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On the Solutions of the Difference Equation $x_{n+1} = \frac{Ax_{n-(2k+1)}}{-A+\prod_{i=0}^{2k+1} x_{n-i}}$

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Abstract : In this paper, we study the solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-(2k+1)}}{-A + \prod_{i=0}^{2k+1} x_{n-i}} \quad \text{for} \quad n = 0, 1, 2, ...$$

where k is a positive number and initial conditions are non zero real numbers with $\prod_{i=0}^{2k+1} x_{-i} \neq A$.

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1 Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Rational difference equations is an important class of difference equations where they have many applications in real life for example the difference equation $x_{n+1} = \frac{a+bx_n}{c+x_n}$ which is known by Riccati Difference Equation has an applications in Optics and Mathematical Biology (see [1]) . Many researchers have investigated the behavior of the solution of rational difference equations. For example see Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Cinar [2] investigated the positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{-1 + bx_n x_{n-1}}.$$

Yalçınkaya [3] investigated the global behaviour of the rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

El-Owaidy et al. [4] studied the dynamics of the recurcive sequence

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}.$$

Elsayed [5] investigated the qualitative behavior of the solution of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}.$$

Hamza et al. [6] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}.$$

Our aim in this paper is to investigate the solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-(2k+1)}}{-A + \prod_{i=0}^{2k+1} x_{n-i}} \text{ for } n = 0, 1, 2, ... \tag{1.1}$$

where k is a positive number and initial conditions are non zero real numbers with $\prod_{i=0}^{2k+1} x_{-i} \neq A$.

Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right), \ n = 0, 1, ... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1. (Periodicity) *A sequence $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) is said to be periodic with period p if*

$$x_{n+p} = x_n \ for\ all\ n \geq -k.$$

2 Main Results

Theorem 2.1. *Assume that* $\prod_{i=0}^{2k+1} x_{-i} = p$ *and* $p \neq A$. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.*(1.1). *Then for* $n = 0, 1, ...$

$$x_{2(k+1)n+1} = \frac{A^{n+1} x_{-(2k+1)}}{(-A+p)^{n+1}}, \; x_{2(k+1)n+2} = \left(\frac{1}{A}\right)^{n+1} x_{-(2k)} \left(-A+p\right)^{n+1},$$
$$x_{2(k+1)n+3} = \frac{A^{n+1} x_{-(2k-1)}}{(-A+p)^{n+1}}, \; x_{2(k+1)n+4} = \left(\frac{1}{A}\right)^{n+1} x_{-(2k-2)} \left(-A+p\right)^{n+1},$$
$$\ldots$$
$$x_{2(k+1)n+2k+1} = \frac{A^{n+1} x_{-1}}{(-A+p)^{n+1}}, \; x_{2(k+1)n+2k+2} = \left(\frac{1}{A}\right)^{n+1} x_0 \left(-A+p\right)^{n+1}.$$

Proof. For $n = 0$ the result holds. Now assume that $n > 0$ and that our assumption holds for $n - 1$. Then

$$x_{2(k+1)n-(2k+1)} = \frac{A^{n} x_{-(2k+1)}}{(-A+p)^{n}}, \quad x_{2(k+1)n-(2k)} = \left(\frac{1}{A}\right)^{n} x_{-(2k)} \left(-A+p\right)^{n},$$
$$x_{2(k+1)n-(2k-1)} = \frac{A^{n} x_{-(2k-1)}}{(-A+p)^{n}}, \quad x_{2(k+1)n-(2k-2)} = \left(\frac{1}{A}\right)^{n} x_{-(2k-2)} \left(-A+p\right)^{n},$$
$$\ldots$$
$$x_{2(k+1)n-1} = \frac{A^{n} x_{-1}}{(-A+p)^{n}}, \quad x_{2(k+1)n} = \left(\frac{1}{A}\right)^{n} x_0 \left(-A+p\right)^{n}.$$

It follows from Eq.(1.1) that

$$x_{2(k+1)n+1} = \frac{A x_{2(k+1)n-(2k+1)}}{-A + \prod_{i=0}^{2k+1} x_{2(k+1)n-i}}$$

and we have from the above equalities

$$\prod_{i=0}^{2k+1} x_{2(k+1)n-i} = p.$$

Then

$$\begin{aligned} x_{2(k+1)n+1} &= \frac{A \frac{A^n x_{-(2k+1)}}{(-A+p)^n}}{-A+p} \\ &= \frac{A^{n+1} x_{-(2k+1)}}{(-A+p)^{n+1}}. \end{aligned}$$

Hence, we have

$$x_{2(k+1)n+1} = \frac{A^{n+1} x_{-(2k+1)}}{(-A+p)^{n+1}}.$$

Also, we get from Eq.(1.1) that

$$x_{2(k+1)n+2} = \frac{Ax_{2(k+1)n-(2k)}}{-A + \prod_{i=-1}^{2k} x_{2(k+1)n-i}}.$$

We have from Eq.(1.1) and the above equalities

$$\prod_{i=-1}^{2k} x_{2(k+1)n-i} = \frac{Ap}{-A+p}.$$

Then

$$\begin{aligned} x_{2(k+1)n+2} &= \frac{A\left(\frac{1}{A}\right)^n x_{-(2k)}\left(-A+p\right)^n}{-A+\frac{Ap}{-A+p}} \\ &= \left(\frac{1}{A}\right)^{n+1} x_{-(2k)}\left(-A+p\right)^{n+1}. \end{aligned}$$

Hence, we have

$$x_{2(k+1)n+2} = \left(\frac{1}{A}\right)^{n+1} x_{-(2k)}\left(-A+p\right)^{n+1}.$$

Similarly, one can obtain the other cases. Thus, the proof is completed. □

Theorem 2.2. *Eq.*(1.1) *has a periodic solutions of period* $(2k+2)$ *iff* $p = 2A$ *and will be take the form* $\left\{x_{-(2k+1)}, x_{-(2k)}, ..., x_{-1}, x_0, x_1, x_2, ..., x_{2k+2}, ...\right\}$.

Proof. Firstly, assume that there exists a prime period $(2k+2)$ solution $x_{-(2k+1)}$, $x_{-(2k)}$,..., x_{-1}, x_0, x_1, x_2,..., x_{2k+2},... of *Eq.*(1.1).

We have from the form of solution of *Eq.*(1.1) that

$$\begin{aligned} x_{-(2k+1)} &= \frac{A^{n+1}x_{-(2k+1)}}{(-A+p)^{n+1}}, \quad x_{-(2k)} = \left(\frac{1}{A}\right)^{n+1} x_{-(2k)}\left(-A+p\right)^{n+1}, \\ x_{-(2k-1)} &= \frac{A^{n+1}x_{-(2k-1)}}{(-A+p)^{n+1}}, \quad x_{-(2k-2)} = \left(\frac{1}{A}\right)^{n+1} x_{-(2k-2)}\left(-A+p\right)^{n+1}, \\ &\cdots \\ x_{-1} &= \frac{A^{n+1}x_{-1}}{(-A+p)^{n+1}}, \quad x_0 = \left(\frac{1}{A}\right)^{n+1} x_0\left(-A+p\right)^{n+1}. \end{aligned}$$

Then $p = 2A$.

Secondly, suppose that $p = 2A$. Then we have

$$\begin{aligned} x_{2(k+1)n+1} &= x_{-(2k+1)}, \quad x_{2(k+1)n+2} = x_{-(2k)}, \\ x_{2(k+1)n+3} &= x_{-(2k-1)}, \quad x_{2(k+1)n+4} = x_{-(2k-2)}, \\ &\cdots \\ x_{2(k+1)n+2k+1} &= x_{-1}, \quad x_{2(k+1)n+2k+2} = x_0. \end{aligned}$$

Thus, we obtain a period $(2k+2)$ solution.
The proof is completed. □

In view of Theorem 2.1 we will give the following corollaries without proofs.

Corollary 2.3. *Assume that* $A = 1$, $x_{-(2k+1)}, x_{-(2k)}, ..., x_0 > 0$ *and* $p > 2$. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.*(1.1). *Then*

$$\lim_{n\to\infty} x_{2(k+1)n+i} = \begin{cases} 0, & i = 1, 3, ..., 2k+1 \\ \infty, & i = 2, 4, ..., 2k+2 \end{cases}$$

Corollary 2.4. *Assume that* $A > 0$, $x_{-(2k+1)}, x_{-(2k)}, ..., x_0 > 0$ *and* $p > A$. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.*(1.1). *Then all solutions of Eq.*(1.1) *are positive.*

Corollary 2.5. *Assume that* $A > 0$, $x_{-(2k+1)}, x_{-(2k)}, ..., x_0 < 0$ *and* $p > A$. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.*(1.1). *Then all solutions of Eq.*(1.1) *are negative.*

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