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Identities in Graph Algebras of Type $(n, n-1, ..., 3, 2, 0)$

T. Poomsa-ard, J. Wetweerapong, C. Khiloukom and T. Musuntei

Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies an identity $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$.

In this paper we generalize the concept of graph algebras of type $\tau = (2, 0)$ to define graph algebras of type $\tau = (n, n-1, n-2, ..., 3, 2, 0), n \geq 2$ and characterize identities in graph algebras. Further we show that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

Keywords : identity, term, normal form term, n-ary algebra, graph algebra.

1 Introduction

Graph algebras have been invented in [9] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G to have the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and two basic operations, a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise,} \end{cases}$$

where $u, v \in V \cup \{\infty\}, \infty \notin V$.

Graph identities were characterized in [3] by using the rooted graph of a term t where the vertices correspond to the variables occurring in t.

In [7], R. Pöschel has shown that any term over the class of all graph algebras of type $\tau = (2, 0)$ can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

We can generalize this concept to define graph algebras of type $\tau = (n, n-1, n-2, ..., 3, 2, 0), n \geq 2$ in the following way :

Let $G = (V, E)$ be a directed graph with vertex set V and set of edges E, an edge in E is an ordered pair of (not necessarily distinct) of vertices of V. For convenient to define the operations on G we will partition E into E_{f_i} such that $E_{f_i} \subseteq V^i, i = 2, 3, 4, ..., n$, where $(v_1, v_2, ..., v_i) \in E_{f_i}$ iff $(v_1, v_2), (v_2, v_3), ..., (v_{i-1}, v_i) \in E$ and if $e_{f_i} = (v_1, v_2, ..., v_i) \in E_{f_i}$ and $e_{f_j} = (v'_1, v'_2, ..., v'_j) \in E_{f_j}, i \neq j$, then $(v_1, v_2), (v_2, v_3), ..., (v_{i-1}, v_i)$ of e_{f_i} and $(v'_1, v'_2), (v'_2, v'_3), ..., (v'_{i-1}, v'_j)$ of e_{f_j} are different edges in E. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and n operations, a nullary operation pointing to ∞, and i-ary operation $f_i, 2 \leq i \leq n$, given for elements of $(V \cup \{\infty\})^i$ by

$$f_i(v_1, v_2, ...v_i) = \begin{cases} v_1, & \text{if} \quad (v_1, v_2, v_3, ..., v_i) \in E_{f_i} \\ \\ \infty & \text{otherwise.} \end{cases}$$

We will write $(v_1, v_2, ..., v_i)$ instead of $f_i(v_1, v_2, ..., v_i)$.

In this paper we will give some basic concepts and characterize identities in graph algebras of type $\tau = (n, n-1, n-2, ..., 3, 2, 0), n \geq 2$. Further we show that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

2 Basic concept

We begin with a more precise definition of terms of the type of graph algebras.

Definition 2.1 The set $W_\tau(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms,

(ii) if $t_1, t_2, ..., t_i$ are terms, then $f_i(t_1, t_2, ..., t_i)$ is a term, where f_i is an i-ary operation such that $i = 2, 3, ..., n$; instead of $f_i(t_1, t_2, ..., t_i)$ for short we will write $(t_1 t_2, ...t_i)$,

(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term in which the symbol ∞ occurs is called a trivial term.

Definition 2.2 To each non-trivial term t of type $\tau = (n, n-1, n-2, ..., 3, 2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set $var(t)$ of all variables occurring in t and where $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1, t_2, \ldots, t_i) = E(t_1) \cup E(t_2) \cup \ldots \cup E(t_i) \cup$$

$$\Big\{(L(t_1), L(t_2), L(t_3)), \ldots, L(t_i))\Big\}, \text{ where } 2 \leq i \leq n,$$

when $t = (t_1, t_2, ..., t_i)$ is a compound term and $L(t_1), L(t_2), ..., L(t_i)$ are the left-most variables in $t_1, t_2, ..., t_i$ respectively.

$L(t)$ is called the *root* of the graph $G(t)$ and the pair $(G(t), L(t))$ is the rooted graph corresponding to t. Formally, to every trivial term t we assign the empty graph $\emptyset$.

Definition 2.3 We say that a graph $G = (V, E)$ satisfies an identity $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \overline{\{\infty\}}$), and in this case, we write $G \models s \approx t$.

Definition 2.4 Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V'$ carrying edges to edges, that is, for which $(v_1, v_2, ..., v_i) \in E_{f_i}$ implies $(h(v_1), h(v_2), ..., h(v_i)) \in E'_{f_i}$.

In [3], it was proved :

Proposition 2.1 *Let $G = (V, E)$ be a graph and let $h : X \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term, then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Further it was proved :

Proposition 2.2 *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property: A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.*

In [3] was proved above two propositions in the case $s, t \in W_\tau(X), \tau = (2, 0)$. We will show that these two propositions still true in the case $s, t \in W_\tau(X), \tau = (n, n-1, n-2, ..., 3, 2, 0), n \geq 2$.

Proposition 2.3 *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ such that $h(\infty) = \infty$ be an evaluation of the variables. Consider the canonical extension of h to the set of all terms. Then there holds: if $t \in W_\tau(X), \tau = (n, n-1, n-2, ..., 3, 2, 0), n \geq 2$ is a trivial term or there exists a variable x in t such that $h(x) = \infty$, then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Proof. Let $t \in W_\tau(X)$. We will prove by induction on the complexity of term.

Suppose that t is a trivial term. We want to prove that $h(t) = \infty$. Let $t = \infty$. Clearly, $h(t) = \infty$. Let $t = (t_1, t_2, ..., t_i)$ where $t_1, t_2, ..., t_i$ are terms and fulfil the equation which we want to prove. Since t is a trivial term. Then there exists $t_j, 1 \leq j \leq n_i$ such that t_j is a trivial term. By assumption $h(t_j) = \infty$. We see that $h(t) = f_i(h(t_1), h(t_2), ..., h(t_i)) = \infty$.

Let t be a non-trivial term. Suppose that there exists a variable x in t such that $h(x) = \infty$. Let $t = x_j,\ x_j \in X$, clearly if $h(x_j) = \infty$, then $h(t) = \infty$. Let $t = (t_1, t_2, ..., t_i)$ where $t_1, t_2, ..., t_i$ are non-trivial terms and fulfil the equation which we want to prove. If there exists a variable x in t_j such that $h(x) = \infty$, then by assumption we have $h(t_j) = \infty$. We see that $h(t) = f_i(h(t_1), h(t_2), ..., h(t_i)) = \infty$.

Suppose that $h(x) \neq \infty$ for all variables x in t and h is a homomorphism from $G(t)$ into G. Let $t = (u_1, u_2, ..., u_i)$ where $u_1, u_2, ..., u_i \in X$. We see that $h(t) = f_i(h(u_1), h(u_2), ..., h(u_i))$. Since $(u_1, u_2, u_3, ..., u_i) \in E_{f_i}$ of $E(t)$. We have $(h(u_1), (h(u_2)), h(u_3), ..., h(u_i)) \in E_{f_i}$ of E. Hence $h(t) = h(u_1) = h(L(t))$. Let $t = (t_1, t_2, ..., t_i)$ where $t_1, t_2, ..., t_i$ are non-trivial terms and fulfil the equation which we want to prove. We see that

$$h(t) = f_i(h(t_1), h(t_2), ..., h(t_i)) = f_i(h(L(t_1)), h(L(t_2)), ..., h(L(t_i))).$$

Since $(L(t_1), L(t_2), L(t_3), ..., L(t_i)) \in E_{f_i}$ of $E(t)$, so we have

$$(h(L(t_1)), h(L(t_2)), h(L(t_3)), ..., h(L(t_i))) \in E_{f_i}$$

of E. Therefore $h(t) = h(L(t_1)) = h(L(t))$. Suppose h is not a homomorphism of graphs. Let $t = (u_1, u_2, ..., u_i)$, where $u_1, u_2, ..., u_i \in X$. We have $h(t) = f_i(h(u_1), h(u_2), ..., h(u_i))$. Since $\{(u_1, u_2), (u_2, u_3), ..., (u_{i-1}, u_i)\} = E(t)$ and h is not a homomorphism of graph. Then $(h(u_1), h(u_2), ..., h(u_i)) \notin E_{f_i}$. Therefore $h(t) = \infty$. Let $t = (t_1, t_2, ..., t_i)$ where $t_1, t_2, ..., t_i$ are non-trivial terms and fulfil the equation which we want to prove. We see that $h(t) = f_i(h(t_1), h(t_2), ..., h(t_i))$. Since for each $j = 1, 2, 3, ..., i, h(t_j) = h(L(t_j))$, if the restriction of h on $V(t_j)$ is a homomorphism and $h(t_j) = \infty$, if the restriction of h on $V(t_j)$ is not a homomorphism. Then $h(t) = \infty$, if there exists j such that the restriction of h on $V(t_j)$ is not a homomorphism. Suppose that the restriction of h on $V(t_j)$ are the homomorphisms for all $j = 1, 2, 3, ..., i$. We have $h(t) = f_i(h(L(t_1)), h(L(t_2)), ..., h(L(t_i)))$. Since h is not a homomorphism of graph. We get that $(h(L(t_1)), h(L(t_2)), ..., h(L(t_i))) \notin E_{f_i}$ of E. Thus $h(t) = \infty$. □

Proposition 2.4 *Let s and t be non-trivial terms from $W_\tau(X), \tau = (n, n-1, ..., 3, 2, 0), n \geq 2$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property: A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.*

Proof. Suppose that a graph $G = (V, E)$ satisfies $s \approx t$ and let h be a restriction of an evaluation of variables. Suppose that h is a homomorphism from $G(s)$ into

G but h is not a homomorphism from $G(t)$ into G. By Proposition 2.3, we have $h(s) = h(L(s))$ and $h(t) = \infty$ which contradicts to the assumption. By the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then h is a homomorphism from $G(s)$ into G.

Conversely, suppose that h is a homomorphism from $G(s)$ into G iff h is a homomorphism from $G(t)$ into G. Let $G = (V, E)$ be a graph and let $h^{'} : V(t) \to V$ be a restriction of an evaluation of variables. If $h^{'}$ is not a homomorphism, then by assumption and Proposition 2.3, we have $h^{'}(s) = \infty = h^{'}(t)$. If $h^{'}$ is a homomorphism, then by assumption and Proposition 2.3 again, we get $h^{'}(s) = h(L(s)) = h(L(t)) = h^{'}(t)$. Hence $\underline{A(G)}$ satisfies $s \approx t$. □

3 Identities in Graph Algebras

Proposition 2.4 gives a method to check whether a graph $G = (V, E)$ satisfies the equation $s \approx t$. We will use this proposition to characterize graph identities. Let $G = (V, E)$ be a graph and $\underline{A(G)}$ be a graph algebra of type $\tau = (2, 0)$. Graph identities were characterized in [3] by the following proposition:

Proposition 3.1 *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Now we will extend this proposition to the case $\underline{A(G)}$ is a graph algebras of type $\tau = (n, n-1, ..., 3, 2, 0), n \geq 2$.

Proposition 3.2 *Let $s, t \in W_\tau(X), \tau = (n, n-1, ..., 3, 2, 0), n \geq 2$ be terms. Then the non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Proof. Suppose that the non-trivial equation $s \approx t$ is an identity in the class of all graph algebras. Let s be a trivial term. Suppose that t is not a trivial term. Consider the graph $G = (V, E)$ such that $G = G(t)$ and $h : V(t) \to V$ is a restriction of an identity evaluation function of variables. By Proposition 2.3, we have $h(s) = \infty \neq L(t) = h(L(t)) = h(t)$, contradict to the assumption.

Suppose that s and t are non-trivial terms. By the assumption and choose G is a complete graph, we can prove that $V(s) = V(t)$ and $L(s) = L(t)$. Now we want to show that $E(s) = E(t)$. Let $(x_1, x_2, ..., x_i) \in E_{f_i}$ of $E(s)$. Suppose that $(x_1, x_2, ..., x_i) \notin E_{f_i}$ of $E(t)$. Consider the graph $G = (V, E)$ such that $G = G(t)$ and $h : V(t) \to V$ is a restriction of an identity evaluation function of variables. We see that h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G. By Proposition 2.4, we get that $\underline{A(G)}$ is not satisfied $s \approx t$ which contradict the assumption. Hence $(x_1, x_2, ..., x_i) \in E_{f_i}$ of $E(t)$. By the same way, we can prove that if $(x_1, x_2, ..., x_i) \in E_{f_i}$ of $E(t)$, then $(x_1, x_2, ..., x_i) \in E_{f_i}$ of $E(s)$. Hence $E(s) = E(t)$.

Conversely, let G be a graph and let h be a restriction of an evaluation of variables. Suppose that s and t are both trivial terms. By Proposition 2.3, we have $h(s) = \infty = h(t)$. Now suppose that s and t are both non-trivial terms and $G(s) = G(t), L(s) = L(t)$. Then by Proposition 2.3, we see that $h(s) = h(L(s)) = h(L(t)) = h(t)$, if h is a homomorphism of graph and $h(s) = \infty = h(t)$, if h is not a homomorphism of graph. Hence $\underline{A(G)}$ satisfies $s \approx t$. □

4 Normal form terms

In [7] it was shown that any non-trivial term t over the class of all graph algebras of type $\tau = (2, 0)$ has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t in the following way. Let t be a non-trivial term. The *normal form term* of t is the term $NF(t)$ constructed by the following algorithm :

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \le j \le k(x)$) ordered by increasing indices $i_1 \le ... \le i_{k(x)}$ and let s_x be the term $(...((xx_{i_1})x_{i_2})...x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \ne \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

Next we will show that any non-trivial term t over the class of all graph algebras of type $\tau = (n, n-1, ..., 3, 2, 0), n \ge 2$ has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now we want to describe how to construct the normal form term. Before to do this we will ordered the elements in $E_{f_i}, i = 2, ..., n$ in the following way: $(x_{k_1}, x_{k_2}, ..., x_{k_i}) < (x_{k'_1}, x_{k'_2}, ..., x_{k'_i})$ iff $k_1 < k'_1$ or $k_1 = k'_1, k_2 < k'_2$ or $k_1 = k'_1, k_2 = k'_2, k_3 < k'_3$ or ... or $k_1 = k'_1, k_2 = k'_2, k_3 = k'_3, ..., k_{i-1} = k'_{i-1}, k_i < k'_i$. Let t be a non-trivial term. The *normal form term* of t is the term $NF(t)$ constructed by the following algorithm :

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list

$$l^i_x = ((x^i_{1p_1}, x^i_{2p_1}, ..., x^i_{(i-1)p_1}), ..., (x^i_{1p_{k^i(x)}}, x^i_{2p_{k^i(x)}}, ..., x^i_{(i-1)p_{k^i(x)}}))$$

such that $(x, x^i_{1p_j}, x^i_{2p_j}, ..., x^i_{(i-1)p_j}) \in E_{f_i}$ of $E(t), 1 \le j \le k^i(x)$ ordered by increasing $i = 2, 3, ..., n$ and let s_x be the term

$((...((...((...((...((((...((x, x^2_{1p_1}), x^2_{1p_2}), ...), x^2_{1p_{k^2(x)}}), x^3_{1p_1}, x^3_{2p_1}), x^3_{1p_2}, x^3_{2p_2}), ...),$

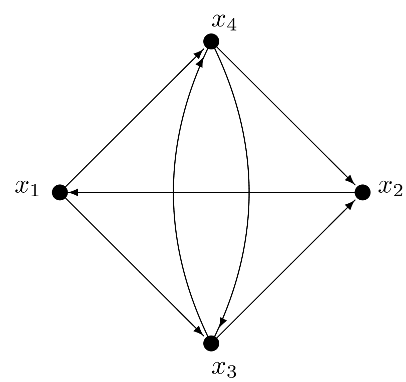
$x^3_{1p_{k^3(x)}}, x^3_{2p_{k^3(x)}}), ...), x^n_{1p_1}, x^n_{2p_1}, ..., x^n_{(n-1)p_1}), ...), x^n_{1p_{k^n(x)}}, ..., x^n_{(n-1)p_{k^n(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph.

Example 4.1 Let $t = ((x_1, (x_3, x_2)), (x_4, (x_2, x_1)), (x_3, x_4))$. Find $NF(t)$
$G(t) = (V(t), E(t)), L(t) = x_1$ where $V(t) = \{x_1, x_2, x_3, x_4\}$, and
$E(t) = \{(x_1, x_3), (x_1, x_4), (x_2, x_1), (x_3, x_2), (x_3, x_4), (x_4, x_2), (x_4, x_3)\}$.

G(t)



$E_{f_2}(t) = \{(x_1, x_3), (x_2, x_1), (x_3, x_2), (x_3, x_4), (x_4, x_2)\}$,
$E_{f_3}(t) = \{(x_1, x_4, x_3)\}$
$l^2_{x_1} = (x_3), l^2_{x_2} = (x_1), l^2_{x_3} = (x_2, x_4), l^2_{x_4} = (x_2)$,
$l^3_{x_1} = ((x_4, x_3)), l^3_{x_2} = \phi, l^3_{x_3} = \phi, l^3_{x_4} = \phi$,
$s_{x_1} = ((x_1, x_3), x_4, x_3), s_{x_2} = (x_2, x_1), s_{x_3} = ((x_3, x_2), x_4), s_{x_4} = (x_4, x_2)$.
$NF(t) = ((x_1, ((x_3, (x_2, x_1)), (x_4, x_2))), x_4, x_3)$.

Next we will prove that for any non-trivial term $t, G(t) = G(NF(t)), L(t) = L(NF(t))$ by the following proposition:

Proposition 4.1 *Let t be any term in* $W_\tau(X), \tau = (n, n-1, ..., 3.2, 0), n \geq 2$. *Then* $G(t) = G(NF(t))$ *and* $L(t) = L(NF(t))$.

Proof. Clearly, $L(NF(t)) = L(t), V(NF(t)) \subseteq V(t)$ and $E(NF(t)) \subseteq E(t)$. Since for any $x \in V(t)$, $x \in V(s_{x'})$ for some $x' \in V(t)$ and $V(NF(t)) = \bigcup_{x \in V(t)} V(s_x)$. Then $V(t) = V(NF(t))$. Suppose that $(x, y) \in E(t)$. Then $(x, y) \in E(s_{x'})$ for some $x' \in V(t)$. Since $E(NF(t)) = \bigcup_{x \in V(t)} E(s_x)$. Hence $E(t) = E(NF(t))$. □

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T. Poomsa-ard, J. Wetweerapong, C. Khiloukom and T. Musuntei
Department of Mathematics
Khon Kaen University
Khon Kaen 40002, Thailand.
e-mail : tiang@kku.ac.th