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Null Biminimal General Helices in the Lorentzian Heisenberg Group

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Abstract : In this paper, we study null biminimal curves in the Lorentzian Heisenberg group Heis^3. We characterize non-geodesic null biminimal general helix in terms of its curvatures and torsions in the Lorentzian Heisenberg group Heis^3.

Keywords : Heisenberg group; Biminimal curve; General helix.
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1 Introduction

Let $f:(M,g)\to(N,h)$ be a smooth function between two Lorentzian manifolds. f is harmonic over compact domain $\Omega\subset M$ if it is a critical point of the energy

$$E(f)=\int_\Omega h(df,df)\,dv_g,$$

where dv_g is the volume form of M. From the first variation formula it follows that is harmonic if and only if its first tension field $\tau(f)=\text{trace}_g\nabla df$ vanishes.

Harmonic maps between Riemannian manifolds were first introduced and established by Eells and Sampson [1] in 1964. Afterwards, there were two reports on harmonic maps by Eells and Lemaire [2, 3] in 1978 and 1988.

The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))\,dv_g, \tag{1.1}$$

where $\tau(f) = trace_g \nabla df$ is the tension field of f. Using the first variational formula one sees that f is a biharmonic function if and only if its bitension field vanishes identically, i.e.

$$\widetilde{\tau}(f) := -\triangle^f(\tau(f)) - \text{trace}_g R^N(df, \tau(f))df = 0, \tag{1.2}$$

where

$$\triangle^f = -\text{trace}_g(\nabla^f)^2 = -\text{trace}_g\left(\nabla^f\nabla^f - \nabla^f_{\nabla^M}\right) \tag{1.3}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N, h) defined by

$$R(X,Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z.$$

Biharmonic maps, which generalized harmonic maps, were first studied by Jiang [4] in 1986.

Recently, there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations [5–9], because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE. In differential geometry, harmonic maps, candidate minimisers of the Dirichlet energy, can be described as constraining a rubber sheet to fit on a marble manifold in a position of elastica equilibrium, i.e. without tension [2]. However, when this scheme falls through, and it can, as corroborated by the case of the two-torus and the two-sphere [10], a best map will minimise this failure, measured by the total tension, called bienergy. In the more geometrically meaningful context of immersions, the fact that the tension field is normal to the image submanifold, suggests that the most effective deformations must be sought in the normal direction [11–22].

An isometric immersion $f : (M, g) \longrightarrow (N, h)$ is called a $\lambda-$biminimal immersion if it is a critical point of the functional:

$$E_{2,\lambda}(f) = E_2(f) + \lambda E(f) \quad , \lambda \in \mathbb{R}.$$

The Euler-Lagrange equation for $\lambda-$biminimal immersions is

$$\widetilde{\tau}(f)^{\perp} = \lambda\tau(f). \tag{1.4}$$

Particularly, f is called a biminimal immersion if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{f_t\}$ through $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is $\widetilde{\tau}(f)^{\perp} = 0$. Here $\widetilde{\tau}(f)^{\perp}$ is the normal component of $\widetilde{\tau}(f)$.

In this paper, we study biminimal curves in Heisenberg group Heis^3. Then we prove that the non-geodesic null biminimal general helices are circular helices. We characterize non-geodesic null biminimal general helix in terms of its curvatures and torsions in the Lorentzian Heisenberg group Heis^3.

2 Preliminaries

The Lorentzian Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x}y + x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by:

$$g = -dx^2 + dy^2 + (xdy + dz)^2,$$

where

$$\omega^1 = dz + xdy, \qquad \omega^2 = dy, \qquad \omega^3 = dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = 0, \quad [e_2, e_1] = 0,$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1, 2, 3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Moreover, we put

$$R_{abc} = R(e_a, e_b)e_c, \;\; R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a, b, c and d take the values $1, 2$ and 3.

Then the non-zero components of the Riemannian curvature tensor field and of the Riemannian curvature tensor are, respectively,

$$R_{121} = -e_2, \quad R_{131} = -e_3, \quad R_{232} = 3e_3,$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.4}$$

3 Null Biminimal Curves in Lorentzian Heisenberg Group

Let $\gamma : I \longrightarrow Heis^3$ be a null curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to Lorentzian Heisenberg group $Heis^3$ along γ defined as follows: T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_T T &= \kappa_1 N, \\ \nabla_T N &= \kappa_2 T - \kappa_1 B, \\ \nabla_T B &= -\kappa_2 N, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} g(T,T) &= g(B,B) = 0, \; g(N,N) = 1, \\ g(T,N) &= g(N,B) = 0, \; g(T,B) = 1, \end{aligned} \tag{3.2}$$

and $\kappa_1 = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ and κ_2 is its torsion.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic null curve parametrized by arc length. γ is a non-geodesic null biminimal curve if and only if*

$$\begin{aligned} \kappa_1'' + 2\kappa_1^2 \kappa_2 &= 4\kappa_1 B_1^2, \\ 2\kappa_1' \kappa_2 + \kappa_2' \kappa_1 &= -4\kappa_1 N_1 B_1. \end{aligned} \tag{3.4}$$

Proof. Using Eq. (1.4) and Eq. (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - \kappa_1 R(T,N)T \\ &= (2\kappa_2' \kappa_1 + \kappa_1' \kappa_2)T + (\kappa_1'' + 2\kappa_1^2 \kappa_2)N + (3\kappa_1 \kappa_1')B + \kappa_1 R(T,N)T \\ &= 0. \end{aligned}$$

By Eq. (3.2), we see that γ is a biharmonic curve if and only if

$$\begin{aligned} \kappa_1'' + 2\kappa_1^2\kappa_2 &= \kappa_1 R(T,N,T,N), \\ 2\kappa_2\kappa_1' + \kappa_1\kappa_2' &= \kappa_1 R(T,N,T,B). \end{aligned} \tag{3.5}$$

A direct computation using Eq. (2.4) yields

$$R(T,N,T,N) = -4B_1^2,$$

and

$$R(T,N,T,B) = 4N_1B_1.$$

These, together with Eq. (3.5), complete the proof of the theorem. □

4 Null Biminimal General Helix in the Lorentzian Heisenberg Group

Definition 4.1. *Let $\gamma : I \longrightarrow Heis^3$ be a curve and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. If κ_1 and κ_2 are positive constant along γ, then γ is called circular helix with respect to Frenet frame [11].*

Definition 4.2. *Let $\gamma : I \longrightarrow Heis^3$ be a curve and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. A curve γ such that*

$$\frac{\kappa_1}{\kappa_2} = constant \tag{4.1}$$

is called a general helix with respect to Frenet frame [11].

Theorem 4.3. *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic null biminimal general helix parametrized by arc length. If $N_1B_1 = constant$, then γ is circular helix.*

Proof. We can use Eq. (3.3) to compute the covariant derivatives of the vector fields T, N and B as:

$$\begin{aligned} \nabla_T T &= T_1'e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3, \\ \nabla_T N &= (N_1' + T_2N_3 - T_3N_2)e_1 + (N_2' + T_1N_3 - T_3N_1)e_2 \\ &\quad + (N_3' + T_2N_1 - T_1N_2)e_3, \\ \nabla_T B &= (B_1' + T_2B_3 - T_3B_2)e_1 + (B_2' + T_1B_3 - T_3B_1)e_2 \\ &\quad + (B_3' + T_2B_1 - T_1B_2)e_3. \end{aligned} \tag{4.2}$$

It follows that the first components of these vectors are given by

$$\begin{aligned} &< \nabla_T T, e_1 >= T_1', \\ &< \nabla_T N, e_1 >= N_1' + T_2N_3 - T_3N_2, \\ &< \nabla_T B, e_1 >= B_1' + T_2B_3 - T_3B_2. \end{aligned} \tag{4.3}$$

On the other hand, using Frenet formulas (3.1), we have

$$\begin{aligned} &< \nabla_T T, e_1 >= \kappa_1 N_1, \\ &< \nabla_T N, e_1 >= \kappa_2 T_1 - \kappa_1 B_1, \\ &< \nabla_T B, e_1 >= -\kappa_2 N_1. \end{aligned} \tag{4.4}$$

These, together with Eq. (4.3) and Eq. (4.4) give

$$\begin{aligned} T_1' &= \kappa_1 N_1, \\ N_1' + T_2 N_3 - T_3 N_2 &= \kappa_2 T_1 - \kappa_1 B_1, \\ B_1' + T_2 B_3 - T_3 B_2 &= -\kappa_2 N_1. \end{aligned} \tag{4.5}$$

From Eq. (4.5), we have

$$B_1' = (1 - \kappa_2) N_1. \tag{4.6}$$

Suppose that γ is a be a non-geodesic biminimal general helix with respect to the Frenet frame $\{T, N, B\}$. Then,

$$\frac{\kappa_1}{\kappa_2} = c. \tag{4.7}$$

We have

$$\kappa_1' \kappa_2 = \kappa_2' \kappa_1. \tag{4.8}$$

We substitue Eq. (4.8) in Eq. (3.3), we obtain

$$\kappa_1' = -\frac{4c}{3} N_1 B_1, \quad \kappa_2' = \frac{4}{3} N_1 B_1. \tag{4.9}$$

From $N_1 B_1$ =constant, it follows that

$$\kappa_1'' = 0. \tag{4.10}$$

We substitue Eq. (4.10) in Eq. (3.4), we obtain

$$\kappa_1 \kappa_2 = 2 B_1^2. \tag{4.11}$$

Next, we replace $\kappa_2 = \frac{\kappa_1}{c}$ in Eq. (4.11)

$$\kappa_1^2 = 2c B_1^2. \tag{4.12}$$

Equation (4.12) derived and taking into account Eq. (4.6) and Eq. (4.9), becomes

$$\kappa_1 = \frac{3c}{3 - 2c}. \tag{4.13}$$

Substituting Eq. (4.7) in Eq. (4.13), we have

$$\kappa_1 = \text{constant}.$$

From Eq. (4.7), we obtain

$$\kappa_2 = \text{constant},$$

which implies γ circular helix.

Here, without loss of generality, we assemed $c > 0$. □

Corollary 4.4. *$\gamma : I \longrightarrow Heis^3$ is non-geodesic null biminimal general helix if and only if*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_2 &= \text{constant} \neq 0, \\ N_1 B_1 &= 0, \\ \kappa_1 \kappa_2 &= 2B_1^2. \end{aligned} \tag{4.14}$$

Proof. Using Theorem 3.1 and Theorem 4.3, we have Eq. (4.14). □

Corollary 4.5.

i) *If $N_1 \neq 0$, then γ is not biminimal general helix.*

ii) *If $N_1 = 0$, then*

$$T(s) = \sinh \Psi_0 e_1 + \sinh \Psi_0 \sinh \Phi(s) e_2 + \sinh \Psi_0 \cosh \Phi(s) e_3, \tag{4.15}$$

where $\Psi_0 \in \mathbb{R}$.

Proof. **i)** We use the third equation of Eq. (4.5) and B_1 =constant, we obtain

$$T_2 B_3 - T_3 B_2 = -\kappa_2 N_1.$$

Note that $T_2 B_3 - T_3 B_2 = N_1$ in above equation, we have

$$(1 + \kappa_2) N_1 = 0. \tag{4.16}$$

Using $N_1 \neq 0$, we have

$$\kappa_2 = -1. \tag{4.17}$$

Assume now that γ is biharmonic. From first Eq. (3.4), we obtain

$$\kappa_1 \kappa_2 = 2B_1^2.$$

If we substitue Eq. (4.17) in above equation, we obtain

$$\kappa_1 = -2B_1^2. \tag{4.18}$$

By multipliying both side of Eq. (4.18) with N_1, we obtain

$$\kappa_1^2 N_1 = -2B_1(B_1 N_1).$$

Using Eq. (3.4) and $N_1 \neq 0$, we conclude that $\kappa_1 = 0$, i.e., γ is a geodesic, a contradiction. These, together with Theorem 3.1 complete the proof of the corollary.

ii) Since γ is s parametrized by arc length, we can write

$$T(s) = \sinh \Psi(s) e_1 + \sinh \phi(s) \sinh \Phi(s) e_2 + \sinh \Psi(s) \cosh \Phi(s) e_3. \tag{4.19}$$

From Eq. (4.5), we obtain

$$T_1' = \kappa_1 N_1.$$

Since $N_1 = 0$, we have

$$T_1' = 0.$$

Then T_1 is constant. Using Eq. (4.19), we get

$$T_1 = \sinh \Psi_0 = \text{constant}.$$

We obtain Eq. (4.15) and corollary is proved. □

Theorem 4.6. *The parametric equations of all null biminimal general helix are:*

$$\begin{aligned} x(s) &= \frac{1}{\delta}\sinh\Psi_0 \sinh(\delta s+\sigma)+c_1, \\ y(s) &= \frac{1}{\delta}\sinh\Psi_0 \cosh(\delta s+\sigma)+c_2, \\ z(s) &= \left[\sinh\Psi_0 - \frac{[\sinh\Psi_0]^2}{\delta}\right]s - \frac{1}{2\delta^2}[\sinh\Psi_0]^2 \sinh 2(\delta s+\sigma) \\ &\quad -\frac{c_1}{\delta}\sinh\Psi_0\cosh(\delta s+\sigma)+c_3, \end{aligned} \tag{4.20}$$

where $\delta = \left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right)$ *and* $\Psi_0, c_1, c_2, c_3, \sigma \in \mathbb{R}$.

Proof. The covariant derivative of the vector field T is:

$$\nabla_T T = T_1' e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3.$$

From Eq. (4.15), we have

$$\begin{aligned} \nabla_T T &= (\Phi'(s)\sinh\Psi_0\cosh\Phi(s) + 2\sinh^2\Psi_0\cosh\Phi(s))e_2 + \\ &\quad (\Phi'(s)\sinh\Psi_0\sinh\Phi(s) + 2\sinh^2\Psi_0\sinh\Phi(s))e_3. \end{aligned}$$

Since $|\nabla_T T| = \kappa_1$, we obtain

$$\Phi(s) = \left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right)s + \sigma, \tag{4.21}$$

where $\sigma \in \mathbb{R}$.

To find equations for null biminimal general helix $\gamma(s) = (x(s), y(s), z(s))$ on the Lorentzian Heisenberg group $Heis^3$, we note that if

$$\frac{d\gamma}{ds} = T = T_1e_1 + T_2e_2 + T_3e_3, \tag{4.22}$$

and our left-invariant vector fields are

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}.$$

Then,

$$\frac{\partial}{\partial x} = e_3, \quad \frac{\partial}{\partial y} = e_2 + xe_3, \quad \frac{\partial}{\partial z} = e_1. \tag{4.23}$$

Therefore, we easily have

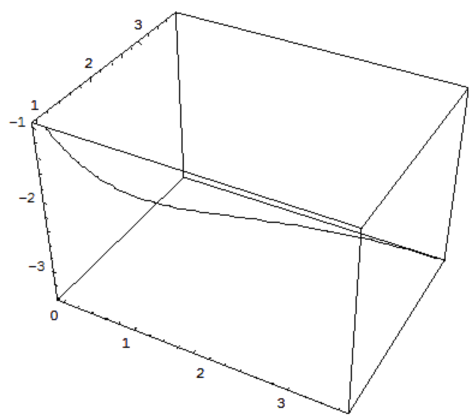
$$\begin{aligned}
\frac{dx}{ds} &= \sinh\Psi_0 \cosh\left[\left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right)s + \sigma\right], \\
\frac{dy}{ds} &= \sinh\Psi_0 \sinh\left[\left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right)s + \sigma\right], \\
\frac{dz}{ds} &= \sinh\Psi_0 \cosh\left[\left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right) + \sigma\right] \\
&\quad -x(s)\sinh\Psi_0 \sinh\left[\left(\pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}} - 2\sinh\Psi_0\right)s + \sigma\right].
\end{aligned} \tag{4.24}$$

If the system Eq. (4.24) is integrated, we obtain Eq. (4.20) and theorem is proved. □

Example 4.7. *Let us consider null biminimal general helix with* $\delta = \sinh\varphi = 1$ *and* $c_1 = c_2 = c_3 = \sigma = 0$. *Then* γ *is given by*

$$\gamma(s) = \left(\sinh(s), \cosh(s), -s + \frac{1}{2}\sinh 2s - \cosh(s)\right). \tag{4.25}$$

We can draw null biminimal general helix with helping the programme of Mathematica as following:



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