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On the Stirling Series of Gamma Function

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Abstract : The aim of this paper is to improve some approximations obtained by truncation of the Stirling series about the gamma function. Finally an open problem is posed.

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1 Introduction

There are many situations in mathematics or in other branches of science when we are forced to deal with large factorials. As a direct computation cannot be made even by the computer programs, approximation formulas were constructed. Maybe the most used is the following formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{1.1}$$

now known as the Stirling formula, after the Scottish mathematician James Stirling (1692-1770).

The gamma function Γ defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

is an extension of the factorial function, since $\Gamma(n+1) = 1{\cdot}2 \cdots n$, for $n = 1, 2, \ldots$.

Although in applied statistics, or statistical physics the formula (1.1) is satisfactory for large values of n, in pure mathematics more precise approximations are necessary. One of the first improvement is of type

$$\Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}, \tag{1.2}$$

with

$$\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}, \tag{1.3}$$

see e.g., [1, 2]. The right-side bound in (1.3) was proved also in [3–6], while successively better values for the left-side bound $1/(12n+1/4)$, or $1/\left(12n+\frac{3}{4n+2}\right)$ were obtained in [3, 4].

We refine here (1.3) and all the above bounds proving the following

Theorem 1.1. *For every integer $n \geq 1$, we have*

$$\frac{1}{12\left(n+\frac{1}{30n}\right)} < \lambda_n < \frac{1}{12\left(n+\frac{1}{30n}-\frac{53}{6300n^3}\right)}. \tag{1.4}$$

Nanjundiah [5] obtained the following stronger result

$$\frac{1}{12n} - \frac{1}{360n^3} < \lambda_n < \frac{1}{12n}, \tag{1.5}$$

thereafter an even better result was given by Shi, Liu and Hu [7, Rel. 10] by

$$\frac{1}{12n} - \frac{1}{360n^3} < \lambda_n < \frac{1}{12n} - \frac{1}{360n(n+1)(n+2)}\ , \quad n \geq 1. \tag{1.6}$$

We improve here (1.5)-(1.6) proving the following

Theorem 1.2. *For every integer $n \geq 1$, we have*

$$\frac{1}{12n} - \frac{1}{360\left(n+\frac{2}{21n}-\frac{47}{882n^3}\right)^3} < \lambda_n < \frac{1}{12n} - \frac{1}{360\left(n+\frac{2}{21n}\right)^3}. \tag{1.7}$$

2 The Proofs

Proof of Theorem 1.1. Taking into account (1.2), we have to prove

$$\frac{1}{12\left(n+\frac{1}{30n}\right)} < \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n - \ln\sqrt{2\pi} + n < \frac{1}{12\left(n+\frac{1}{30n}-\frac{53}{6300n^3}\right)}.$$

In this sense we define for $t \in \left\{0, \frac{53}{6300}\right\}$ the sequence

$$x_n^{(t)} = \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n - \ln\sqrt{2\pi} + n - \frac{1}{12\left(n+\frac{1}{30n}-\frac{t}{n^3}\right)}, \quad n \geq 1,$$

and denote $x_{n+1}^{(t)} - x_n^{(t)} = f_t(n)$, where

$$\begin{aligned} f_t(x) &= \ln(x+1) - \left(x+\frac{3}{2}\right)\ln(x+1) + \left(x+\frac{1}{2}\right)\ln x + 1 \\ &\quad -\frac{1}{12\left(x+1+\frac{1}{30(x+1)}-\frac{t}{(x+1)^3}\right)} + \frac{1}{12\left(x+\frac{1}{30x}-\frac{t}{x^3}\right)}. \end{aligned}$$

For $t = 0$ case we have

$$f_0''(x) = -\frac{P(x)}{2x^2(x+1)^2(30x^2+1)^3(30x^2+60x+1)^3},$$

where

$$\begin{aligned} P(x) &= 29\,791 + 172\,980x + 3110\,580x^2 + 42\,973\,200x^3 + 328\,881\,600x^4 \\ &\quad + 969\,894\,000x^5 + 1324\,998\,000x^6 + 858\,600\,000x^7 + 214\,650\,000x^8. \end{aligned}$$

Now f_0 is strictly concave on $[1,\infty)$ with $f_0(\infty) = 0$, so $f_0(x) < 0$, for every $x \in [1,\infty)$. As a consequence, $x_n^{(0)}$ is strictly decreasing, convergent to zero. Thus $x_n^{(0)} > 0$ and the left-hand side of (1.4) is proved.

In $t = \frac{53}{6300}$ case, we have

$$f''_{\frac{53}{6300}}(x) = \frac{Q(x)}{2x^2(x+1)^2(210x^2+6300x^4-53)^3(25\,620x+38\,010x^2+25\,200x^3+6300x^4+6457)^3},$$

where

$$Q(x) = 34374032103348000000000x^{18} + \cdots + 40079285253659861$$

is a 18th degree polynomial with all coefficients positive.

Now $f_{\frac{53}{6300}}$ is strictly convex on $[1,\infty)$ with $f_{\frac{53}{6300}}(\infty) = 0$, so $f_{\frac{53}{6300}}(x) > 0$, for every $x \in [1,\infty)$. As a consequence, $x_n^{\left(\frac{53}{6300}\right)}$ is strictly increasing, convergent to zero. Thus $x_n^{\left(\frac{53}{6300}\right)} < 0$ and the right-hand side of (1.4) is proved. □

Proof of Theorem 1.2. Taking into account (1.2), we have to prove

$$\begin{aligned} \frac{1}{12n} - \frac{1}{360\left(n+\frac{2}{21n}-\frac{47}{882n^3}\right)^3} &< \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n - \ln\sqrt{2\pi} + n \\ &< \frac{1}{12n} - \frac{1}{360\left(n+\frac{2}{21n}\right)^3}. \end{aligned}$$

In this sense we define for $s \in \left\{0, \frac{47}{882}\right\}$ the sequence

$$y_n^{(s)} = \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n - \ln\sqrt{2\pi} + n - \left(\frac{1}{12n} - \frac{1}{360\left(n+\frac{2}{21n}-\frac{s}{n^3}\right)^3}\right),$$

and denote $y_{n+1}^{(s)}-y_n^{(s)}=g_s(n)$, where

$$g_s(x)=\ln(x+1)-\left(x+\frac{3}{2}\right)\ln(x+1)+\left(x+\frac{1}{2}\right)\ln x+1$$

$$-\frac{1}{12(x+1)}+\frac{1}{12x}+\frac{1}{360\left(x+1+\frac{2}{21(x+1)}-\frac{s}{(x+1)^3}\right)^3}-\frac{1}{360\left(x+\frac{2}{21x}-\frac{s}{x^3}\right)^3}.$$

In $s=0$ case we have

$$g_0''(x)=\frac{R(x)}{30x^3(x+1)^3(21x^2+2)^5(21x^2+42x+23)^5},$$

where $R(x)=111\,993\,486\,567\,921x^{16}+\cdots+1029\,814\,880$ is a 16th degree polynomial with all coefficients positive.

Now g_0 is strictly convex on $[1,\infty)$ with $g_0(\infty)=0$, so $g_0(x)>0$, for every $x\in[1,\infty)$. As a consequence, $y_n^{(0)}$ is strictly increasing, convergent to zero. Thus $y_n^{(0)}<0$ and the right-hand side of (1.7) is proved.

In $s=\frac{47}{882}$ case, we have

$$g''_{\frac{47}{882}}(x)=-\frac{T(x)}{30x^3(x+1)^3(84x^2+882x^4-47)^5(3696x+5376x^2+3528x^3+882x^4+919)^5},$$

where

$$T(x)=548795630736394078777581639168 0x^{34}+\cdots+751686673758358176995965$$

is a 34th degree polynomial. Although some coefficients of T are negative, the polynomial $T(x+1)$ has all coefficients positive, so $T(x)>0$, for every $x\in[1,\infty)$.

Now $g_{\frac{47}{882}}$ is strictly concave on $[1,\infty)$ with $g_{\frac{47}{882}}(\infty)=0$, so $g_{\frac{47}{882}}(x)<0$, for every $x\in[1,\infty)$. As a consequence, $y_n^{\left(\frac{47}{882}\right)}$ is strictly decreasing, convergent to zero. Thus $y_n^{\left(\frac{47}{882}\right)}>0$ and the left-hand side of (1.7) is proved. □

3 Concluding remarks

It is to be noticed that our new results (1.4) and (1.7) improve the estimates

$$\frac{1}{12n}-\frac{1}{360n^3}<\ln\frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}<\frac{1}{12n} \tag{3.1}$$

obtained by truncation of the Stirling asymptotic series

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\exp\left(\sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right). \tag{3.2}$$

See [8, p. 257, Rel. 6.1.40]. Such inequalities are true even if we truncate (3.2) at any mth term. We mean that for every integer $m \geq 1$, we have

$$\sum_{i=1}^{2m} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} < \ln \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} < \sum_{i=1}^{2m-1} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} \tag{3.3}$$

(inequality (3.1) is $m=1$ case). Inequality (3.3) is consequence of the following

Lemma 3.1 ([9, Theorem 8])**.** *For every $m \geq 1$, the functions*

$$F_m(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln 2\pi - \sum_{i=1}^{2m} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}}$$

and

$$G_m(x) = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{i=1}^{2m-1} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}}$$

are completely monotonic on $(0,\infty)$.

Inequalities (3.3) were used to obtain estimates of gamma function or to construct asymptotic expansions in [10–13] and they are suitable for refinement and obtaining other results.

A function $z : I \to \mathbb{R}$ is said to be completely monotonic on interval I if it is indefinite derivable on I such that

$$(-1)^n z^{(n)}(x) \geq 0\,, \quad \text{for all } x \in I \text{ and } n = 0,1,2,3\dots. \tag{3.4}$$

Dubourdieu [14] proved that if a non-constant function z is completely monotonic, then strict inequalities hold in (3.4). Completely monotonic functions appear naturally in various fields, like, for example, probability theory and potential theory. The main properties of these functions are given in [15, Chapter IV].

Inequality (3.3) follows now from $F_m > 0$ and $G_m > 0$.

As we can see from (1.4) and (1.7), by modifying the last term of the truncated series (3.2), better estimates than (3.3) can be obtained. In order to reinforce our study, remark that the following estimates involving the third term of (3.2)

$$\begin{aligned} \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260\left(n + \frac{3}{20n}\right)^5} &< \ln \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \\ &< \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260\left(n + \frac{3}{20n} - \frac{1909}{13\,200n^3}\right)^5} \end{aligned} \tag{3.5}$$

can be similarly established as (1.4)-(1.7).

Finally we propose the following open problem. Let m be a positive integer. Do exist $a,b,c,d > 0$ (depending on m) such that

$$\sum_{i=1}^{2m-1} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} + \frac{B_{4m}}{4m\,(4m-1)\left(x + \frac{a}{x} - \frac{b}{x^3}\right)^{4m-1}} < \ln \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}$$

$$< \sum_{i=1}^{2m-1} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} + \frac{B_{4m}}{4m\,(4m-1)\left(x+\frac{a}{x}\right)^{4m-1}} \tag{3.6}$$

and

$$\sum_{i=1}^{2m-2} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} + \frac{B_{4m-2}}{(4m-2)\,(4m-3)\left(x+\frac{c}{x}\right)^{4m-3}} < \ln \frac{\Gamma\,(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}}$$

$$< \sum_{i=1}^{2m-2} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}} + \frac{B_{4m-2}}{(4m-2)\,(4m-3)\left(x+\frac{c}{x}-\frac{d}{x^3}\right)^{4m-3}} \tag{3.7}$$

for every integer $x > 0$? Remark that (3.6) improves the first inequality (3.3) and (3.7) improves the second inequality (3.3). By Theorem 1.1 and (3.5), the inequality (3.7) is true for $m = 1$ with $c = \frac{1}{30}$, $d = \frac{53}{6300}$, for $m = 2$ with $c = \frac{3}{20}$, $d = \frac{1909}{13200}$ and by Theorem 1.2, the inequality (3.6) is true for $m = 1$ with $a = \frac{2}{21}$, $b = \frac{47}{882}$.

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