarity problem is denoted by $CP(K, W)$; see [27], for more detail.

Theorem 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let K be a nonempty closed convex subset of E. Let Θ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : K \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function and let $B : K \longrightarrow E^*$ be a continuous and monotone mappings, let $A : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(A) \subset K$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let W be an α-inverse-strongly monotone operator of K into E^*. Let T and S are closed relatively quasi-nonexpansive from K into itself such that $F := F(T) \cap F(S) \cap VI(K, W) \cap A^{-1}(0) \cap \Omega \neq \emptyset$ and $\|Wy\| \leq \|Wy - Wu\|$ for all $y \in K$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = K$,*

$$\begin{cases} w_n = \Pi_K J^{-1}(Jx_n - \lambda_n W x_n), \\ z_n = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) JT(J_{r_n} w_n)), \\ y_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n) JS(z_n)), \\ u_n \in K \ \ \text{such that} \ \ \Theta(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \ \forall y \in K, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1 - \beta_n)\phi(z, z_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{4.3}$$

for all $n \in \mathbb{N}$, where Π_K is the generalized projection from E onto K, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\limsup_{n \longrightarrow \infty} \alpha_n < 1$, $\limsup_{n \longrightarrow \infty} \beta_n < 1$, $\liminf_{n \longrightarrow \infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2 \alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If T and S are uniformly continuous, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. As in the proof Lemma 7.1.1 of Takahashi in [27], we have $VI(K, W) = CP(K, W)$. So, we obtain the desired result. ◻

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