Thai Journal of Mathematics
Volume 9 (2011) Number 1 : 95–119
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A Hybrid Method for Generalized Equilibrium, Variational Inequality and Fixed Point Problems of Finite Family of Nonexpansive Mappings

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Abstract : In this paper, we introduce a new iterative method for finding a common element of the set of solutions of a generalized equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping in real Hilbert spaces. Furthermore, we prove that the proposed iterative method converges strongly to a common element of the above three sets. Our result improve and extend the corresponding results of Kangtunyakarn and Suantai [A. Kangtunyakarn, S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Theory, Methods & Applications 71 (2009) 4448–4460], Takahashi and Takahashi [S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515] and many others.

Keywords : Fixed point; Generalized equilibrium problem; Variational inequality; Nonexpansive mapping; Inverse-strongly monotone mapping.
2010 Mathematics Subject Classification : 46C05; 47H09; 47H10.

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1 Introduction

Let C be a nonempty closed convex subset of real Hilbert space H and $G : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real number. Let $\Psi : C \to H$ be a nonlinear mapping. The generalized equilibrium problem for G and Ψ is to find $u \in C$ such that

$$G(u,v) + \langle \Psi u, v-u \rangle \geq 0, \ \forall v \in C. \tag{1.1}$$

The set of solutions for the problem (1.1) is denoted by $GEP(G,\Psi)$, i.e.,

$$GEP(G,\Psi) = \{u \in C : G(u,v) + \langle \Psi u, v-u \rangle \geq 0, \ \forall v \in C\}.$$

The equilibrium problem for G is to find $u \in C$ such that

$$G(u,v) \geq 0, \ \forall v \in C. \tag{1.2}$$

The set of solutions (1.2) is denoted by $EP(G)$. Many problems in physics, optimization, and economics require some elements of $EP(G)$, see [1, 2, 3]. Several iterative methods have been proposed to solve the equilibrium problem, see [2, 4, 5, 6, 7, 8, 9]. The variational inequality problem is to find $u \in C$ such that

$$\langle \Psi u, v-u \rangle \geq 0, \ \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $VI(\Psi, C)$.

If the case of $\Psi \equiv 0$, then the problem (1.1) is reduced to the problem (1.2). In the case of $G \equiv 0$, the problem (1.1) reduces to the variational inequality problem (1.3).

It is well known that (1.1) contains as special cases for instance minimax problems, optimization problems, Nash equilibrium problems in noncooperative games, complementarity problems, fixed point problems and variational inequalities and others, see for instance [1, 2, 7, 9, 10, 11, 12].

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2$. A mapping $A : C \to H$ is called *α-inverse-strongly monotone*, see [13], if there exists a positive real number α such that $\langle x-y, Ax-Ay \rangle \geq \alpha \|Ax-Ay\|^2$ for all $x, y \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. Goebel and Kirk [14] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory. Recall that a mapping $f : C \to C$ is *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x)-f(y)\| \leq \alpha \|x-y\|$, for all $x, y \in C$.

In 2007, Tada and Takahashi [15] and Takahashi and Takahashi [7] considered iterative methods for finding a common element of a equilibrium problem and the set of fixed points of a nonexpansive mapping. On the other hand, Takahashi and Toyoda [16] and Yao et al. [17] introduced an iterative method for finding a common element of the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping. Further, Moudafi [18] and Takahashi and Takahashi [12] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping. Recently, Colao et al. [19] introduced a new general iterative method for finding a common element of the set of solutions of a equilibrium problem and the set of common fixed points of finite family of nonexpansive mappings in a Hilbert space.

Very recently, Kangtunyakarn and Suantai [4] introduced a new mapping and the iteration method to obtain strong convergence to a common element of the set of solutions of a equilibrium problem and the set of common fixed points of finite family of nonexpansive mappings under some sufficient suitable conditions, as follows:

For a finite family of nonexpansive mappings $T_1, T_2, ..., T_N$ and sequence $\{\xi_{n,i}\}_{i=1}^N$ in $[0,1]$, they defined the mapping $K_n : C \to C$ as follows:

$$\begin{aligned} U_{n,1} &= \xi_{n,1} T_1 + (1-\xi_{n,1}) I, \\ U_{n,2} &= \xi_{n,2} T_2 U_{n,1} + (1-\xi_{n,2}) U_{n,1}, \\ U_{n,3} &= \xi_{n,3} T_3 U_{n,2} + (1-\xi_{n,3}) U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \xi_{n,N-1} T_{N-1} U_{n,N-2} + (1-\xi_{n,N-1}) U_{n,N-2}, \\ K_n = U_{n,N} &= \xi_{n,N} T_N U_{n,N-1} + (1-\xi_{n,N}) U_{n,N-1}. \end{aligned}$$

For $x_1 \in C$, let $\{u_n\}$ and $\{x_n\}$ be the sequences defined by

$$\begin{cases} G(u_n, v) + \dfrac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C; \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta) I - \alpha_n A) K_n u_n, \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

In this paper, motivated by above results, we introduce a general iterative method (3.1) below for finding a common element of the set of solutions of a generalized equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping in real Hilbert spaces. We obtain a strong convergence theorem which improves and extends the corresponding results of Kangtunyakarn and Suantai [4], Takahashi and Takahashi [12], and many others.

2 Preliminaries

Let C be closed convex subset of a Hilbert space H, let P_C be the metric projection of H onto C, i.e., for $x \in H$, P_C satisfies the property $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. It is well known that P_C is a nonexpansive mapping of H onto

C and satisfies $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$ for all $x, y \in H$. Moreover, $P_C x \in C$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \text{ and } \|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2$$

for all $x \in H$ and $y \in C$. It is easy to see that

$$u \in VI(A, C) \text{ if and only if } u = P_C(u - \lambda Au)$$

where $\lambda > 0$. It is also known that Hilbert space H satisfies *Opial's condition* [20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $p \in Tx$ and $q \in Ty$ imply $\langle x - y, p - q\rangle \geq 0$. A monotone mapping T is maximal if the graph $G(T)$ of T is not properly contained in the graph of any monotone mappings. It is known that a monotone mapping T is maximal if and only if for $(x, p) \in H \times H, \langle x - y, p - q\rangle \geq 0$ for all $(y, q) \in G(T)$ implies $p \in Tx$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mapping and let $N_C u$ be the normal cone to C at $u \in C$, i.e., $N_C u = \{w \in H : \langle u - v, w\rangle \geq 0,\ \forall v \in C\}$. Define

$$Tu = \begin{cases} Au + N_C u, & u \in C; \\ \emptyset, & u \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tu$ if and only if $u \in VI(A, C)$; see [21].

Lemma 2.1. *Let H be a real Hilbert space. Then, for all $x, y \in H$,*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.

(iii) $\|\lambda x + (1 - \lambda) y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, *for* $\lambda \in [0, 1]$.

Lemma 2.2. ([22]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \alpha_n \beta_n,\ \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. ([23]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ *for all integer* $n \ge 0$ *and*

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.4. ([6]) *Let* A *be a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\bar{\gamma}$ *and* $0 < \rho \le \|A\|^{-1}$. *Then* $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.

For solving the generalized equilibrium problem, we assume that the bifunction $G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $G(x,x) = 0$, $\forall x \in C$;

(A2) G is monotone, i.e. $G(x,y) + G(y,x) \le 0$, $\forall x, y \in C$;

(A3) $\forall x, y, z \in C$, $\lim_{t\to 0^+} G(tz + (1-t)x, y) \le G(x,y)$;

(A4) $\forall x \in C$, $y \mapsto G(x,y)$ is convex and lower semicontinuous.

Lemma 2.5. ([1]) *Let* C *be a nonempty closed convex subset of* H *and let* G *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1) - (A4)$. *Let* $r > 0$ *and* $x \in H$. *Then, there existe* $z \in C$ *such that*

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall y \in C. \tag{2.1}$$

Lemma 2.6. ([2]) *Assume that* $G : C \times C \to \mathbb{R}$ *satisfies* $(A1) - (A4)$. *For* $x \in H$ *and* $r > 0$, *define a mapping* $T_r : H \to C$ *as follows :*

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall y \in C \right\}$$

for all $z \in H$. *Then, the following hold:*

(B1) T_r *is single-valued;*

(B2) T_r *is firmly nonexpansive, i.e.* $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$, $\forall x, y \in H$;

(B3) $F(T_r) = EP(G)$;

(B4) $EP(G)$ *is closed and convex.*

Remark 2.7. *Replacing* x *with* $x - r\Psi x \in H$ *in* (2.1), *then there exists* $z \in C$ *such that* $G(z,y) + \langle \Psi x, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \ge 0$, $\forall y \in C$. *It follows by Lemma 2.6 that* $z = T_r(I - r\Psi)(x)$ *and it is easy to see that* $F(T_r(I - r\Psi)) = GEP(G, \Psi)$.

Definition 2.8. ([4]) *Let C be a nonempty convex subset of a real Banach space. Let $T_1, T_2, ..., T_N$ be a finite family of nonexpansive mappings of C into itself, and let $\xi_1, \xi_2, ..., \xi_N$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i = 1, 2, ..., N$. Define a mapping $K : C \to C$ as follows:*

$$\begin{aligned} U_1 &= \xi_1 T_1 + (1 - \xi_1) I, \\ U_2 &= \xi_2 T_2 U_1 + (1 - \xi_2) U_1, \\ U_3 &= \xi_3 T_3 U_2 + (1 - \xi_3) U_2, \\ &\vdots \\ U_{N-1} &= \xi_{N-1} T_{N-1} U_{N-2} + (1 - \xi_{N-1}) U_{N-2}, \\ K = U_N &= \xi_N T_N U_{N-1} + (1 - \xi_N) U_{N-1}. \end{aligned}$$

Such a mapping K is called the K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_1, \xi_2, ..., \xi_N$.

Lemma 2.9. ([4]) *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $T_1, T_2, ..., T_N$ be a finite family of nonexpansive mappings of C into itself, with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\xi_1, \xi_2, ..., \xi_N$ be real numbers such that $0 < \xi_i < 1$ for every $i = 1, 2, ..., N-1$ and $0 < \xi_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_1, \xi_2, ..., \xi_N$. Then $F(K) = \bigcap_{i=1}^{N} F(T_i)$.*

Lemma 2.10. ([4]) *Let C be a nonempty closed convex subset of a Banach space. Let $T_1, T_2, ..., T_N$ be a finite family of nonexpansive mappings of C into itself and $\{\xi_{n,i}\}_{i=1}^{N}$ sequence in $[0, 1]$ such that $\xi_{n,i} \to \xi_i$, as $n \to \infty$, $(i = 1, 2, ..., N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\xi_1, \xi_2, ..., \xi_N$, and $T_1, T_2, ..., T_N$ and $\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,N}$ respectively. Then, for every $x \in C$, we have $\lim_{n\to\infty} \|K_n x - Kx\| = 0$.*

Lemma 2.11. ([4]) *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T_1, T_2, ..., T_N$ be a finite family of nonexpansive mappings of H into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For every $n \in \mathbb{N}$, let K_n be a K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,N}$ with $\{\xi_{n,i}\}_{i=1}^{N} \subset [a, b]$ where $0 < a \leq b < 1$. For a sequence $\{r_n\}$ in $(0, \infty)$, let $T_{r_n} : H \to C$ be defined by*

$$T_{r_n}(x) = \left\{ z \in C : G(z, y) + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}.$$

If $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$ and $\lim_{n\to\infty} |\xi_{n,i} - \xi_{n-1,i}| = 0, \forall i \in \{1, 2, ..., N\}$, then

(i) $\lim_{n\to\infty} \left\| K_{n+1} T_{r_{n+1}} w_n - K_{n+1} T_{r_n} w_n \right\| = 0,$

(ii) $\lim_{n\to\infty} \|K_{n+1} w_n - K_n w_n\| = 0,$

for every bounded sequence $\{w_n\}$ in H.

3 Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of a generalized equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping in real Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi : C \to H$ a β-inverse-strongly monotone mapping, $A : C \to H$ a ρ-inverse-strongly monotone mapping, $f : C \to C$ a contraction mapping with constant $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\xi_{n,i}\}_{i=1}^{N-1} \subset (0,1)$, $\{\xi_{n,N}\} \subset (0,1]$, $\{\xi_i\}_{i=1}^{N-1} \subset (0,1)$, $\xi_N \in (0,1]$ be such that $\xi_{n,i} \to \xi_i$ for all $i = 1,2,...,N$. Let $K_n : C \to C$ be a K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,N}$. Suppose that $\Omega := \bigcap_{i=1}^N F(T_i) \bigcap GEP(G,\Psi) \bigcap VI(A,C) \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $\|B\| = 1$ and let $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} G(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \dfrac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C; \\ y_n = P_C(u_n - \lambda_n A u_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n B) K_n y_n \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset [a,b]$ for some $0 < a < b < 2\rho$ and $\{r_n\} \subset [c,d]$ for some $0 < c < d < 2\beta$ satisfying:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to the point $z_0 \in \Omega$, where $z_0 = P_\Omega(I - (B - \gamma f)) z_0$.

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality that $\alpha_n \leq (1-\beta_n)\|B\|^{-1}$ and $1 - \alpha_n(\bar{\gamma} - \alpha\gamma) > 0$ for all $n \in \mathbb{N}$. Since B is a strongly positive bounded linear operator on H, we have $\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}$. Observe that

$$\begin{aligned} \langle ((1-\beta_n) I - \alpha_n B) x, x \rangle &= 1 - \beta_n - \alpha_n \langle Bx, x \rangle \\ &\geq 1 - \beta_n - \alpha_n \|B\| \\ &\geq 0, \ \forall x \in H. \end{aligned}$$

By Lemma 2.4, we have $\|(1-\beta_n)I-\alpha_n B\| \le 1-\beta_n-\alpha_n\bar{\gamma}$.

We shall divide our proof into 7 steps.

Step 1. We will show that $\{x_n\}$ is bounded. For any $x, y \in C$ and $r_n \in (0, 2\beta)$, we have

$$\begin{aligned}
\|(I-r_n\Psi)x-(I-r_n\Psi)y\|^2 &= \|(x-y)-r_n(\Psi x-\Psi y)\|^2 \\
&= \|x-y\|^2 - 2r_n\langle x-y, \Psi x-\Psi y\rangle + r_n^2\|\Psi x-\Psi y\|^2 \\
&\le \|x-y\|^2 - 2r_n\left(\beta\left\|\|\Psi x-\Psi y\|^2\right\|\right) + r_n^2\|\Psi x-\Psi y\|^2 \\
&= \|x-y\|^2 - r_n(2\beta-r_n)\|\Psi x-\Psi y\|^2 \\
&\le \|x-y\|^2 \qquad (3.2)
\end{aligned}$$

which implies that $I-r_n\Psi$ is nonexpansive. The same as in (3.2), for $\lambda_n \in (0, 2\rho)$, we have $I-\lambda_n A$ is nonexpansive. Remark 2.7 implies that the sequence $\{u_n\}$ and $\{x_n\}$ are well defined. Inview of the iterative sequence (3.1), we have

$$\begin{aligned}
0 &\le G(u_n, v) + \langle \Psi x_n, v-u_n\rangle + \frac{1}{r_n}\langle v-u_n, u_n-x_n\rangle \\
&= G(u_n, v) + \frac{1}{r_n}\langle v-u_n, u_n-(x_n-r_n\Psi x_n)\rangle
\end{aligned}$$

for all $v \in C$. It follows from Lemma 2.6 that $u_n = T_{r_n}(x_n-r_n\Psi x_n)$ for all $n \in \mathbb{N}$.

Let $z^* \in \Omega$. For each $n \in \mathbb{N}$, we have $z^* = K_n z^* = T_{r_n}(z^*-r_n\Psi z^*)$. By Lemma 2.6, we have

$$\begin{aligned}
\|u_n-z^*\|^2 &= \|T_{r_n}(x_n-r_n\Psi x_n) - T_{r_n}(z^*-r_n\Psi z^*)\|^2 \\
&\le \langle u_n-z^*, (x_n-r_n\Psi x_n)-(z^*-r_n\Psi z^*)\rangle \\
&= \frac{1}{2}\Big(\|u_n-z^*\|^2 + \|(x_n-r_n\Psi x_n)-(z^*-r_n\Psi z^*)\|^2 \\
&\quad - \|(u_n-z^*)-((x_n-r_n\Psi x_n)-(z^*-r_n\Psi z^*))\|^2\Big) \\
&= \frac{1}{2}\|u_n-z^*\|^2 + \frac{1}{2}\Big(\|(x_n-r_n\Psi x_n)-(z^*-r_n\Psi z^*)\|^2 \\
&\quad - \|(u_n-x_n)-r_n(\Psi z^*-\Psi x_n)\|^2\Big),
\end{aligned}$$

and it follows by nonexpansiveness of $I-r_n\Psi$ that

$$\begin{aligned}
\|u_n-z^*\|^2 &\le \|x_n-z^*\|^2 - \|(u_n-x_n)-r_n(\Psi z^*-\Psi x_n)\|^2 \\
&\le \|x_n-z^*\|^2. \qquad (3.3)
\end{aligned}$$

For $z^* \in VI(A, C)$, we have $z^* = P_C(z^*-\lambda_n Az^*)$. Since P_C and $I-\lambda_n A$ are nonexpansive mappings, by (3.1), we have

$$\begin{aligned}
\|y_n-z^*\|^2 &= \|P_C(u_n-\lambda_n Au_n) - P_C(z^*-\lambda_n Az^*)\|^2 \\
&\le \|u_n-z^*\|^2. \qquad (3.4)
\end{aligned}$$

Thus, (3.3) and (3.4) imply that

$$\|y_n - z^*\| \leq \|u_n - z^*\| \leq \|x_n - z^*\| \tag{3.5}$$

and so

$$\begin{aligned}
\|x_{n+1} - z^*\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n B) K_n y_n - z^*\| \\
&= \|\alpha_n (\gamma f(x_n) - Bz^*) + \beta_n (x_n - z^*) + ((1-\beta_n) I - \alpha_n B)(K_n y_n - z^*)\| \\
&\leq \alpha_n \|\gamma f(x_n) - Bz^*\| + \beta_n \|x_n - z^*\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - z^*\| \\
&\leq \alpha_n \|\gamma f(x_n) - Bz^*\| + \beta_n \|x_n - z^*\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - z^*\| \\
&= \alpha_n \|\gamma f(x_n) - Bz^*\| + (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\| \\
&\leq \alpha_n \|\gamma f(x_n) - \gamma f(z^*)\| + \alpha_n \|\gamma f(z^*) - Bz^*\| + (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\| \\
&\leq \alpha_n \gamma \alpha \|x_n - z^*\| + \alpha_n \|\gamma f(z^*) - Bz^*\| + (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\| \\
&= (1 - \alpha_n (\bar{\gamma} - \gamma\alpha)) \|x_n - z^*\| + (\bar{\gamma} - \gamma\alpha) \alpha_n \frac{\|\gamma f(z^*) - Bz^*\|}{(\bar{\gamma} - \gamma\alpha)} \\
&\leq \max\left\{\|x_n - z^*\|, \frac{\|\gamma f(z^*) - Bz^*\|}{(\bar{\gamma} - \gamma\alpha)}\right\}.
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\|x_{n+1} - z^*\| \leq \max\left\{\|x_1 - z^*\|, \frac{\|\gamma f(z^*) - Bz^*\|}{(\bar{\gamma} - \gamma\alpha)}\right\} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is bounded, so $\{u_n\}$, $\{y_n\}$, $\{K_n y_n\}$, $\{\Psi x_n\}$, $\{f(x_n)\}$ and $\{Au_n\}$ are also bounded.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$ and $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. From $u_n = T_{r_n}(x_n - r_n \Psi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} \Psi x_{n+1})$, we have

$$G(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C \tag{3.7}$$

and

$$G(u_{n+1}, v) + \langle \Psi x_{n+1}, v - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle v - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \ \forall v \in C. \tag{3.8}$$

Putting $v = u_{n+1}$ in (3.7) and $v = u_n$ in (3.8), we get

$$G(u_n, u_{n+1}) + \langle \Psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$G(u_{n+1}, u_n) + \langle \Psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

Adding the above two inequalities, the monotonicity of G implies that

$$\langle \Psi x_{n+1} - \Psi x_n, u_n - u_{n+1} \rangle + \left\langle u_n - u_{n+1}, \frac{u_{n+1} - x_{n+1}}{r_{n+1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0.$$

This together with nonexpansiveness of $I - r_n \Psi$, we have

$$\begin{aligned}
0 &\leq \left\langle u_n - u_{n+1}, r_n \left(\Psi x_{n+1} - \Psi x_n\right) + \frac{r_n}{r_{n+1}} \left(u_{n+1} - x_{n+1}\right) - \left(u_n - x_n\right) \right\rangle \\
&= \left\langle u_{n+1} - u_n, -r_n \left(\Psi x_{n+1} - \Psi x_n\right) - \frac{r_n}{r_{n+1}} \left(u_{n+1} - x_{n+1}\right) + \left(u_n - x_n\right) \right\rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1} \rangle + \\
&\quad \left\langle u_{n+1} - u_n, \left(x_{n+1} - r_n \Psi x_{n+1}\right) - \left(x_n - r_n \Psi x_n\right) + \left(1 - \frac{r_n}{r_{n+1}}\right) \left(u_{n+1} - x_{n+1}\right) \right\rangle \\
&\leq - \left\| u_{n+1} - u_n \right\|^2 + \left\| u_{n+1} - u_n \right\| \times \\
&\quad \left(\left\| \left(x_{n+1} - r_n \Psi x_{n+1}\right) - \left(x_n - r_n \Psi x_n\right) \right\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| u_{n+1} - x_{n+1} \right\| \right) \\
&\leq - \left\| u_{n+1} - u_n \right\|^2 + \left\| u_{n+1} - u_n \right\| \left(\left\| x_{n+1} - x_n \right\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| u_{n+1} - x_{n+1} \right\| \right),
\end{aligned}$$

which implies

$$\left\| u_{n+1} - u_n \right\|^2 \leq \left\| u_{n+1} - u_n \right\| \left(\left\| x_{n+1} - x_n \right\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| u_{n+1} - x_{n+1} \right\| \right)$$

and hence

$$\left\| u_{n+1} - u_n \right\| \leq \left\| x_{n+1} - x_n \right\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| u_{n+1} - x_{n+1} \right\|. \tag{3.9}$$

By nonexpansiveness of P_C and $I - \lambda_{n+1} A$, we have

$$\begin{aligned}
\left\| y_{n+1} - y_n \right\| &= \left\| P_C \left(u_{n+1} - \lambda_{n+1} A u_{n+1}\right) - P_C \left(u_n - \lambda_n A u_n\right) \right\| \\
&\leq \left\| \left(u_{n+1} - \lambda_{n+1} A u_{n+1}\right) - \left(u_n - \lambda_n A u_n\right) \right\| \\
&= \left\| \left(u_{n+1} - \lambda_{n+1} A u_{n+1}\right) - \left(u_n - \lambda_{n+1} A u_n\right) - \left(\lambda_{n+1} - \lambda_n\right) A u_n \right\| \\
&\leq \left\| \left(u_{n+1} - \lambda_{n+1} A u_{n+1}\right) - \left(u_n - \lambda_{n+1} A u_n\right) \right\| + \left| \lambda_{n+1} - \lambda_n \right| \left\| A u_n \right\| \\
&\leq \left\| u_{n+1} - u_n \right\| + \left| \lambda_{n+1} - \lambda_n \right| \left\| A u_n \right\|. \qquad (3.10)
\end{aligned}$$

Putting $z_n = \frac{1}{1-\beta_n} \left(x_{n+1} - \beta_n x_n\right)$, we have $x_{n+1} = \beta_n x_n + \left(1 - \beta_n\right) z_n$. Since

$$z_n = \frac{1}{1 - \beta_n} \left(x_{n+1} - \beta_n x_n\right) = \frac{\alpha_n \gamma f\left(x_n\right) + \left(\left(1 - \beta_n\right) I - \alpha_n B\right) K_n y_n}{1 - \beta_n},$$

we have

$$\begin{aligned} z_{n+1}-z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}B)K_{n+1}y_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nB)K_ny_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-BK_{n+1}y_{n+1})+\frac{\alpha_n}{1-\beta_n}(BK_ny_n-\gamma f(x_n)) \\ &\quad +K_{n+1}y_{n+1}-K_ny_n. \end{aligned} \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we get

$$\begin{aligned} \|z_{n+1}-z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|BK_{n+1}y_{n+1}\|) \\ &\quad \frac{\alpha_n}{1-\beta_n}(\|BK_ny_n\|+\|\gamma f(x_n)\|)+\|K_{n+1}y_{n+1}-K_ny_n\| \\ &\le \|K_{n+1}y_{n+1}-K_ny_n\|+M(\alpha_n+\alpha_{n+1}) \\ &\le \|K_{n+1}y_{n+1}-K_ny_{n+1}\|+\|K_ny_{n+1}-K_ny_n\|+M(\alpha_n+\alpha_{n+1}) \\ &\le \|K_{n+1}y_{n+1}-K_ny_{n+1}\|+\|y_{n+1}-y_n\|+M(\alpha_n+\alpha_{n+1}) \\ &\le \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\,\|Au_n\|+\|K_{n+1}y_{n+1}-K_ny_{n+1}\| \\ &\quad +M(\alpha_n+\alpha_{n+1}) \\ &\le \left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\,\|Au_n\| \\ &\quad +\|K_{n+1}y_{n+1}-K_ny_{n+1}\|+M(\alpha_n+\alpha_{n+1}) \end{aligned}$$

where

$$M=\max\left\{\sup_n\frac{\|\gamma f(x_{n+1})\|+\|BK_{n+1}y_{n+1}\|}{1-\beta_{n+1}},\sup_n\frac{\|BK_ny_n\|+\|\gamma f(x_n)\|}{1-\beta_n}\right\},$$

it follows that

$$\begin{aligned} \|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\le \left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|+|\lambda_{n+1}-\lambda_n|\,\|Au_n\| \\ &\quad +\|K_{n+1}y_{n+1}-K_ny_{n+1}\|+M(\alpha_n+\alpha_{n+1}). \end{aligned}$$

This together with the conditions $(i)-(iv)$ we obtain by Lemma 2.11 that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2.3, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Consequence,

$$\begin{aligned} \lim_{n\to\infty}\|x_{n+1}-x_n\| &= \lim_{n\to\infty}\|\beta_nx_n+(1-\beta_n)z_n-x_n\| \\ &= \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \end{aligned}$$

By (3.9) and (3.10), we have $\lim_{n\to\infty}\|y_{n+1}-y_n\| = \lim_{n\to\infty}\|u_{n+1}-u_n\| = 0$.

Step 3. We will show that $\lim_{n\to\infty}\|x_n - K_n y_n\| = 0$. Since

$$\begin{aligned}\|x_n - K_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - K_n y_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n(\gamma f(x_n) - BK_n y_n) + \beta_n(x_n - K_n y_n)\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - BK_n y_n\| + \beta_n\|x_n - K_n y_n\|,\end{aligned}$$

we have

$$\|x_n - K_n y_n\| \le \frac{1}{1-\beta_n}\left(\|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - BK_n y_n\|\right).$$

By (i) and **Step 2**, we obtain $\lim_{n\to\infty}\|x_n - K_n y_n\| = 0$.

Step 4. We will show that $\lim_{n\to\infty}\|u_n - y_n\| = 0$, $\lim_{n\to\infty}\|u_n - x_n\| = 0$, $\lim_{n\to\infty}\|Ky_n - y_n\| = 0$ and $\lim_{n\to\infty}\|Kx_n - x_n\| = 0$. Set $w_n = \gamma f(x_n) - BK_n y_n$ and let $\delta > 0$ be a constant such that $\delta > \max\{\sup_n\|w_n\|, \sup_n\|x_n - z^*\|\}$. By nonexpansiveness of K_n and P_C and ρ-inverse-strongly monotonicity of A together with (3.3), we have

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)K_n y_n - z^*\|^2 \\ &= \|(1-\beta_n)(K_n y_n - z^*) + \beta_n(x_n - z^*) + \alpha_n(\gamma f(x_n) - B(K_n y_n))\|^2 \\ &\le \|(1-\beta_n)(K_n y_n - z^*) + \beta_n(x_n - z^*)\|^2 + 2\langle\alpha_n(\gamma f(x_n) - B(K_n y_n)), \\ &\qquad (1-\beta_n)(K_n y_n - z^*) + \beta_n(x_n - z^*) + \alpha_n(\gamma f(x_n) - B(K_n y_n))\rangle \\ &= \|(1-\beta_n)(K_n y_n - z^*) + \beta_n(x_n - z^*)\|^2 + 2\alpha_n\langle w_n, x_{n+1} - z^*\rangle \\ &\le (1-\beta_n)\|K_n y_n - z^*\|^2 + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\|w_n\|\|x_{n+1} - z^*\| \\ &\le (1-\beta_n)\|y_n - z^*\|^2 + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2 \qquad (3.12) \\ &\le (1-\beta_n)\left(\|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*)\|^2\right) + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2,\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\le (1-\beta_n)\left\{\|u_n - z^*\|^2 - \lambda_n(2\rho - \lambda_n)\|Au_n - Az^*\|^2\right\} \\ &\quad + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2 \\ &\le (1-\beta_n)\left\{\|x_n - z^*\|^2 - \lambda_n(2\rho - \lambda_n)\|Au_n - Az^*\|^2\right\} \\ &\quad + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2 \\ &= \|x_n - z^*\|^2 - (1-\beta_n)\lambda_n(2\rho - \lambda_n)\|Au_n - Az^*\|^2 + 2\alpha_n\delta^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n)\lambda_n(2\rho - \lambda_n)\|Au_n - Az^*\|^2 &\le \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + 2\alpha_n\delta^2 \\ &\le \|x_n - x_{n+1}\|(\|x_n - z^*\| + \|x_{n+1} - z^*\|) \\ &\quad + 2\alpha_n\delta^2.\end{aligned}$$

Since $\{\lambda_n\} \subset [a,b]$ for some $0 < a < b < 2\rho$, we have

$$(1-\beta_n)\, a\,(2\rho - b)\,\|Au_n - Az^*\|^2 \le \|x_n - x_{n+1}\|\,(\|x_n - z^*\| + \|x_{n+1} - z^*\|) + 2\alpha_n\delta^2. \tag{3.13}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, $\limsup_{n\to\infty}\beta_n < 1$ and $\{x_n\}$ is bounded, (3.13) implies that $\lim_{n\to\infty}\|Au_n - Az^*\| = 0$. By nonexpansiveness of $I - \lambda_n A$ and firmly nonexpansiveness of P_C, we have

$$\begin{aligned}
\|y_n - z^*\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(z^* - \lambda_n Az^*)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*), y_n - z^* \rangle \\
&= \frac{1}{2}\Big(\|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*)\|^2 + \|y_n - z^*\|^2 \\
&\quad - \|(u_n - y_n) - \lambda_n(Au_n - Az^*)\|^2 \Big) \\
&\le \frac{1}{2}\Big(\|u_n - z^*\|^2 + \|y_n - z^*\|^2 \\
&\quad - \Big(\|u_n - y_n\|^2 - 2\lambda_n\langle u_n - y_n, Au_n - Az^*\rangle + \lambda_n^2\|Au_n - Az^*\|^2 \Big)\Big) \\
&= \frac{1}{2}\Big(\|u_n - z^*\|^2 + \|y_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Az^*\rangle \\
&\quad - \lambda_n^2\|Au_n - Az^*\|^2 \Big)
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n - z^*\|^2 &\le \|u_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Az^*\rangle \\
&\quad - \lambda_n^2\|Au_n - Az^*\|^2 \\
&\le \|x_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Az^*\rangle \\
&\quad - \lambda_n^2\|Au_n - Az^*\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &\le (1-\beta_n)\|y_n - z^*\|^2 + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2 \\
&\le (1-\beta_n)\Big(\|x_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Az^*\rangle \\
&\quad - \lambda_n^2\|Au_n - Az^*\|^2 \Big) + \beta_n\|x_n - z^*\|^2 + 2\alpha_n\delta^2 \\
&\le \|x_n - z^*\|^2 - (1-\beta_n)\|u_n - y_n\|^2 \\
&\quad + 2\lambda_n(1-\beta_n)\langle u_n - y_n, Au_n - Az^*\rangle + 2\alpha_n\delta^2,
\end{aligned}$$

which implies that

$$\begin{aligned}(1-\beta_n)\|u_n-y_n\|^2 &\le \|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2\\ &\quad +2\lambda_n(1-\beta_n)\langle u_n-y_n, Au_n-Az^*\rangle+2\alpha_n\delta^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-z^*\|+\|x_{n+1}-z^*\|)\\ &\quad +2\lambda_n(1-\beta_n)\|u_n-y_n\|\|Au_n-Az^*\|+2\alpha_n\delta^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|Au_n-Az^*\|=0$, $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, $\limsup_{n\to\infty}\beta_n<1$, and the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded, it follows that $\lim_{n\to\infty}\|u_n-y_n\|=0$. By nonexpansiveness of $T_{r_n}(I-r_n\Psi)$, we have

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &\le (1-\beta_n)\|y_n-z^*\|^2+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &\le (1-\beta_n)\|u_n-z^*\|^2+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &= (1-\beta_n)\|T_{r_n}(x_n-r_n\Psi x_n)-T_{r_n}(z^*-r_n\Psi z^*)\|^2+\beta_n\|x_n-z^*\|^2\\ &\quad +2\alpha_n\delta^2\\ &\le (1-\beta_n)\|(x_n-r_n\Psi x_n)-(z^*-r_n\Psi z^*)\|^2+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &= (1-\beta_n)\|(x_n-z^*)-r_n(\Psi x_n-\Psi z^*)\|^2+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &= (1-\beta_n)\Big(\|x_n-z^*\|^2-2r_n\langle x_n-z^*,\Psi x_n-\Psi z^*\rangle\\ &\quad +r_n^2\|\Psi x_n-\Psi z^*\|^2\Big)+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &\le (1-\beta_n)\Big(\|x_n-z^*\|^2-r_n(2\beta-r_n)\|\Psi x_n-\Psi z^*\|^2\Big)\\ &\quad +\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &= \|x_n-z^*\|^2-r_n(2\beta-r_n)(1-\beta_n)\|\Psi x_n-\Psi z^*\|^2+2\alpha_n\delta^2\end{aligned}$$

and so

$$\begin{aligned}r_n(2\beta-r_n)&(1-\beta_n)\|\Psi x_n-\Psi z^*\|^2\\ &\le \|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2+2\alpha_n\delta^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-z^*\|+\|x_{n+1}-z^*\|)+2\alpha_n\delta^2.\end{aligned}$$

Since $\{r_n\}\subset[c,d]$ for some $0<c<d<2\beta$, we have

$$\begin{aligned}c(2\beta-d)&(1-\beta_n)\|\Psi x_n-\Psi z^*\|^2\\ &\le \|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2+2\alpha_n\delta^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-z^*\|+\|x_{n+1}-z^*\|)+2\alpha_n\delta^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, $\limsup_{n\to\infty}\beta_n<1$ and $\{x_n\}$ is bounded, it implies that $\lim_{n\to\infty}\|\Psi x_n-\Psi z^*\|=0$. By (3.3), (3.4) and (3.12), we have

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &\le (1-\beta_n)\|u_n-z^*\|^2+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &\le (1-\beta_n)\left(\|x_n-z^*\|^2-\|(u_n-x_n)-r_n(\Psi z^*-\Psi x_n)\|^2\right)\\ &\quad+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &\le (1-\beta_n)\left(\|x_n-z^*\|^2-\|u_n-x_n\|^2+2r_n\langle u_n-x_n,\Psi z^*-\Psi x_n\rangle\right.\\ &\quad\left.-r_n^2\|\Psi z^*-\Psi x_n\|^2\right)+\beta_n\|x_n-z^*\|^2+2\alpha_n\delta^2\\ &\le \|x_n-z^*\|^2-(1-\beta_n)\|u_n-x_n\|^2\\ &\quad+2r_n(1-\beta_n)\langle u_n-x_n,\Psi z^*-\Psi x_n\rangle+2\alpha_n\delta^2\end{aligned}$$

and so

$$\begin{aligned}(1-\beta_n)\|u_n-x_n\|^2 &\le \|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2\\ &\quad+2r_n(1-\beta_n)\langle u_n-x_n,\Psi z^*-\Psi x_n\rangle+2\alpha_n\delta^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-z^*\|+\|x_{n+1}-z^*\|)\\ &\quad+2r_n(1-\beta_n)\langle u_n-x_n,\Psi z^*-\Psi x_n\rangle+2\alpha_n\delta^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-z^*\|+\|x_{n+1}-z^*\|)\\ &\quad+2r_n(1-\beta_n)\|u_n-x_n\|\|\Psi z^*-\Psi x_n\|+2\alpha_n\delta^2. \qquad (3.14)\end{aligned}$$

Since $\lim_{n\to\infty}\|\Psi x_n-\Psi z^*\|=0$, $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, $\limsup_{n\to\infty}\beta_n<1$ and the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded, it follows from (3.14) that $\lim_{n\to\infty}\|u_n-x_n\|=0$. Since

$$\begin{aligned}\|x_n-K_nx_n\| &\le \|x_n-K_ny_n\|+\|K_ny_n-K_nu_n\|+\|K_nu_n-K_nx_n\|\\ &\le \|x_n-K_ny_n\|+\|y_n-u_n\|+\|u_n-x_n\|\end{aligned}$$

and

$$\begin{aligned}\|K_ny_n-y_n\| &\le \|K_ny_n-K_nx_n\|+\|K_nx_n-x_n\|+\|x_n-y_n\|\\ &\le \|K_nx_n-x_n\|+2\|x_n-y_n\|\\ &\le \|K_nx_n-x_n\|+2(\|x_n-u_n\|+\|u_n-y_n\|),\end{aligned}$$

it follows by **Step 3** and **Step 4** that

$$\lim_{n\to\infty}\|x_n-K_nx_n\|=0 \text{ and } \lim_{n\to\infty}\|K_ny_n-y_n\|=0.$$

Let K be the K-mapping generated by $T_1,T_2,...,T_N$ and $\xi_1,\xi_2,...,\xi_N$. From

$$\|Ky_n-y_n\|\le\|Ky_n-K_ny_n\|+\|K_ny_n-y_n\|$$

and

$$\|Kx_n-x_n\|\le\|Kx_n-K_nx_n\|+\|K_nx_n-x_n\|,$$

it follows by Lemma 2.10 that

$$\lim_{n\to\infty} \|Ky_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|Kx_n - x_n\| = 0.$$

Step 5. We will show that there exists a unique element $z_0 \in H$ such that $z_0 = P_\Omega (I - (B - \gamma f)) z_0$. Observe that $P_\Omega (I - (B - \gamma f))$ is a contraction of H into itself and from Lemma 2.4, we have that $\|I - B\| \leq 1 - \bar{\gamma}$. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}
&\|P_\Omega (I - (B - \gamma f))(x) - P_\Omega (I - (B - \gamma f))(y)\| \\
&\quad \leq \|(I - (B - \gamma f))(x) - (I - (B - \gamma f))(y)\| \\
&\quad = \|(I - B)(x - y) + \gamma (f(x) - f(y))\| \\
&\quad \leq \|I - B\| \|x - y\| + \gamma \|(f(x) - f(y))\| \\
&\quad \leq (1 - \bar{\gamma}) \|x - y\| + \gamma\alpha \|x - y\| \\
&\quad = (1 - (\bar{\gamma} - \gamma\alpha)) \|x - y\|.
\end{aligned}$$

Hence $P_\Omega (I - (B - \gamma f))$ is a contraction. Since H is complete, there exists a unique element $z_0 \in H$ such that $z_0 = P_\Omega (I - (B - \gamma f)) z_0$.

Step 6. We will show that $\limsup_{n\to\infty} \langle (\gamma f - B) z_0, x_n - z_0 \rangle \leq 0$. We choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\limsup_{n\to\infty} \langle (\gamma f - B) z_0, Ky_n - z_0 \rangle = \lim_{i\to\infty} \langle (\gamma f - B) z_0, Ky_{n_i} - z_0 \rangle$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup z$. From $\lim_{n\to\infty} \|Ky_n - y_n\| = 0$, we obtain $Ky_{n_i} \rightharpoonup z$. Therefore we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\gamma f - B) z_0, Ky_n - z_0 \rangle &= \lim_{i\to\infty} \langle (\gamma f - B) z_0, Ky_{n_i} - z_0 \rangle \\
&= \langle (\gamma f - B) z_0, z - z_0 \rangle.
\end{aligned}$$

Next we prove that $z \in \Omega$. First, we show that $z \in GEP(G, \Psi)$. Indeed, we observe that $u_n = T_{r_n}(x_n - r_n \Psi x_n)$ and

$$G(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C.$$

By $(A2)$, we deduce that

$$\langle \Psi x_{n_i}, v - u_{n_i} \rangle + \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq G(v, u_{n_i}), \ \forall v \in C. \tag{3.15}$$

From $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and $y_{n_i} \rightharpoonup z$, we get $u_{n_i} \rightharpoonup z$.

Put $z_t = tv + (1-t)z$ for all $t \in (0, 1]$ and $v \in C$. Consequently, we get $z_t \in C$. From (3.15), it follows that

$$\langle \Psi x_{n_i}, z_t - u_{n_i} \rangle + \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle - G(z_t, u_{n_i}) \geq 0$$

and hence

$$\begin{aligned}\langle \Psi z_t, z_t - u_{n_i}\rangle &\geq \langle \Psi z_t, z_t - u_{n_i}\rangle - \langle \Psi x_{n_i}, z_t - u_{n_i}\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \\ &\quad + G\left(z_t, u_{n_i}\right) \\ &= \langle \Psi z_t - \Psi u_{n_i}, z_t - u_{n_i}\rangle + \langle \Psi u_{n_i} - \Psi x_{n_i}, z_t - u_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + G\left(z_t, u_{n_i}\right).\end{aligned}$$

Since $\langle \Psi z_t - \Psi u_{n_i}, z_t - u_{n_i}\rangle \geq 0$, above inequality implies

$$G\left(z_t, u_{n_i}\right) \leq \langle \Psi z_t, z_t - u_{n_i}\rangle + \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle - \langle \Psi u_{n_i} - \Psi x_{n_i}, z_t - u_{n_i}\rangle.$$

By **Step 4**, we know that $\lim_{n\to\infty} \|u_n - x_n\| = 0$, it follows by Lipschitz continuity of Ψ that $\lim_{i\to\infty} \|\Psi u_{n_i} - \Psi x_{n_i}\| = 0$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$, it follows from $(A4)$ that

$$\begin{aligned}G\left(z_t, z\right) &\leq \lim_{i\to\infty} G\left(z_t, u_{n_i}\right) \\ &\leq \lim_{i\to\infty} \langle \Psi z_t, z_t - u_{n_i}\rangle \\ &= \langle \Psi z_t, z_t - z\rangle.\end{aligned}$$

Owing to $(A1)$ and $(A4)$, we get that

$$\begin{aligned}0 = G\left(z_t, z_t\right) &\leq tG\left(z_t, v\right) + (1-t)\, G\left(z_t, z\right) \\ &\leq tG\left(z_t, v\right) + (1-t)\langle \Psi z_t, z_t - z\rangle \\ &= tG\left(z_t, v\right) + (1-t)\, t\langle \Psi z_t, v - z\rangle,\end{aligned}$$

and hence $G\left(z_t, v\right) + (1-t)\langle \Psi z_t, v - z\rangle \geq 0$. Letting $t \to 0$, we have

$$G\left(z, v\right) + \langle \Psi z, v - z\rangle \geq 0.$$

This implies that $z \in GEP(G, \Psi)$.

Next, assume that there exists $j \in \{1, 2, ..., N\}$ such that $z \neq T_j z$, by Lemma 2.9 we have $z \neq Kz$. Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we have $x_{n_i} \rightharpoonup z$. From $\lim_{n\to\infty} \|Kx_n - x_n\| = 0$ and Opial's condition, we get

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - z\| &< \liminf_{i\to\infty} \|x_{n_i} - Kz\| \\ &\leq \liminf_{i\to\infty} \left(\|x_{n_i} - Kx_{n_i}\| + \|Kx_{n_i} - Kz\|\right) \\ &= \liminf_{i\to\infty} \|Kx_{n_i} - Kz\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - z\|.\end{aligned}$$

This is a contradiction. Hence, $z \in \bigcap_{i=1}^{N} F\left(T_i\right)$.

Let T be the maximal monotone mapping define by

$$Tu = \begin{cases} Au + N_C u, & u \in C \\ \emptyset, & u \notin C. \end{cases}$$

For any given $(u, w) \in G(T)$, hence $w - Au \in N_C u$. Since $y_n \in C$, by the definition of N_C, we have $\langle u - y_n, w - Au \rangle \geq 0$. From $y_n = P_C(u_n - \lambda_n A u_n)$, we have

$$\langle u - y_n, y_n - (u_n - \lambda_n A u_n) \rangle \geq 0,$$

and so

$$\left\langle u - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \geq 0.$$

By the ρ-inverse monotonicity of A, we have $\langle u - y_{n_i}, w \rangle - \langle u - y_{n_i}, Au \rangle \geq 0$ and so

$$\begin{aligned}
\langle u - y_{n_i}, w \rangle &\geq \langle u - y_{n_i}, Au \rangle \\
&\geq \langle u - y_{n_i}, Au \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\
&= \left\langle u - y_{n_i}, Au - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle u - y_{n_i}, Au - Ay_{n_i} \rangle + \langle u - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle u - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - y_n\| = 0$, $y_{n_i} \rightharpoonup z$ and by Lipschitz continuity of A, we obtain $\langle u - z, w \rangle \geq 0$. Since T is maximal monotone, hence $0 \in Tz$. This shows that $z \in VI(A, C)$. Thus, $z \in \Omega$. By **Step 5**, we have $\langle (\gamma f - B) z_0, z - z_0 \rangle \leq 0$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\gamma f - B) z_0, x_n - z_0 \rangle &= \limsup_{n\to\infty} \langle (\gamma f - B) z_0, (x_n - K_n y_n) + (K_n y_n - z_0) \rangle \\
&= \limsup_{n\to\infty} \langle (\gamma f - B) z_0, K_n y_n - z_0 \rangle \\
&= \limsup_{n\to\infty} \langle (\gamma f - B) z_0, K y_n - z_0 \rangle \\
&= \lim_{i\to\infty} \langle (\gamma f - B) z_0, K y_{n_i} - z_0 \rangle \\
&= \langle (\gamma f - B) z_0, z - z_0 \rangle \leq 0.
\end{aligned}$$

Step 7. Finally we show that $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to z_0.

Indeed, from (3.1), we have

$$
\begin{aligned}
\|x_n - z_0\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n B) K_n y_n - z_0\|^2 \\
&= \|((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0) + \beta_n (x_n - z_0) + \alpha_n (\gamma f(x_n) - Bz^*)\|^2 \\
&\le \|((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0) + \beta_n (x_n - z_0)\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - Bz_0, x_{n+1} - z_0 \rangle, \\
&= \left\| \frac{(1-\beta_n)((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0)}{(1-\beta_n)} + \beta_n (x_n - z_0) \right\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - Bz_0, x_{n+1} - z_0 \rangle \\
&\le (1-\beta_n) \left\| \frac{((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0)}{(1-\beta_n)} \right\|^2 + \beta_n \|x_n - z_0\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - Bz_0, x_{n+1} - z_0 \rangle \\
&= (1-\beta_n) \left\| \frac{((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0)}{(1-\beta_n)} \right\|^2 + \beta_n \|x_n - z_0\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \gamma f(z_0), x_{n+1} - z_0 \rangle + 2\alpha_n \langle \gamma f(z_0) - Bz^*, x_{n+1} - z_0 \rangle \\
&\le \frac{\|((1-\beta_n) I - \alpha_n B)(K_n y_n - z_0)\|^2}{1-\beta_n} + \beta_n \|x_n - z_0\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \|x_n - z_0\| \|x_{n+1} - z_0\| + 2\alpha_n \langle \gamma f(z_0) - Bz_0, x_{n+1} - z_0 \rangle \\
&\le \frac{(1-\beta_n - \alpha_n \bar{\gamma})^2 \|x_n - z_0\|^2}{1-\beta_n} + \beta_n \|x_n - z_0\|^2 \\
&\quad + \alpha_n \gamma \alpha \left(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \right) + 2\alpha_n \langle \gamma f(z_0) - Bz_0, x_{n+1} - z_0 \rangle \\
&= \left(1 - \alpha_n (2\bar{\gamma} - \gamma\alpha) + \frac{\alpha_n^2 \bar{\gamma}^2}{1-\beta_n} \right) \|x_n - z_0\|^2 + \alpha_n \gamma \alpha \|x_{n+1} - z_0\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(z_0) - Bz_0, x_{n+1} - z_0 \rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \frac{1}{1-\alpha_n \gamma \alpha} \left(1 - \alpha_n (2\bar{\gamma} - \gamma\alpha) + \frac{\alpha_n^2 \bar{\gamma}^2}{1-\beta_n} \right) \|x_n - z_0\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n \gamma \alpha} \langle \gamma f(z^*) - Bz_0, x_{n+1} - z_0 \rangle \\
&= \frac{1}{1-\alpha_n \gamma \alpha} (1 - \alpha_n (2\bar{\gamma} - \gamma\alpha)) \|x_n - z_0\|^2 \\
&\quad + \frac{\alpha_n}{1-\alpha_n \gamma \alpha} \left(2 \langle \gamma f(z_0) - Bz_0, x_{n+1} - z_0 \rangle + \frac{\alpha_n \bar{\gamma}^2}{1-\beta_n} \|x_n - z_0\|^2 \right) \\
&= \frac{1}{1-\alpha_n \gamma \alpha} (1 - \alpha_n \gamma \alpha - 2\alpha_n (\bar{\gamma} - \gamma\alpha)) \|x_n - z_0\|^2 \\
&\quad + \frac{\alpha_n}{1-\alpha_n \gamma \alpha} \left(2 \langle \gamma f(z_0) - Bz_0, x_{n+1} - z_0 \rangle + \frac{\alpha_n \bar{\gamma}^2}{1-\beta_n} \|x_n - z_0\|^2 \right),
\end{aligned}
$$

and so

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\le \left(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)\|x_n-z_0\|^2\\
&\quad+\frac{\alpha_n}{1-\alpha_n\gamma\alpha}\left(2\langle\gamma f(z_0)-Bz_0,x_{n+1}-z_0\rangle+\frac{\alpha_n\bar{\gamma}^2}{1-\beta_n}\|x_n-z_0\|^2\right)\\
&=\left(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)\|x_n-z_0\|^2+\frac{2(\bar{\gamma}-\gamma\alpha)}{2(\bar{\gamma}-\gamma\alpha)}\frac{\alpha_n}{1-\alpha_n\gamma\alpha}\\
&\quad\times\left(2\langle\gamma f(z_0)-Bz_0,x_{n+1}-z_0\rangle+\frac{\alpha_n\bar{\gamma}^2}{1-\beta_n}\|x_n-z_0\|^2\right)\\
&=\left(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)\|x_n-z_0\|^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\\
&\quad\times\left(\frac{\langle\gamma f(z_0)-Bz_0,x_{n+1}-z_0\rangle}{\bar{\gamma}-\gamma\alpha}+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\alpha)(1-\beta_n)}\|x_n-z_0\|^2\right).
\end{aligned}\tag{3.16}$$

Set $\kappa_n=\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}$ and $\delta_n=\frac{\langle\gamma f(z_0)-Bz_0,x_{n+1}-z_0\rangle}{\bar{\gamma}-\gamma\alpha}+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\alpha)(1-\beta_n)}\|x_n-z_0\|^2$ for all $n\in\mathbb{N}$. We can rewrite (3.16) as $\|x_{n+1}-z_0\|^2\le(1-\kappa_n)\|x_n-z_0\|^2+\kappa_n\delta_n$. By our hypotheses it is easily verified that $\sum_{n=1}^{\infty}\kappa_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. Therefore, by Lemma 2.2, we can conclude that $\|x_n-z_0\|\to 0$. Since $\|u_n-x_n\|\to 0$ and $\|u_n-y_n\|\to 0$, it follows that $\|u_n-z_0\|\to 0$ and $\|y_n-z_0\|\to 0$. This completes the proof. □

As direct consequences of Theorem 3.1, we have the following three corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, $f:C\to C$ a contraction mapping with constant $\alpha\in(0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\xi_{n,i}\}_{i=1}^{N-1}\subset(0,1)$, $\{\xi_{n,N}\}\subset(0,1]$, $\{\xi_i\}_{i=1}^{N-1}\subset(0,1)$, $\xi_N\in(0,1]$ be such that $\xi_{n,i}\to\xi_i$ for all $i=1,2,...,N$. Let $K_n:C\to C$ be a K-mapping generated by $T_1,T_2,...,T_N$ and $\xi_{n,1},\xi_{n,2},...,\xi_{n,N}$. Suppose that $\Omega:=\bigcap_{i=1}^N F(T_i)\bigcap EP(G)\neq\emptyset$. Let B be a strongly positive bounded linear operator on H with coefficient $\bar{\gamma}>0$ and $\|B\|=1$ and let $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. For $x_1\in C$, let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} G(u_n,v)+\dfrac{1}{r_n}\langle v-u_n,u_n-x_n\rangle\ge 0,\ \forall v\in C;\\ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)K_n u_n\end{cases}$$

for all $n\in\mathbb{N}$, where $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfying:

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty,\lim_{n\to\infty}\alpha_n=0,$

(ii) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}\frac{r_n}{r_{n+1}}=1,$

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to the point $z_0 \in \Omega$, where $z_0 = P_\Omega (I - (B - \gamma f)) z_0$.

Proof. Let $\lambda_n = 1$ for all $n \in \mathbb{N}$ and $\Psi x = 0$ and $Ax = 0$ for all $x \in C$ in Theorem 3.1. Since $u_n \in C$, we get that $u_n = P_C u_n$. Then $y_n = u_n$. Therefore the conclusion follows. □

Remark 3.3. *Putting $\beta_n = \beta$, $\forall n \in \mathbb{N}$ in Corollary 3.2, we obtained Theorem 3.1 in [4].*

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Psi : C \to H$ a β-inverse-strongly monotone mapping, $A : C \to H$ a ρ-inverse-strongly monotone mapping and $f : C \to C$ a contraction mapping with constant $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\xi_{n,i}\}_{i=1}^{N-1} \subset (0,1)$, $\{\xi_{n,N}\} \subset (0,1]$, $\{\xi_i\}_{i=1}^{N-1} \subset (0,1)$, $\xi_N \in (0,1]$ be such that $\xi_{n,i} \to \xi_i$ for all $i = 1,2,...,N$. Let $K_n : C \to C$ be a K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,N}$. Suppose that $\Omega := \bigcap_{i=1}^N F(T_i) \bigcap VI(\Psi, C) \bigcap VI(A, C) \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $\|B\| = 1$ and let $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} u_n = P_C (x_n - r_n \Psi x_n); \\ y_n = P_C (u_n - \lambda_n A u_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) K_n y_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset [a,b]$ for some $0 < a < b < 2\rho$ and $\{r_n\} \subset [c,d]$ for some $0 < c < d < 2\beta$ satisfying:

(i) $\sum_{n=1}^\infty \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to the point $z_0 \in \Omega$, where $z_0 = P_\Omega (I - (B - \gamma f)) z_0$.

Proof. In Theorem 3.1, put $G(x,y) = 0$ for all $x, y \in C$. Then, we obtain that $\langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \ge 0$, $\forall v \in C$, $\forall n \in \mathbb{N}$. This implies that $\langle v - u_n, (x_n - r_n \Psi x_n) - u_n \rangle \le 0$, $\forall v \in C$. So, we get that $P_C (x_n - r_n \Psi x_n) = u_n$ for all $n \in \mathbb{N}$. Then, we obtain the desired result from Theorem 3.1. □

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, $f : C \to C$ a contraction mapping with constant $\alpha \in (0,1)$. Let T be a nonexpansive mapping* *of C into itself. Suppose that $\Omega := F(T) \bigcap EP(G) \neq \emptyset$. For $x_1 \in C$, let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} G(u_n, v) + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) T u_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset [c,d]$ for some $0 < c < d < \infty$ satisfying:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,$

(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1,$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to the point $z_0 \in \Omega$, where $z_0 = P_\Omega f(z_0)$.

Proof. In Theorem 3.1, put $\lambda_n = 1$ for all $n \in \mathbb{N}$ and $\Psi x = 0$ and $Ax = 0$ for all $x \in C$. Since $u_n \in C$, we get that $u_n = P_C u_n$. Then $y_n = u_n$. Put $N = 1$, $T_1 = T$ and $\xi_{n,1} = 1$ for all $n \in \mathbb{N}$. Then $K_n = T$. Hence, we obtain the desired result from Theorem 3.1. □

Recall that a mapping $T : C \to C$ is called *strictly k-pseudocontractive mapping*, see [13], if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \quad \text{for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive mapping. Putting $A = I - T$, where $T : C \to C$ is a strictly k-pseudocontractive mapping. We know that

$$\langle x - y, Ax - Ay \rangle \geq \frac{1-k}{2} \|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$

That is A is $\frac{1-k}{2}$-inverse-strongly monotone mapping. Now, using Theorem 3.1 we state a strong convergence theorem for strictly k-pseudocontractive mapping as follows.

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi : C \to C$ be a strictly k-pseudocontractive mapping, $A : C \to C$ be a strictly l-pseudocontractive mapping, $f : C \to C$ a contraction mapping with constant $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\xi_{n,i}\}_{i=1}^{N-1} \subset (0,1)$, $\{\xi_{n,N}\} \subset (0,1]$, $\{\xi_i\}_{i=1}^{N-1} \subset (0,1)$, $\xi_N \in (0,1]$ be such that $\xi_{n,i} \to \xi_i$ for all $i = 1, 2, ..., N$. Let $K_n : C \to C$ be a K-mapping generated by $T_1, T_2, ..., T_N$ and $\xi_{n,1}, \xi_{n,2}, ..., \xi_{n,N}$. Suppose that $\Omega := \bigcap_{i=1}^N F(T_i) \bigcap GEP(G, \Psi') \bigcap VI(A', C) \neq \emptyset$, where $\Psi' = I - \Psi$ and $A' = I - A$. Let B be a strongly positive bounded linear* *operator on H with coefficient $\bar{\gamma} > 0$ and $\|B\| = 1$ and let $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} G(u_n, v) + \langle \Psi' x_n, v - u_n \rangle + \dfrac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0, \ \forall v \in C; \\ y_n = P_C (u_n - \lambda_n A' u_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) K_n y_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset [a,b]$ for some $0 < a < b < 1 - l$ and $\{r_n\} \subset [c,d]$ for some $0 < c < d < 1 - k$ satisfying:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to the point $z_0 \in \Omega$, where $z_0 = P_\Omega (I - (B - \gamma f)) z_0$.

Acknowledgements : The authors would like to thank Professor Suthep Suantai and Assistant Professor Poom Kumam for drawing our attention to the subject and for many useful discussions. The research was supported by Nakhon Pathom Rajabhat University Research Fund for financial support.

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(Received 27 October 2010)
(Accepted 28 December 2010)