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# Existence of Solutions of Quasilinear Integrodifferential Evolution Equations with Impulsive Conditions

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**Abstract :** We prove the existence of solutions of quasilinear integrodifferential evolution equations with impulsive condition. The results are obtained by using the fractional powers of operators and the Schauder fixed point theorem. An example is provided to illustrate the theory.

**Keywords :** Existence of solution; Quasilinear integrodifferential equation; Analytic semigroup; Fixed point theorem; Impulsive condition.
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## 1 Introduction

In this paper we study the existence solution to quasilinear impulsive evolution integrodifferential equation of the form

$$v'(t) + A(t,v)v(t) = f(t, v(t), \int_0^t g(t,s,v(s)))ds \quad t \in J, \quad t \neq t_i, \tag{1.1}$$

$$v(0) = u_0, \tag{1.2}$$

$$\Delta v(\theta_i) = I_i(v(\theta_i)), \; i = 1, 2, ..., m, \; 0 < \theta_1 < \cdots < \theta_m < T, \tag{1.3}$$

where $A(t,v)$ is the infinitesimal generator of an analytic semigroup in a Banach space $X$. Here $u_0 \in X$; $f : J \times X \times X \to X$ is uniformly bounded and continuous

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in all of its arguments and $g : \Delta \times X \to X$ is continuous. Here $J = [0, T]$ and $\Delta = \{(t, s) : 0 \le s \le t \le T\}$. Let $\mathcal{PC}([0, T] : X)$ consist of functions $u$ from $[0, T]$ into $X$, such that $v(t)$ is continuous at $t \neq \theta_i$ and left continuous at $t = \theta_i$, and the right limit $v({\theta_i}^+)$ exists for $i = 1, 2, 3, \ldots, m$. Evidently $\mathcal{PC}([0, T] : X)$ is a Banach space with the norm

$$\|v\|_{\mathcal{PC}} = \sup_{t \in [0,T]} \|v(t)\|.$$

The problem of existence of solutions of quasilinear evolution equations in Banach spaces has been studied by many authors [1–9]. Hayden and Massey [10], have considered analyticity for semilinear equations

$$\frac{du}{dt} + A(t)u = f(t, u).$$

Pazy [1] considered the following quasilinear equation

$$\begin{aligned} u'(t) + A(t, u)u(t) &= 0, \quad 0 < t \le T, \\ u(0) &= u_0, \end{aligned}$$

and the mild and classical solutions by using fixed point argument. Kato [11] studied the nonhomogeneous evolution equations. Oka [7] and Oka and Tanaka [8] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. An equation of this type occurs in a nonlinear conversation law with memory

$$u(t, x) + \psi(u(t, x))_x = \int_0^t b(t - s)\psi(u(t, x))_x \, ds + f(t, x), \; t \in [0, T], \quad (1.4)$$

$$u(0, x) = \Psi(x), \quad x \in \mathbb{R}. \quad (1.5)$$

It is clear that if nonlocal condition (1.2) is introduced to (1.4), then it will also have better effect than the classical condition $u(0, x) = \Psi(x)$. It is interesting to investigate the existence problem for these of equations in Banach spaces.Recently Balachandran and Park [4] have studied the existence of solutions of quasilinear integrodifferential evolution equations by using the Schauder fixed point theorem.

On the other hand, the study of the impulsive differential equations has attracted a great deal of attention. The theory of impulsive differential and integrodifferential equations become an important area of invetigation in recent years. Many evolution process are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus differential equations involving impulsive effects appear as a natural description of observed evolution phenomena of several real world problems. The theory of impulsive differential and integrodifferential equations has been studied by several authors [5, 12–15].

## 2 Preliminaries

Consider the intial value problem

$$\begin{aligned} x'(t) + A(t)x(t) &= f(t), \quad 0 \le s < t \le T, \\ x(s) &= y, \end{aligned} \tag{2.1}$$

with the following assumptions:

$(E_1)$ The domain $D(A(t)) = D$ of $A(t),\ 0 \le t \le T$ is dense in $X$ and independent of $t$.

$(E_2)$ For $t \in J$, the resolvent $R(\lambda; A(t)) = (\lambda I - A(t))^{-1}$ of $A(t)$ exists for all $\lambda$ with Re $\lambda \le 0$ and there is a constant $C$ such that

$$\|R(\lambda; A(t))\| \le C[\|\lambda\| + 1]^{-1} \text{ for } Re\lambda \le 0,\ t \in J.$$

$(E_3)$ There exists constants $L$ and $0 \le \alpha \le 1$ such that

$$\|(A(t) - A(s))A(\tau)\| \le L|t - s|^{\alpha} \text{ for } t, s, \tau \in J.$$

**Theorem 2.1.** *Under the assumptions $(E_1) - (E_3)$ there is a unique evolution system $U_v(t, s)$ on $0 \le s \le t \le T$, satisfying*

(i) $\|U_v(t, s)\| \le M_0$ for $0 \le s \le t \le T$.

(ii) For $0 \le s \le t \le T$, $U_v(t, s) : X \to D$ and $t \to U_v(t, s)$ is strongly differentiable in $X$. The derivative $\frac{\partial}{\partial t}U_v(t, s) \in B(X)$ and it is strongly continuous on $0 \le s \le t \le T$. Moreover,

$$\frac{\partial}{\partial t}U_v(t, s) + A(t)U_v(t, s) = 0,$$
$$\|\frac{\partial}{\partial t}U_v(t, s)\| = \|A(t)U_v(t, s)\| \le M_0(t - s)^{-1} \text{ and}$$
$$\|A(t)U_v(t, s)A^{-1}(s)\| < M_0 \text{ for } 0 < s < t < T.$$

(iii) For every $v \in D$ and $t \in J, U_v(t, s)v$ is differential with respect to $s$ on $0 \le s \le t \le T$ and

$$\frac{\partial}{\partial t}U_v(t, s) = U_v(t, s)A(s)v.$$

(iv) $U_v(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$ and

$$U_v(t,r) = U_v(t,s)U_v(s,r), \quad r \leq s \leq t$$

$$U_v(t,t) = I.$$

Note that $(E_2)$ and the fact that $D$ is dense in $X$ imply that for every $t \in J$, $-A(t)$ is the infinitesimal generator of an analytic semigroup. We define the classical solutions (2.1) as functions $x : [s,T] \to X$ which are continuous for $s \leq t \leq T$, continuously differentiable for $s < t \leq T$, $x(t) \in D$ for $s < t \leq T$, $x(s) = y$ and $x'(t) + A(t)x(t) = f(t)$ holds for $s < t \leq T$. We will call a function $x(t)$ is a solution of the initial value problem (2.1) if it is a classical solution of this problem.

**Theorem 2.2.** *Let $A(t), 0 \leq t \leq T$ satisfy the conditions $(E_1) - (E_3)$ and let $U_v(t,s)$ be the evolution system in Theorem 2.1. If $f$ is Holder continuous on $[0,T]$ when the initial value problem (2.1) has, for every $y \in X$, a unique solution $x(t)$ given by*

$$x(t) = U_v(t,s)y + \int_s^t U_v(t,\tau)f(\tau)d\tau.$$

The proofs of the above theorems can be found in [1].

# 3 Existence Results

In this paper we discuss, the existence of solutions of quasilinear integrodifferential equations with impulsive condition by using fractional powers of operators and the Schauder fixed-point theorem. The results generalize the results of [1, 4–6, 15]. Throughout the paper $C_i$'s are positive constants. Let $r > 0$ and take $B_r = \{v \in X; \|v\|_{\mathcal{PC}} < r\}$ and assume the following conditions.

(A1) The operator $A_0 = A(0,u_0)$ is a closed operator with domain $D$ dense in $X$ and

$$\|(\lambda I - A_0)^{-1}\| \leq C[\|\lambda\| + 1]^{-1} \quad \text{for all } \lambda \text{ with } \mathrm{Re}\lambda \leq 0.$$

(A2) The operator $A_0^{-1}$is completely continuously operator on $X$.

(A3) For some $\alpha \in [0,1)$ and for any $v \in B_r$ the operator $A(t, A_0^{-\alpha}v)$ is well defined on $D$ for all $t \in J$. Furthermore for any $t,\tau \in J$ and for $v,w \in B_r$.

$$\|[A(t,A_0^{-\alpha}v) - A(\tau, A_0^{-\alpha}w)]A(\tau, A_0^{-\alpha}w)\| \leq C_1[|t-\tau|^\epsilon + \|v-w\|^\rho],$$

where $0 < \epsilon \leq 1, \quad 0 < \rho \leq 1$.

(A4) For every $t,\tau \in J$ and $v,w \in B_r$

$$\|f(t,A_0^{-\alpha}v_1, A_0^{-\alpha}w_1) - f(\tau, A_0^{-\alpha}v_2, A_0^{-\alpha}w_2)\| \leq C_2[|t-\tau|^\epsilon + \|v_1-v_2\|^\rho + \|w_1-w_2\|^\rho].$$

(A5) For every $s, t, \in J$ and $v_1, v_2 \in B_r$

$$\|g(t, s, A_0^{-\alpha} v_1) - g(t, s, A_0^{-\alpha} v_2)\| \leq C_3 \|v_1 - v_2\|^\rho.$$

(A6) For every $v_1, v_2 \in B_r$.

$$\|I_i(A_0^{-\alpha} v_1) - I_i(A_0^{-\alpha} v_2)\| \leq C_4 \|v_1 - v_2\|^\rho.$$

(A7) $u_0 \in D(A_0^\beta)$ for some $\beta > \alpha$ and

$$\|A_0^\alpha u_0\| < r.$$

Under these assumptions, we get the following lemma, They are due to Kato [16–18]

$(K_1)$ $\|A(t)^\alpha U_v(t, s)\| \leq (\beta - \alpha)^{-1} N_1 (t - s)^{-\alpha};\quad N_1 > 0,\ 0 \leq \alpha < \beta,$

$(K_2)$ $\|A(0)^\alpha A(t)^{-\alpha}\| \leq M_\alpha,\quad$ for any $\ 0 \leq t \leq T.$

In proposition [19], we have

$$A_0^\alpha [U_v(t, 0) - U_v(s, 0)] A(0)^{-\beta}\| \leq C_5 |t - s|^{\beta - \alpha} \tag{3.1}$$

and

$$A_0^\alpha [U_v(t, \theta_i) - U_v(s, \theta_i)]\| \leq C_6 |t - s|^{1 - \alpha} (s - \theta_i)^{-1}\ \forall\ i = 1, 2, ..., m. \tag{3.2}$$

Let us take

$$f_v(t) = f(t, A_0^{-\alpha} v(t), \int_0^t g(t, A_0^{-\alpha} v(s)) ds)$$

$$\text{and}\quad I_i(v(\theta_i)) = v(\theta_i^+) - v(\theta_i^-).$$

Then if follows that the function $f_v(t)$ is Holder continuous such that

$$\|f_v(t) - f_v(\tau)\| \leq C_7 |t - \tau|^\mu, \quad \text{where } \mu = min\{\epsilon, \eta\rho\}.$$

**Lemma 3.1** ([19]). *Let the functions $f_v(t)$ is continuous on $[0, T]$. Then for any $0 \leq t_2 \leq t_1 \leq T$, $\quad 0 \leq \alpha < \beta$, the following inequality holds*

$$\begin{aligned} \|A_0^\alpha [\int_0^{t_1} U_v(t_1, s) f_v(s) ds - \int_0^{t_2} U_v(t_2, s) f_v(s) ds]\| \\ \leq C_\alpha |t_1 - t_2|^{1 - \alpha} (|log(t_1 - t_2)| + 1). \end{aligned} \tag{3.3}$$

**Theorem 3.2.** *Let the assumptions* (A1)–(A7) *are satisfied, then there exists at least one continuously differential solution of the equation* (1.1)–(1.3).

*Proof.* To study the existence problem, we must introduce a set $\mathcal{S}$ of function $v(t),\ t \in J$ and a transformation $w_v = \Psi v$ defined by $w_v = A_0^{\alpha} w$, where $w$ is the unique solution of

$$\begin{aligned} \frac{dw}{dt} + A_v(t)w &= f(t, A_0^{-\alpha}v(t), \int_0^t g(t,s,A_0^{-\alpha}v(s))ds), && (3.4)\\ w(0) &= u_0, && (3.5)\\ \Delta w(\theta_i) &= I_i(v(\theta_i)),\ i = 1,2,3,\dots,m. && (3.6) \end{aligned}$$

We show that $\Psi$ has a fixed point, that is, there is a function $y \in \mathcal{S}$ such that $\Psi y = y$ and so $v = A_0^{\alpha} y$ is the required solution of our problem (1.1)–(1.3).

Define the set

$$\mathcal{S} = \{v \in Y : \|v(t) - v(\tau)\|_{\mathcal{PC}} \leq K|t-\tau|^{\eta} \ \text{ for } t, \tau \in J,\ v(0) = A_0^{\alpha}u_0\}, \qquad (3.7)$$

where $K$ is a positive constant and $\eta$ is any number satisfying $0 < \eta < \beta - \alpha$ and $Y$ is a Banach space $\mathcal{PC}(J : X)$ with usual sup norm. From assumption (A7), and the definition of $\mathcal{S}$ it follows that if $T$ is sufficiently small (depending on $K, \eta, \|A_0^{\alpha}u_0\|$ then

$$\|v(t)\|_{\mathcal{PC}} < r \ \text{ for } t \in J.$$

Hence the operator $A_v(t) = A(t, A_0^{-\alpha}v(t))$ is well defined and satisfies the conditions

$$\begin{aligned} \|(A_v(t) - A_v(\tau))A_0^{-1}\| &= C_8[|t-\tau|^{\epsilon} + \|v(t) - v(\tau)\|_{\mathcal{PC}}^{\rho}],\\ &= C_8[|t-\tau|^{\epsilon} + K|t-\tau|^{\eta\rho}],\\ &= C_9|t-\tau|^{\mu}, \end{aligned}$$

where $\mu = \min\{\epsilon, \eta\rho\}$. Further, if $v(0) = A_0^{\alpha}u_0$,

$$A_v(0) = A(0, A_0^{-\alpha}v(0)) = A(0, A_0^{-\alpha}A_0^{\alpha}u_0) = A(0, u_0) = A_0,$$

and it follows that for every $t \in J$ and $\lambda$ with $Re\lambda \leq 0$,

$$\begin{aligned} \|[\lambda I - A_v(t)]^{-1}\| &\leq C_{10}[|\lambda| + 1]^{-1},\\ \|[A_v(t) - A_v(\tau)]A_v^{-1}(s)\| &\leq C_{11}|t-\tau|^{\mu} \end{aligned}$$

for every $t, \tau, s \in J$.

By the assumptions (i)–(iv) there exists a fundamental solution $U_v(t,s)$ corresponding to $A_v(t)$, and all estimates for fundamental solutions derived in Theorem 2.1 hold uniformly with respect to $v \in \mathcal{S}$.

Since, $f_v(0) = f(0, A_0^{-\alpha}v(0), 0)$ is independent of $v$, we have from the above inequalities

$$\|f_v(t)\| \leq \mathcal{M}_1, \ \text{ and } \ \|I_i(v(\theta_i))\| \leq \mathcal{M}_2,$$

where $\mathcal{M}_1 > 0$ and $\mathcal{M}_2 > 0$ from Lemma 3.1 and using (3.1)-(3.2) we get

$$\left\| A_0^{\alpha}\left[\int_0^{t_1} U_v(t_1,s)f_v(s)ds - \int_0^{t_2} U_v(t_2,s)f_v(s)ds\right]\right\| + \left\| A_0^{\alpha}\sum_{0<\theta_i<t_i}\left[U_v(t_1,\theta_i) - U_v(t_2,\theta_i)\right] I_i(v(\theta_i))\right\|$$
$$\leq \mathcal{M}_1 C_{12}|t_1-t_2|^{1-\alpha}(|log(t_1-t_2)|+1) + \mathcal{M}_2 C_{13}|t_1-t_2|^{1-\alpha}(t_2-\theta_i)^{-1}.$$

We shall show that the operator $\Psi : \mathcal{S} \to Y$ defined by

$$\Psi v(t) = A_0^{\alpha}U_v(t,0)u_0 + A_0^{\alpha}\left[\int_0^t U_v(t,s)f_v(s)ds + \sum_{0<\ \theta_i<t} U_v(t,\theta_i)I_i(v(\theta_i))\right] \quad (3.8)$$

has a fixed point. This fixed point is the solution of equation (1.1)–(1.3). Clearly $\mathcal{S}$ is closed convex and bounded set of $Y$. First we show that $\Psi$ maps $\mathcal{S}$ into itself. Obviously $\Psi v(0) = A_0^{\alpha}u_0$. For any $0 \leq \alpha < \beta \leq 1$ and $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned}\|\Psi v(t_1) - \Psi v(t_2)\| \quad \leq \quad & \|A_0^{\alpha}[U_v(t_1,0) - U_v(t_2,0)]A_0^{-\beta}\|\|A_0^{\beta}u_0\| \\ & + \left\| A_0^{\alpha}\int_0^{t_1} U_v(t_1,s)f_v(s)ds + A_0^{\alpha}\sum_{0<\ \theta_i<t_1} U_v(t_1,\theta_i)I_i(v(\theta_i)) \right. \\ & \left. - \left[A_0^{\alpha}\int_0^{t_2} U_v(t_2,s)f_v(s)ds + A_0^{\alpha}\sum_{0<\ \theta_i<t_2} U_v(t_2,\theta_i)I_i(v(\theta_i))\right]\right\|.\end{aligned}$$

Thus, for T sufficiently small,

$$\begin{aligned}\|\Psi v(t_1) - \Psi v(t_2)\| \quad \leq \quad & rC_{14}|t_1-t_2|^{\beta-\alpha} + C_{15}\mathcal{M}_1 T|t_1-t_2|^{1-\alpha} \\ & + C_{16}\mathcal{M}_2|t_1-t_2|^{1-\alpha}(t_2-\theta_i)^{-1} \end{aligned} \quad (3.9)$$

for $\eta < \beta - \alpha$ and $i = 1, 2, ..., m$. Hence $\Psi$ maps $\mathcal{S}$ into itself.

Next we show that this operator is continuous on the space $Y$. Let $v_1, v_2 \in \mathcal{S}$ and set $w_1 = A_0^{-\alpha}\Psi v_1,\ w_2 = A_0^{-\alpha}\Psi v_2$, then

$$\frac{dw_j}{dt} + A_{v_j}(t)w_j = f_{v_j}(t), \quad (3.10)$$

$$w_j(0) = u_0 \quad j = 1, 2., \quad (3.11)$$

$$\Delta w(\theta_i) = I_i(v(\theta_i)),\ i = 1, 2, 3, \ldots, m. \quad (3.12)$$

Therefore,

$$\frac{d(w_1 - w_2)}{dt} + A_{v_1}(t)(w_1 - w_2) = [A_{v_2}(t) - A_{v_1}(t)]w_2 + f_{v_1}(t) - f_{v_2}(t).$$

It is easy to see that the function $A_{v_2}(t)w_2(t)$ and $A_0A_{v_2}^{-1}$ are uniformly Holder continuous, and so $A_0w_2(t) = [A_0A_{v_2}^{-1}]A_{v_2}(t)w_2(t)$ is uniformly Holder continuous.

Similarly the functions

$$f_{v_1}(t) - f_{v_2}(t) \quad \text{and} \quad I_i(v_1(\theta_i)) - I_i(v_2(\theta_i)) \ \ \forall \ \ i = 1, 2, ..., m$$

are also uniformly Holder continuous in $[\tau, T], \ \tau > 0$. Hence we have

$$\begin{aligned}[w_1(t) - w_2(t)] &= U_{v_1}(t,\tau)[w_1(\tau) - w_2(\tau)] + \int_0^t U_{v_1}(t,s)\Big([A_{v_2}(s) - A_{v_1}(s)]w_2(s) \\ &\quad +[f_{v_1}(s) - f_{v_2}(s)]\Big)ds + \sum_{0<\ \theta_i<t} U_{v_1}(t,\theta_i)[I_i(v_1(\theta_i)) - I_i(v_2(\theta_i))].\end{aligned}$$

Since $A_0^\alpha \int_0^t U_{v_2}(t,s) f_{v_2}(s)ds + \sum_{0<\ \theta_i<t} U_{v_2}(t,\theta_i)I_i(v_2(\theta_i))$ is a bounded function, it follows that

$$\|A_0 w_2(t)\| \le C_{17} t^{\beta-1}.$$

Hence we can take $\tau \to 0$ in the above equation, we get

$$\begin{aligned}[w_1(t) - w_2(t)] &= \int_0^t U_{v_1}(t,s)\Big([A_{v_2}(s) - A_{v_1}(s)]w_2(s) + [f_{v_1}(s) - f_{v_2}(s)]\Big)ds \\ &\quad + \sum_{0<\ \theta_i<t} U_{v_1}(t,\theta_i)[I_i(v_1(\theta_i)) - I_i(v_2(\theta_i))].\end{aligned}$$

Since $w_1 = A_0^{-\alpha}\Psi v_1$ and $w_2 = A_0^{-\alpha}\Psi v_2$ and from (A3)–(A6), it follows that

$$\begin{aligned}[\Psi v_1(t) - \Psi v_2(t)] &\le \int_0^t \|A_0^\alpha U_{v_1}(t,s)\|\Big([\|A_{v_2}(s) - A_{v_1}(s)]w_2(s)\| + \|f_{v_1}(s) - f_{v_2}(s)\|\Big)ds \\ &\quad + \sum_{0<\ \theta_i<t} \|A_0^\alpha U_{v_1}(t,\theta_i)\|\|[I_i(v_1(\theta_i)) - I_i(v_2(\theta_i))]\| \\ &\le \int_0^t C_{18}|t-s|^{-\alpha}\Big[C_{19}\|v_1(s) - v_2(s)\|^\rho s^{\beta-1} + C_{20}\|v_1(s) - v_2(s)\|^\rho\Big]ds \\ &\quad + \mathcal{M}_2 \sum_{i=1}^m |t-\theta_i|^{-\alpha}\|v_1(\theta_i)) - v_2(\theta_i))\|^\rho.\end{aligned}$$

Hence

$$\begin{aligned}\|\Psi v_1(t) - \Psi v_2(t)\| &\le \Big[C_{21}|t-s|^{-\alpha}s^{\beta-1} + C_{22} \\ &\quad + \mathcal{M}_2 \sum_{i=1}^m |t-\theta_i|^{-\alpha}\Big] \max\{\|v_1 - v_2\|^\rho\} . \qquad (3.13)\end{aligned}$$

This shows that $\Psi : \mathcal{S} \to Y$ is continuous. This completes the proof. □

**Theorem 3.3.** *Let the assumptions* (A1), (A3)-(A6) *hold with* $\rho = 1$*. Then the assertion of Theorem 3.1 is valid and the solution is unique.*

*Proof.* If $\rho = 1$, then from (3.13) shows that for $T$ sufficiently small $\Psi$ is a contraction, that is $\|\Psi v_1(t) - \Psi v_2(t)\| \leq \mathcal{K}\|v_1 - v_2\|$ for some $\mathcal{K} < 1$. Hence by the Banach fixed point theorem $\Psi$ has a unique fixed point. □

In section 4, we discuss the case when $U_u(t, s)$ is compact by using Schauder fixed point thorem. We need to prove that $\Psi$ is a compact operator (or completely continuous). We claim that the set $\Psi\mathcal{S}$ is contained in a compact subset of $Y$. Indeed, the function $v(t)$ of $\mathcal{S}$ are uniformly bounded and equicontinuous. By Arzela-Ascoli's theorem it is sufficient to show that for each $t$ the set $\{\Psi v(t) : v \in \mathcal{S}\}$ is contained in a compact subset of $X$.

# 4 $U_u(t, s)$ is compact

Note that a compact (or completely continuous) operator is a continuous operator which maps a bounded set into a precompact set. We shall make the following assumptions.

(H1) $f$ is continuous and maps a bounded set into a bounded set.

(H2) $I_i : X \to X, i = 1, 2, 3, ..., m$ are compact operator, and $U_v(\cdot, \cdot)$ is also compact ($U_v(t, s)$ is a compact operator for any $t > 0$).

Next we show that $\Psi$ is continuous on the space $Y$. Let $v \in \mathcal{S}$ and set $w = A_0^{-\alpha}\Psi v$. Then

$$\frac{dw}{dt} + A_v(t)w = f_v(t), \tag{4.1}$$

$$w(0) = u_0, \tag{4.2}$$

$$\Delta w(t) = I_i(v(\theta_i)),\ i = 1, 2, \ldots, m,\ 0 < \theta_1 <, \ldots \theta_m < T. \tag{4.3}$$

Let $\mathcal{S}$ be a closed convex set in Banach space $Y$ and let $\Psi$ be a continuous operator from $\mathcal{S}$ into $Y$ such that $\Psi\mathcal{S}$ is contained in $\mathcal{S}$. To show that the closure of $\Psi\mathcal{S}$ is compact. Let

$$\mathcal{S} = \{v \in Y : \|v(t) - v(\tau)\|_{\mathcal{PC}} \leq K|t - \tau|^\eta \text{ for } t, \tau \in [0, T],\ v(0) = A_0^\alpha u_0\}.$$

Consider an operator $\Psi$ on $\mathcal{S}$ defined by

$$\begin{aligned}\Psi v(t) &= A_0^\alpha U_v(t, 0)u_0 + A_0^\alpha \left[\int_0^t U_v(t, s)f_v(s)ds + \sum_{0< \ \theta_i<t} U_v(t, \theta_i)I_i(v(\theta_i))\right] \\ &= \Psi_1 v(t) + \Psi_2 v(t)\end{aligned}$$

where

$$\Psi_1 v(t) = A_0^\alpha \left[U_v(t, 0)u_0 + \int_0^t U_v(t, s)f_v(s)ds\right], \quad 0 \leq s \leq t \leq T$$

and

$$\Psi_2 v(t) = A_0^\alpha \sum_{0< \theta_i<t} U_v(t,\theta_i)I_i(v(\theta_i)), \quad 0 \le \theta_i < t \le T.$$

From our assumptions, $\Psi$ is a coninuous mapping from $\mathcal{S}$ to $\mathcal{S}$. Thus we are able to apply Schauder's fixed point theorem to obtain a fixed point. For that we need to prove that $\Psi$ is a compact operator or $\Psi_1$ and $\Psi_2$ are both compact operators.

To prove the compactness of $\Psi_2$, note that

$$\begin{aligned}
\Psi_2 v(t) &= A_0^\alpha \sum_{0< \theta_i< t} U_v(t,\theta_i)I_i(v(\theta_i)) \\
&= A_0^\alpha \left\{ \begin{array}{ll} 0 & \text{if} \quad t \in [0,\theta_1] \\ U_v(t,\theta_1)I_1(v(\theta_1)) & \text{if} \ \ t \in (\theta_1,\theta_2] \\ \cdots & \\ \sum_{i=1}^m U_v(t,\theta_i)I_i(v(\theta_i)) & \text{if} \ \ t \in (\theta_m, T] \end{array} \right\}
\end{aligned}$$

and that interval $[0,T]$ is divided into finite subintervals by $\theta_i, \ i = 1,2,3,...,m$, so that we only need to prove that

$$\mathcal{Z} = \{U(\cdot,\theta_1)I_1(v(\theta_1)) : \cdot \in [\theta_1,\theta_2], v \in Y\} \tag{4.4}$$

is precompact in $\mathcal{PC}([\theta_1,\theta_2],X)$, as the cases for other subintervals are the same.

From the above assumption, we see that for each $t \in [\theta_1,\theta_2]$, the set

$$\{U_v(t,\theta_1)I_1(v(\theta_1)) : v \in \mathcal{S}\}$$

is precompact in $X$.

Using the semigroup property, we get

$$\begin{aligned}
A_0^\alpha \|U_v(t,\theta_1)I_1(v(\theta_1)) - U_v(s,\theta_1)I_1(v(\theta_1))\| &\le \|A_0^\alpha[U_v(t,\theta_1) - U_v(s,\theta_1)]\|\|I_1(v(\theta_1))\| \\
&\le M_5 C_{19}|t-s|^{1-\alpha} \qquad (4.5)
\end{aligned}$$

for any $t,s \in [0,T]$.

Thus, the functions in $\mathcal{Z}$ are equicontinuous due to the compactness of $I_1$ and the strong continuity of $U_v(\cdot,\cdot)$. An application of the Arzela-Ascoli's theorem justifies the precompactness of $\mathcal{Z}$. Therefore, $\Psi_2$ is a compact operator.

The same argument can be used to prove that the compactness of $\Psi_1$. That is, for any $0 \le \alpha < \beta \le 1$. The set $\{A_0^\alpha U_v(t,0)u_0 : v \in \mathcal{S}, \ \|A_0^\alpha u_0\| < r\}$ is precompact in $X$, since $U_v(\cdot,\cdot)$ is compact.

$$\|A_0^\alpha U_v(t,0)A_0^{-\beta}\|\|A_0^\beta u_0\| \le rt^{\beta-\alpha}.$$

For each $t \in (0,T]$ and $\epsilon \in (0,t)$,

$$\begin{aligned}
\left\{A_0^\alpha \int_0^{t-\epsilon} U_v(t,s)f_v(s)ds\right\} &= \left\{A_0^\alpha \int_0^{t-\epsilon} U_v(t,t-\epsilon)U_v(t-\epsilon,s)f_v(s)ds\right\} \\
&= \left\{U_v(t,t-\epsilon)\int_0^{t-\epsilon} A_0^\alpha[U_v(t-\epsilon,s)]f_v(s)ds\right\}
\end{aligned}$$

is precompact in $X$ since $U_v(\cdot,\cdot)$ is compact. Then, as

$$U_v(t,t-\epsilon)\int_0^{t-\epsilon} A_0^\alpha[U_v(t-\epsilon,s)]f_v(s)ds \to A_0^\alpha\int_0^t U_v(t,s)f_v(s)ds,\ as\ \epsilon\to 0,$$

we conclude that $\left\{A_0^\alpha\int_0^t U_v(t,s)f_v(s)ds : v\in\mathcal{S}\right\}$ is precompact in $X$, using the total boundedness. Therefore, for each $t\in[0,T]$, $\{(\Psi_1 v)(t) : v\in\mathcal{S}\}$ is precompact in $X$.

Next, we show that the equicontinuity of

$$\mathcal{P}=\{(\Psi_1 v)(\cdot) : \cdot\in[0,T],\ v\in\mathcal{S}\}. \tag{4.6}$$

The equicontinuity of $\{A_0^\alpha U_v(\cdot,s)u_0 : \cdot\in[0,T],\ v\in\mathcal{S}\}$ can be shown using the condition (4.1), for the secondterm in $\mathcal{P}$, we let $0\le\alpha<\beta\le 1$ and $0\le t_1\le t_2\le T$, we have

$$\begin{aligned}
&\left\|A_0^\alpha\int_0^{t_2} U_v(t_2,s)f_v(s)ds - A_0^\alpha\int_0^{t_1} U_v(t_1,s)f_v(s)ds\right\|\\
= &\left\|A_0^\alpha\left(\int_0^{t_1}[U_v(t_2,s)-U_v(t_1,s)]f_v(s)ds+\int_{t_1}^{t_2} U_v(t_2,s)f_v(s)ds\right)\right\|\\
\le &\left\|A_0^\alpha\int_0^{t_1}[U_v(t_2,s)-U_v(t_1,s)]f_v(s)ds\right\|+\int_{t_1}^{t_2}\|A_0^\alpha U_v(t_2,s)\|\|f_v(s)\|ds\\
\le &\left\|A_0^\alpha\int_0^{t_1}[U_v(t_2,s)-U_v(t_1,s)]f_v(s)ds\right\|\\
&+(\beta-\alpha)^{-1}N_1(t_2-s)^{-\alpha}\int_{t_1}^{t_2}\|f_v(s)\|ds.
\end{aligned}\tag{4.7}$$

If $t_1=0$, then the right-hand side of (4.7) can be made small when $t_2$ is small independently of $v\in\mathcal{S}$. If $t_1>0$, then we can find a small number $\xi>0$ so that if $t_1\le\xi$, then the right-hand side of (4.7) can be estimated as

$$\begin{aligned}
&\left\|A_0^\alpha\int_0^{t_1}[U_u(t_2,s)–U_u(t_1,s)]f_v(s)ds\right\|+(\beta-\alpha)^{-1}N_1(t_2-s)^{-\alpha}\int_{t_1}^{t_2}\|f_v(s)\|ds\\
&\quad\le\ \xi(\beta-\alpha)^{-1}N_1\Big[(t_2-s)^{-\alpha}+(t_1-s)^{-\alpha}\Big]\max\|f_v(s)\|\\
&\qquad+(\beta-\alpha)^{-1}N_1(t_2-s)^{-\alpha}\int_{t_1}^{t_2}\|f_v(s)\|ds
\end{aligned}$$

which can be made small when $t_2-t_1$ is small independently of $v\in\mathcal{S}$. If $t_1>\xi$, then the right-hand side of (4.7) can be estimated as

$$\left\|A_0^\alpha \int_0^{t_1} [U_v(t_2,s) - U_u(t_1,s)] f_v(s) ds\right\| + (\beta-\alpha)^{-1} N_1 (t_2-s)^{-\alpha} \int_{t_1}^{t_2} \|f_v(s)\| ds$$
$$\begin{aligned} \leq & \int_0^{t_1-\xi} \|A_0^\alpha [U_v(t_2,s) - U_v(t_1,s)] f_v(s)\| ds \\ & + \int_{t_1-\xi}^{t_1} \|A_0^\alpha [U_u(t_2,s) - U_u(t_1,s)] f_v(s)\| ds + (\beta-\alpha)^{-1} N_1 (t_2-s)^{-\alpha} \int_{t_1}^{t_2} \|f_v(s)\| ds \\ \leq & \int_0^{t_1-\xi} \|A_0^\alpha [U_v(t_2,s) - U_v(t_1,s)] f_v(s)\| ds \\ & + \xi(\beta-\alpha)^{-1} N_1 \Big[(t_2-s)^{-\alpha} + (t_1-s)^{-\alpha}\Big] \max \|f_v(s)\| \\ & + (\beta-\alpha)^{-1} N_1 (t_2-s)^{-\alpha} \int_{t_1}^{t_2} \|f_v(s)\| ds. \end{aligned}$$

Now, as $U_v(\cdot,\cdot)$ is compact, $U_u(t,s)$ is operators norm continuous for $t > 0$. Thus $U_u(t,s)$ is operator norm continuous uniformly for $t \in [\xi, T]$. Therefore, $\|A_0^\alpha [U_v(t_2,s) - U_u(t_1,s)]\|$ and hence

$$\int_0^{t_1-\xi} \|A_0^\alpha [U_v(t_2,s) - U_v(t_1,s)] f_v(s)\| ds$$

can be made small when $t_2 - t_1$ is small independently of $v \in \mathcal{S}$. Accordingly, we see that the Arzela-Ascoli theorem, and hence $\Psi$ is also a compact operator. Now, Schauder fixed point theorem implies that $\Psi$ has a fixed point. This completes the proof.

**Remark 4.1.** *When there is no impulsive initial condition (1.3), Theorem 3.1 reduces to the results proved in [4]. However the Banach space here in $\mathcal{PC}$ whereas in [4] it is $\mathcal{C}$. This is the main difference from that paper.*

# 5 Example

Consider the following nonlinear parabolic integrodifferential equation

$$\begin{aligned} \frac{\partial z}{\partial t} + \Sigma_{|\alpha|} &= {}_{2m} a_\alpha(v,t,z,Dz,...,D^{2m-1}z) D^\alpha z \\ &= f\Big(v,t,z,Dz,...,D^{2m-1}z, \int_0^t g(v,t,s,z,Dz,...,D^{2m-1}z)\Big) ds, \end{aligned} \tag{5.1}$$

$$\frac{\partial^j z}{\partial x^j} = 0 \text{ on } \ S_T = \{(v,t) : v \in \partial\Omega,\ 0 \leq t \leq T\},\ 0 \leq j \leq m-1, \tag{5.2}$$

$$u(v,0) = 0 \text{ on } \Omega_0 = \{(v,0) : v \in \partial\Omega\}, \tag{5.3}$$

$$\Delta z|_{t=t_i} = I_i(z) = \int_\Omega d_i(q,s) cos^2 v(s) ds, \quad 1 \leq i \leq n \tag{5.4}$$

in a cylinder $Q_T = \Omega \times (0, T)$ with coefficients in $\overline{Q_T}$, where $\Omega$ is a bounded domain in $\mathcal{R}^n$, $\partial\Omega$ the boundary of $\Omega, x$ is the outward normal and $d_i \in \mathcal{C}(\overline{\Omega} \times \overline{\Omega}; \mathcal{R}^n)$ for each $i = 1, 2, ..., n$. Here the parabolicity means that for any vector $y \neq 0$ and for arbitrary values of $z, Dz, ..., D^{2m-1}z$,

$$(-1)^m Re\{\Sigma_{|\alpha|} =_{2m} a_\alpha(v, t, z, Dz, ..., D^{2m-1}z)y^\alpha\} \geq C|y|^{2m}, \ C > 0.$$

If $z_0(v) \in C^{2m-1}(\bar{Q})$, then $A_0 z = \Sigma_{|\alpha|} = {}_{2m} a_\alpha(v, t, z, Dz, ..., D^{2m-1}z)D^\alpha z$ is a strongly elliptic operator with continuous coefficients. So the condition (i) holds. Let us take $X$ to be $L^p(\overline{\Omega})$, $1 < p < \infty$. Then $A_0^{-1}$ maps bounded subsets of $L^p(\Omega)$ in to bounded subsets of $W^{2m,p}(\Omega)$, so it is a completely continuous operator in $L^p(\Omega)$.

Further, if $(2m-1)/2m < \alpha < 1$, then [6]

$$|D^\beta A_0 - \alpha z|_{0,p}^\Omega \leq C|z|_{0,p}^\Omega \ , \ \ 0 \leq |\beta| \leq 2m-1,$$

where $C$ depends only on a bound on the coefficients $A_0$, on a module of strong ellipticity and on a modulus of continuity of the leading coefficients. Here the norm is defined as

$$|z|_{j,p}^\Omega = \left\{ \sum_{|\alpha| \leq j} \int_\Omega |D^\alpha z(v)|^p dv \right\}^{\frac{1}{p}}$$

for any nonnegative integer $j$ and a real number $p$, $1 \leq p < \infty$. It follows that if $f$ and $a_\alpha$ are continuously differentiable in all variables, then (A4) and (A3) hold with $\sigma = \rho = 1$. Hence there exist fundamental operator solution $U_v(t, s)$ for the equation (5.1)–(5.4). The nonlinear functions $f, g$ satisfy the conditions (A4), (A5) and $I_i$ also satisfy the condition (A6). Hence by the above theorem there exist a local solution for the equation (5.1)–(5.4).

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