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# Bayesian Estimation Distribution and Survival Function of Records and Inter-Record Times and Numerical Computation for Weibull Model

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**Abstract :** In this paper, we consider an experiment from a sample whose size is determined by the values smaller than all previous ones which is record values. We suppose that the data available are lower record values such as $L_1$, $K_1$, $L_2$, $K_2$,..., $L_r$, $K_r$,..., where $L_1$, $L_2$,... are successive and $K_1, K_2, ...$ are the numbers of trials needed to obtain new records. Bayesian estimation and survival function are obtained based on record values under square error and linear exponential loss functions or briefly (SEL, LLF). We consider weibull distribution with unknown two parameters $\alpha$ and $\beta$. Estimation of both parameters and numerical computations under square error loss function are investigated.

**Keywords :** Admissibility; Bayes estimation; Bayes prediction; Squared error loss; Linex loss function; Linear-exponential loss; Maximum likelihood prediction.
**2010 Mathematics Subject Classification :** 62N02; 62F15; 62C10.

## 1 Introduction

Let $L_1, L_2, L_3, ...$ be a sequence of continuous random variables. $L_k$ is a lower record value if its value is smaller than all preceding values $L_1, L_2, ..., L_{k-1}$. There has been general interest in record values for centuries, particularly for sporting events like the Olympic Games. Motivated by the reported frequency of record weather conditions, Chandler [1] began studying the distributions of lower records, record times, and inter-record times for independent and identically distributed *(i.i.d)* sequences of random variables. Interested readers may refer to Glick [2], Ahsanullah [3], Arnold and Balakrishnan [4], Al-Hussaini [5, 6], Neutsand [7], for a review of developments in this area of research. There are also some papers done on statistical inference based on record values. See for instance, Berred [8], Ahmadi et al. [9].

# 2 Main Results

## 2.1 Bayesian estimation densities and survival functions

Through this paper, we assume that the data available for study are lower record values. Such data may be rewritten as the following :

$$L_1, K_1, L_2, K_2, ..., L_r, K_r \tag{2.1}$$

where $L_i$ is the $i$th record value or new minimum and $K_i$ is the number of trials following the observation of $L_i$ needed to obtain a new record. Inference with record values would seem to provide a good opportunity for Bayesian techniques. From a sequence of $n$ *(i.i.d)* continuous random variables only about $\log(n)$ records are expected. We expect to have little data, hence any prior information is welcome. We adopt the natural conjugate prior distribution for parameters. This leads to a posterior distribution in the same family as the prior. The form of the natural conjugate prior can often be identified by interchanging the role of the data and the parameter in the likelihood function. The natural conjugate for the record distribution is often the same as that for the distribution generating the original *(i.i.d)* sequence from which the records were taken.

The classical decision theory approach to point estimation hinges on choice of the loss function. Clearly, the choice of the loss function may be crucial. It has always been recognized that the most commonly used SEL function is inappropriate in many situations. Under SEL a measure of inaccuracy, i.e., $R(\theta,\delta) = E_\theta\{L(\theta,\delta(X))\}$ (Risk of $\theta$ and $\delta$ ) is often too sensitive to the assumptions about the behavior of the tail of the probability distribution of $X$. In practice, overestimation and underestimation of the same magnitude often have different economics and the actual loss function is asymmetric. There are numerous such examples in the literature. A useful alternative to the SEL is a convex but asymmetric loss function, called the LINEX (Linear-Exponential) loss function was proposed by Vatutin [10]. LINEX loss function (LLF) is defined as following:

$$L(\theta,\delta) = b[e^{v(\delta-\theta)} - v(\delta-\theta) - 1], \quad v \neq 0, \; b > 0, \tag{2.2}$$

where '$v$' and '$b$' are the shape and scale parameters of the loss function (2.2). Obviously, the nature of LLF changes according to the choice of $v$. Without loss of generality we will assume that $b = 1$ in (2.2), in what follows. The sign of $v$ represents the direction of penalty and its magnitude represents the degree of symmetry. For $v = 1$, loss is quite asymmetric about zero with overestimation being more costly than underestimation. In general when $v > 0$, this loss increase almost linearly for negative error $\delta - \theta$, and almost exponential for positive error. Therefore, overestimation is a more serious mistake than underestimation. When $v < 0$, the linear-exponential increase are interchanged, where underestimation is more serious than overestimation.

The magnitude of a reflects the degree of asymmetry, so the proposed loss function allows for an asymmetric penalty. The loss function is strictly convex and for small, positive values of $v$, i.e. $v^j \simeq 0$ for $j \geq 3$, the loss function is almost symmetric and not far from a squared error loss function. Indeed, on expanding $e^{v(\delta-\theta)} \simeq 1 + v(\delta - \theta) + \frac{v^2(\delta-\theta)^2}{2}$, $L(\theta, \delta) \simeq \frac{v^2(\delta-\theta)^2}{2}$, a squared error loss function. Thus for small values of $|v|$, optimal estimates and prediction are not far different from those obtained with a squared error loss function. Writing $M_{\theta|X}(t) := E_{\theta|X}[e^{t\theta}]$ for the moment-generating function of the posterior distribution of $\theta$, it is easy to verify that the value of $\delta(X)$ that minimizes $E_{\theta|X}[L(\theta, \delta(X)]$ in (2.2) as following:

$$\delta_B(X) = -\frac{1}{v} \ln M_{\theta|X}(-v), \tag{2.3}$$

where, $M_{\theta|X}$ exists and is finite.

**Theorem 2.1.** *Suppose that the data available $\{L_1, K_1, ..., L_r, K_r\}$, are a sequence of variable $X$ with distribution function $F_\theta(x) = 1 - e^{-\lambda_\theta(x)}, x > 0$, where, $\lambda_\theta(x)$ is nonnegative continuous differentiable function of $x$ such that $\lambda_\theta(x) \to 0$ as $x \to 0^+$, and $\lambda_\theta(x) \to +\infty$ as $x \to +\infty$, then estimator of $\theta$ under SEL and LLF function, as following:*

$$\hat{\theta}_{BS} = \frac{\int_\Theta \theta C_1(\theta; \boldsymbol{l}, \delta) e^{-D_1(\theta; \boldsymbol{l}, \boldsymbol{k}, \delta)} d\theta}{\int_\Theta C_1(\theta; \boldsymbol{l}, \delta) e^{-D_1(\theta; \boldsymbol{l}, \boldsymbol{k}, \delta)} d\theta},$$

$$\hat{\theta}_{BL} = -\frac{1}{v} \ln \left( \frac{\int_\Theta e^{-v\theta} C_1(\theta; \boldsymbol{l}, \delta) e^{-D_1(\theta; \boldsymbol{l}, \boldsymbol{k}, \delta)} d\theta}{\int_\Theta C_1(\theta; \boldsymbol{l}, \delta) e^{-D_1(\theta; \boldsymbol{l}, \boldsymbol{k}, \delta)} d\theta} \right).$$

*Proof.* The joint probability function or likelihood associated as following:

$$L(\mathbf{l,k}) = \prod_{i=1}^{r} f(l_i)[1 - F(l_i)]^{k_i - 1} I_{(0, l_{i-1})}, \tag{2.4}$$

where $l_0 \equiv \infty$, $k_r \equiv 1$ , $I_A(x)$ is the indicator function of the set $A$ and $(\mathbf{l,k}) := (l_1, k_1, ..., l_r, k_r)$.

We know that:

$$F_\theta(x) = 1 - e^{-\lambda_\theta(x)}, \quad x > 0, \tag{2.5}$$

then the corresponding density function is given by:

$$f_\theta(x) = \lambda'_\theta(x) e^{-\lambda_\theta(x)}, \quad x > 0 \tag{2.6}$$

By substituting (2.6) and (2.5) in (2.4) the likelihood function of $(\mathbf{l}, \mathbf{k})$ is given by:

$$L(\theta; \mathbf{l}, \mathbf{k}) \equiv A(\theta, \mathbf{l}) e^{-B(\theta, \mathbf{l}, \mathbf{k})}, \tag{2.7}$$

where $A(\theta, \mathbf{l}) = \prod_{i=1}^{r} \lambda_\theta^{'}(l_i)$, and $B(\theta, \mathbf{l}, \mathbf{k}) = \sum_{i=1}^{r} k_i \lambda_\theta(l_i)$.

We suggest that conjugate prior density function to be given by

$$\pi(\theta; \delta) \propto C_0(\theta; \delta) e^{-D_0(\theta;\delta)}, \quad \theta \in \Theta, \ \delta \in \Omega, \tag{2.8}$$

where $\Omega$ is the hyperparameter space. From (2.7) and (2.8), the posterior density function is given by:

$$\pi(\theta | \mathbf{l}, \mathbf{k}) \propto C_1(\theta; \mathbf{l}, \delta) e^{-D_1(\theta; \mathbf{l}, \mathbf{k}, \delta)}, \tag{2.9}$$

where

$$C_1(\theta; \mathbf{l}, \delta) = C_0(\theta; \delta) A(\theta; \mathbf{l}), \text{ and } \ D_1(\theta; \mathbf{l}, \mathbf{k}, \delta) = D_0(\theta; \delta) B(\theta; \mathbf{l}, \mathbf{k}).$$

Assuming a squared error loss function (SEL), the Bayes estimate of a parameter is it' s posterior mean. Therefore, by (2.9) the Bayes estimate of the parameter $\theta$ as following:

$$\hat{\theta}_{BS} = \frac{\int_\Theta \theta C_1(\theta; \mathbf{l}, \delta) e^{-D_1(\theta; \mathbf{l}, \mathbf{k}, \delta)} d\theta}{\int_\Theta C_1(\theta; \mathbf{l}, \delta) e^{-D_1(\theta; \mathbf{l}, \mathbf{k}, \delta)} d\theta}. \tag{2.10}$$

From (2.3), the Bayes estimator of $\theta$ under LINEX loss function

$$\hat{\theta}_{BL} = -\frac{1}{v} \ln \left( \frac{\int_\Theta e^{-v\theta} C_1(\theta; \mathbf{l}, \delta) e^{-D_1(\theta; \mathbf{l}, \mathbf{k}, \delta)} d\theta}{\int_\Theta C_1(\theta; \mathbf{l}, \delta) e^{-D_1(\theta; \mathbf{l}, \mathbf{k}, \delta)} d\theta} \right). \tag{2.11}$$

□

**Remark 2.2.** *$\hat{\theta}_{BS}$ and $\hat{\theta}_{BL}$ in* (2.10) *and* (2.11) *are the unique Bayes estimates of $\theta$ under SEL and LLF functions, respectively. Hence, they are admissible provided that the prior density* (2.8) *be proper.*

## 2.2 Weibull model

In this section we assume that data available are a sequnce of random variable with Weibull model such that both parameters are unknown. Baysian estimator of parameters under SEL and numerical computation are proposed.

**Theorem 2.3.** *Suppose that the data available $\{L_1, K_1, ..., L_r, K_r\}$, are a sequence of random variable $X$ with distribution function $W(\alpha, \beta)$, such that both $\alpha$ and $\beta$ are unknown, and prior density function of $(\alpha, \beta)$ is given by*

$$\pi(\alpha, \beta) = \frac{d^c b^a}{\Gamma(c)\Gamma(a)} \alpha^{c+a-1} \beta^{a-1} e^{-\alpha(d+b\beta)}, \tag{2.12}$$

*then*

$$\hat{\alpha}_{BS} = \frac{(r+c+a)\psi(r+a,r+c+a+1)}{\psi(r+a,r+c+a)}, \quad \hat{\beta}_{BS} = \frac{\psi(r+a+1,r+c+a)}{\psi(r+a,r+c+a)} \tag{2.13}$$

*where* $\psi(r+a,r+c+a) = \int_0^\infty \frac{\beta^{r+a-1}[\eta(l)]^\beta}{[\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta]^{r+c+a}} d\beta$.

*Proof.* Since

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, \quad x>0,\ \alpha>0, \quad \beta>0, \tag{2.14}$$

then

$$F(x;\alpha,\beta) = 1 - e^{-\alpha x^\beta}, \quad x>0,\ \alpha>0, \quad \beta>0. \tag{2.15}$$

By substituting (2.14) and (2.15) in (2.4) we obtain:

$$\begin{aligned} f(\mathbf{l},\mathbf{k};\alpha,\beta) &= \prod_{i=1}^r \alpha\beta l_i^{\beta-1} e^{-\alpha l_i^\beta} (e^{-\alpha l_i^\beta})^{k_i-1} \\ &= (\alpha\beta)^r \prod_{i=1}^r l_i^{\beta-1} e^{-\alpha k_i l_i^\beta} \\ &= (\alpha\beta)^r [\eta(l)]^{\beta-1} e^{-\alpha \Sigma_{i=1}^r k_i l_i^\beta} \end{aligned} \tag{2.16}$$

where $\eta(\mathbf{l}) = \prod_{i=1}^r l_i$.
The posterior density function $\pi(\alpha,\beta|l,k)$ is given by

$$\begin{aligned} \pi(\alpha,\beta|l,k) &\propto f(\mathbf{l},\mathbf{k}|\alpha,\beta)\pi(\alpha,\beta) \\ &= mf(l,k|\alpha,\beta)\pi(\alpha,\beta) \\ &= m(\alpha\beta)^r[\eta(l)]^{\beta-1} e^{-\alpha\Sigma_{i=1}^r k_i l_i^\beta} \times \frac{d^c b^a}{\Gamma(c)\Gamma(a)} \alpha^{c+a-1}\beta^{a-1} e^{-\alpha(d+b\beta)} \\ &= M\alpha^{r+c+a-1}\beta^{r+a-1}[\eta(l)]^\beta e^{-\alpha(\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta)}. \end{aligned} \tag{2.17}$$

We know that, $\int_0^\infty \int_0^\infty \pi(\alpha,\beta|l,k) d\alpha d\beta = 1$, so we can write as following:

$$\begin{aligned} M &= \left[\int_0^\infty \int_0^\infty \beta^{r+a-1}[\eta(l)]^\beta \alpha^{r+c+a-1} e^{-\alpha(\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta)} d\alpha d\beta\right]^{-1} \\ &= \left[\int_0^\infty \beta^{r+a-1}[\eta(l)]^\beta \left(\int_0^\infty \alpha^{r+c+a-1} e^{-\alpha(\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta)} d\alpha\right) d\beta\right]^{-1} \\ &= \left[\int_0^\infty \beta^{r+a-1}[\eta(l)]^\beta \left(\frac{\Gamma(r+c+a)}{[\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta]^{r+c+a}}\right) d\beta\right]^{-1} \\ &= \left[\Gamma(r+c+a)\int_0^\infty \frac{\beta^{r+a-1}[\eta(l)]^\beta}{[\Sigma_{i=1}^r k_i l_i^\beta + d + b\beta]^{r+c+a}} d\beta\right]^{-1} \\ &= [\Gamma(r+c+a)\psi(r+a,r+c+a)]^{-1} \end{aligned} \tag{2.18}$$

by substituting (2.18) in (2.17) we obtain:

$$\pi(\alpha,\beta|k,l)=\frac{\alpha^{r+c+a-1}\beta^{r+a-1}[\eta(l)]^{\beta}e^{-\alpha[\Sigma_{i=1}^{r}k_il_i^{\beta}+d+b\beta]}}{\Gamma(r+c+a)\psi(r+a,r+c+a)} \tag{2.19}$$

so we can write as following:

$$\begin{aligned}\hat{\alpha}_{BS} &= E[\alpha|k,l]\\ &= M\int_0^{\infty}\int_0^{\infty}\beta^{r+a-1}[\eta(l)]^{\beta}\alpha^{r+c+a}e^{-\alpha[\Sigma_{i=1}^{r}k_il_i^{\beta}+d+b\beta]}d\alpha d\beta\\ &= M\int_0^{\infty}\beta^{r+a-1}[\eta(l)]^{\beta}\frac{\Gamma(r+c+a+1)}{[\Sigma_{i=1}^{r}k_il_i^{\beta}+d+b\beta]^{r+c+a+1}}d\beta\\ &= M\Gamma(r+c+a+1)\psi(r+a,r+c+a+1) \end{aligned}\tag{2.20}$$

by substituting $M$ from (2.18) in (2.20)we obtain:

$$\hat{\alpha}_{BS}=\frac{(r+c+a)\psi(r+a,r+c+a+1)}{\psi(r+a,r+c+a)} \tag{2.21}$$

and similarly

$$\begin{aligned}\hat{\beta}_{BS} &= E[\beta|\mathbf{l},\mathbf{k}]\\ &= M\int_0^{\infty}\int_0^{\infty}\beta^{r+a}[\eta(l)]^{\beta}\alpha^{r+c+a-1}e^{-\alpha[\Sigma_{i=1}^{r}k_il_i^{\beta}+d+b\beta]}d\alpha d\beta\\ &= M\int_0^{\infty}\int_0^{\infty}\beta^{r+a}[\eta(l)]^{\beta}\frac{\Gamma(r+c+a)}{[\Sigma_{i=1}^{r}k_il_i^{\beta}+d+b\beta]^{r+c+a}}d\beta \end{aligned}\tag{2.22}$$

by substituting $M$ from (2.18) in (2.22) we obtain:

$$\hat{\beta}_{BS}=\frac{\psi(r+a+1,r+c+a)}{\psi(r+a,r+c+a)} \tag{2.23}$$

□

## 2.3 Numerical computations

In this section, we using MATLAB software for simulation of estimated parameters in last section. We generate sample of size $m=12$ from the $W(2,3)$ model given by (2.14) and written in order form as: $(\mathbf{l},\mathbf{k})=0.7936$, 1, 0.2995, 5, 0.2202, 31, 0.2176, 1. It can be seen that with prior parameters $a=3$, $b=1$, $c=4$ and $d=3$ for the joint prior density given in (2.12), the Bayes estimators of $\alpha$ and $\beta$ under SEL are $\hat{\alpha}_{BS}=1.5218$ and $\hat{\beta}_{BS}=2.5182$, respectively.

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