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Spectrum and Fine Spectrum of Generalized Second Order Difference Operator Δ^2_{uv} on Sequence Space c_0

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Abstract : The purpose of this paper is to determine spectrum and fine spectrum of newly introduced operator Δ^2_{uv} on sequence space c_0. The operator Δ^2_{uv} on sequence space c_0 is defined by $\Delta^2_{uv}x = (u_nx_n - v_{n-1}x_{n-1} + u_{n-2}x_{n-2})^\infty_{n=0}$ with $x_{-1}, x_{-2} = 0$, where $x = (x_n) \in c_0$, $u = (u_k)$ is a either constant or strictly decreasing sequence of positive real numbers with $U = \lim_{k\to\infty} u_k \neq 0$, $v = (v_k)$ is a sequence of positive real numbers such that $v_k \neq 0$ for each $k \in \mathbb{N}_0$ with $V = \lim_{k\to\infty} v_k \neq 0$. In this paper we have obtained the results on spectrum and point spectrum for the operator Δ^2_{uv} over sequence space c_0. We have also obtained the results on continuous spectrum $\sigma_c(\Delta^2_{uv}, c_0)$, residual spectrum $\sigma_r(\Delta^2_{uv}, c_0)$ and fine spectrum of the operator Δ^2_{uv} on sequence space c_0.

Keywords : Generalized second order difference operator; Sequence space c_0; Spectrum of an operator.
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1 Introduction

The study of spectrum and fine spectrum for various operators are made by various authors. Wenger [1] examined the fine spectrum of the integer power of

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the Cesàro operator in c and Rhoades [2] generalized this result to the weighted mean methods. The spectra of Cesàro operator on the sequence space c_0 have also been investigated by Reade [3]. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c has been examined by Yildirim [4]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c is determined by Altay and Basar [5]. Complete study of the spectrum such as point spectrum, continuous spectrum, residual spectrum of the operator Δ on sequence spaces c_0 and c made by these authors. The fine spectrum of the generalized difference operator $B(r, s)$ over sequence spaces c_0 and c is established by Altay and Basar [6]. The fine spectrum of the generalized difference operator $B(r, s, t)$ over sequence spaces c_0 and c is established by Furkan, Bilgic and Altay [7], where r, s, t are taken as scalars.

The present work is in a continuation of the previous works which gives the characterization of fine spectrum of the operator Δ^2_{uv} for various real sequences $u = (u_k)$ and $v = (v_k)$ under certain restrictions over the sequence space c_0. If $u = (1)$ and $v = (2)$ are constant sequences, then the operator Δ^2_{uv} reduces to second order forward difference operator Δ^2. Thus, the results of this paper unifies the corresponding results of many authors on operators whose matrix representation is a triple-band matrix.

2 Preliminaries and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ be a bounded linear operator. We denote the range of T as $R(T)$, where $R(T) = \{y \in Y : y = Tx,\ x \in X\}$, and the set of all bounded linear operators on X into itself is denoted by $B(X)$. Further, the adjoint T^* of T is a bounded linear operator on the dual space X^* of X defined by

$$(T^*\phi)(x) = \phi(Tx) \text{ for all } \phi \in X^* \text{ and } x \in X.$$

Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\alpha = (T - \alpha I)$, where α is a complex number and I is the identity operator on $D(T)$. The inverse of T_α (if exists) is denoted by T_α^{-1}, where $T_\alpha^{-1} = (T - \alpha I)^{-1}$ and known as the resolvent operator of T. It is easy to verify that T_α^{-1} is linear, if T_α is linear. Since the spectral theory is concerned with many properties of T_α and T_α^{-1} which depend on α, so we are interested the set of those α in the complex plane for which T_α^{-1} exists or T_α^{-1} is bounded or domain of T_α^{-1} is dense in X. For this, we need some definitions and known results given below which will be used in the sequel.

Definition 2.1. ([8], pp. 371) Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of T is a complex number α such that

(R1) T_α^{-1} exists,

(R2) T_α^{-1} is bounded,

(R3) T_α^{-1} is defined on a set which is dense in X.

Resolvent set $\rho(T, X)$ of T is the set of all regular values α of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called spectrum of T. The spectrum $\sigma(T, X)$ is further partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual spectrum as follows:

Point Spectrum $\sigma_p(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} does not exist, i.e., condition $(R1)$ fails. An element of $\sigma_p(T, X)$ is called an eigenvalue of T.

Continuous spectrum $\sigma_c(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that conditions $(R1)$ and $(R3)$ hold but condition $(R2)$ fails, i.e., T_α^{-1} exists, domain of T_α^{-1} is dense in X but T_α^{-1} is unbounded.

Residual Spectrum $\sigma_r(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} exists but do not satisfy condition $(R3)$, i.e., domain of T_α^{-1} is not dense in X. The condition $(R2)$ may or may not holds good.

Goldberg's Classification of Operator T_α([9], pp. 58): Let X be a Banach space and $T_\alpha \in B(X)$, where α is a complex number. Again let $R(T_\alpha)$ and T_α^{-1} denote the range and inverse of the operator T_α, respectively. Then the following possibilities may occur;

(A) $R(T_\alpha) = X$,

(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,

(C) $\overline{R(T_\alpha)} \neq X$,

and

(1) T_α is injective and T_α^{-1} is continuous,

(2) T_α is injective and T_α^{-1} is discontinuous,

(3) T_α is not injective.

Remark 2.2. *Combining (A), (B), (C) and (1),(2), (3); we get nine different cases. These are labelled by* $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ *and* C_3. *The notation* $\alpha \in A_2\sigma(T, X)$ *means the operator* $T_\alpha \in A_2$, *i.e.,* $R(T_\alpha) = X$ *and* T_α *is injective but* T_α^{-1} *is discontinuous. Similarly others.*

Remark 2.3. *If* α *is a complex number such that* $T_\alpha \in A_1$ *or* $T_\alpha \in B_1$, *then* α *belongs to the resolvent set* $\rho(T, X)$ *of* T *on* X. *The other classification gives rise to the fine spectrum of* T.

Definition 2.4. ([10], pp. 220–221) Let λ, μ be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. For every $x = (x_k) \in \lambda$ and every integer n, we write

$$A_n(x) = \sum_k a_{nk} x_k,$$

where the sum without limits is always taken from $k = 0$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (\lambda, \mu)$ if and only if $Ax \in \mu$ whenever $x \in \lambda$.

Lemma 2.5. ([11], pp. 129) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if

(1) the rows of A in l_1 and their l_1 norms are bounded,

(2) the columns of A are in c_0.

Note: The operator norm of T is the supremum of the l_1 norms of the rows.

Lemma 2.6. ([9], pp. 59) T has a dense range if and only if T^* is one to one, where T^* denotes the adjoint operator of the operator T.

Lemma 2.7. ([9], pp. 60) The adjoint operator T^* of T is onto if and only if T has a bounded inverse.

3 Spectrum and Point Spectrum of the Operator Δ^2_{uv} on Sequence Space c_0

In this section we introduce the new second order forward difference operator Δ^2_{uv} and compute spectrum and point spectrum of the operator Δ^2_{uv} over space c_0.

Let $u = (u_k)$ is a either constant or strictly decreasing sequence of positive real numbers with $U = \lim_{k\to\infty} u_k \neq 0$, and $v = (v_k)$ be a sequence of positive real numbers such that $v_k \neq 0$ for each $k \in \mathbb{N}_0$ with $V = \lim_{k\to\infty} v_k \neq 0$. We define the operator Δ^2_{uv} on sequence space c_0 as

$$\Delta^2_{uv} x = (u_n x_n - v_{n-1} x_{n-1} + u_{n-2} x_{n-2})_{n=0}^{\infty} \text{ with } x_{-1}, x_{-2} = 0,$$

where $x = (x_n) \in c_0$.

It is easy to verify that the operator Δ^2_{uv} can be represented by the matrix

$$\Delta^2_{uv} = \begin{pmatrix} u_0 & 0 & 0 & 0 & \dots \\ -v_0 & u_1 & 0 & 0 & \dots \\ u_0 & -v_1 & u_2 & 0 & \dots \\ 0 & u_1 & -v_2 & u_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.1. $\Delta^2_{uv} : c_0 \to c_0$ *is a bounded linear operator and* $\|\Delta^2_{uv}\|_{(c_0,c_0)} = \sup_k(|u_k| + |v_{k-1}| + |u_{k-2}|)$.

Proof. Proof is simple. So we omit. □

Note: Through out this work, we consider $\sqrt{z}$, where z is a complex number, as the square root of z with non-negative real part. If $\text{Re}(\sqrt{z}) = 0$ then $\sqrt{z}$ represents the square root of z with $\text{Im}(\sqrt{z}) \geq 0$.

Theorem 3.2. *Assume $\sqrt{V^2} = V$ and define the set $\mathcal{S}$ by*

$$\mathcal{S} = \left\{ \alpha \in \mathbb{C} : \frac{2|(U - \alpha)|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} \leq 1 \right\}.$$

Then spectrum of the operator Δ^2_{uv} on sequence space c_0 is given by $\sigma(\Delta^2_{uv}, c_0) = \mathcal{S}$.

Proof. The proof of the theorem is divided into two parts.
In the first part, we show that $\sigma(\Delta^2_{uv}, c_0) \subseteq \mathcal{S}$, which we prove by contradiction. That is assuming $\alpha \in \mathbb{C}$ with$\left|\frac{2(U-\alpha)}{V+\sqrt{V^2-4U(U-\alpha)}}\right| > 1$, we will show that $\alpha \in \rho(\Delta^2_{uv}, c_0)$. In second part, we establish the reverse inequality, i.e., $\mathcal{S} \subseteq \sigma(\Delta^2_{uv}, c_0)$.
Part I: Let $\alpha \in \mathbb{C}$ with $\left|\frac{2(U-\alpha)}{V+\sqrt{V^2-4U(U-\alpha)}}\right| > 1$. Clearly, $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$ as it does not satisfy the condition. Further, $(\Delta^2_{uv} - \alpha I)$ reduces to a triangle and hence has an inverse. Thus, $(\Delta^2_{uv} - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{u_0-\alpha} & 0 & 0 & 0 & \dots \\ \frac{v_0}{(u_0-\alpha)(u_1-\alpha)} & \frac{1}{u_1-\alpha} & 0 & 0 & \dots \\ \frac{v_0 v_1}{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)} - \frac{u_0}{(u_0-\alpha)(u_2-\alpha)} & \frac{v_1}{(u_1-\alpha)(u_2-\alpha)} & \frac{1}{u_2-\alpha} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$b_{n,k} = \frac{v_{n-1}b_{n-1,k} - u_{n-2}b_{n-2,k}}{(u_n - \alpha)}, \quad k = 0, 1, 2, \dots, n.$$

By Lemma 2.5, the operator $(\Delta^2_{uv} - \alpha I)^{-1} \in (c_0, c_0)$ if

(1) series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$ and $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$.

(2) $\lim_{n\to\infty} |b_{nk}| = 0$ for each $k \in \mathbb{N}_0$.

In order to show that $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$, first we prove that the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$.

For this consider $S_n = \sum_{k=0}^{n} |b_{nk}| = |b_{n,n}| + |b_{n,n-1}| + \dots + |b_{n,0}|$. Clearly, for n is even, the series

$$\begin{aligned} S_n &= \left|\frac{1}{(u_n-\alpha)}\right| + \left|\frac{v_{n-1}}{(u_n-\alpha)(u_{n-1}-\alpha)}\right| + \left|\frac{v_{n-1}v_{n-2}}{(u_n-\alpha)(u_{n-1}-\alpha)(u_{n-2}-\alpha)}\right. \\ &\quad \left. - \frac{u_{n-2}}{(u_n-\alpha)(u_{n-2}-\alpha)}\right| + \dots + \left|\frac{v_0 v_1 \dots v_{n-1}}{(u_0-\alpha)(u_1-\alpha)\dots(u_n-\alpha)}\right. \\ &\quad \left. - \frac{u_0 v_2 \dots v_{n-1}}{(u_0-\alpha)(u_2-\alpha)\dots(u_n-\alpha)} - \dots + \frac{u_0 u_2 \dots u_{n-2}}{(u_0-\alpha)\dots(u_n-\alpha)}\right| \end{aligned}$$

is convergent. Similarly for n is odd, S_n is also convergent. Next we show that $\sup_n S_n$ is finite. Now let

$$w_1 = \frac{V \, + \, \sqrt{V^2 - \, 4U(U - \, \alpha)}}{2(U - \, \alpha)} \quad \text{and} \quad w_2 = \frac{V \, - \, \sqrt{V^2 - \, 4U(U - \, \alpha)}}{2(U - \, \alpha)}.$$

We can observe,

$$\lim_{n\to\infty} \frac{1}{(u_n - \alpha)} = \frac{1}{U - \alpha} = a_1 = \frac{1}{\sqrt{V^2 - \, 4U(U - \, \alpha)}}[(w_1) - (w_2)]$$

$$\lim_{n\to\infty} \frac{v_{n-1}}{(u_n - \alpha)(u_{n-1} - \alpha)} = \frac{V}{(U - \alpha)^2} = a_2 = \frac{1}{\sqrt{V^2 - \, 4U(U - \, \alpha)}} \left[(w_1)^2 - (w_2)^2\right]$$

$$\lim_{n\to\infty} \frac{v_{n-1}v_{n-2}}{(u_n - \alpha)(u_{n-1} - \alpha)(u_{n-2} - \alpha)} - \frac{u_{n-2}}{(u_n - \alpha)(u_{n-2} - \alpha)}$$
$$= \frac{V^2}{(U - \alpha)^3} - \frac{U}{(U - \alpha)^2} = a_3 = \frac{1}{\sqrt{V^2 - \, 4U(U - \, \alpha)}} \left[(w_1)^3 - (w_2)^3\right]$$

Clearly, $a_n = \frac{1}{\sqrt{V^2 - \, 4U(U - \, \alpha)}} \, [(w_1)^n - (w_2)^n]$.
Suppose $V^2 = 4U(U - \alpha)$ then

$$a_n = \left(\frac{2n}{V}\right)\left[\frac{V}{2(U - \alpha)}\right]^n,$$

which gives,

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left|\frac{2k}{V}\right| \left|\frac{V}{2(U - \alpha)}\right|^k < \infty,$$

since $\left|\frac{V}{2(U-\alpha)}\right| < 1$ and it follows from the ratio test. Therefore $\alpha \notin \mathcal{S}$ implies $a_n \to 0$. So, we may assume that $V^2 \neq 4U(U - \alpha)$. Since α is not in $\mathcal{S}$, we have $|w_1| < 1$. Now we show that $|w_2| < 1$. Since $|w_1| < 1$, we have

$$\left|1 \, + \, \sqrt{1 - \, 4U(U - \, \alpha)/V^2}\right| < \left|\frac{2(U - \alpha)}{V}\right|.$$

Since $|1 - \sqrt{z}| \leq |1 + \sqrt{z}|$ for any $z \in \mathbb{C}$, we must have

$$\left|1 \, - \, \sqrt{1 - \, 4U(U - \, \alpha)/V^2}\right| < \left|\frac{2(U - \alpha)}{V}\right|,$$

which leads us to the fact that $|w_2| < 1$. Taking limit both sides of S_n and since $|w_1| < 1$ and $|w_2| < 1$, we get

$$\begin{aligned} \lim_{n\to\infty} S_n &= \sum_{k=1}^{\infty} |a_k| \\ &\leq \frac{1}{|\sqrt{V^2 - \, 4U(U - \, \alpha)}|} \left(\sum_{k=1}^{\infty} |w_1|^k + \sum_{k=1}^{\infty} |w_2|^k\right) < \infty. \end{aligned}$$

Since (S_n) is a sequence of positive real numbers and $\lim_{n\to\infty} S_n < \infty$, so $\sup_n S_n < \infty$. For n is odd,

$$\begin{aligned}
&\lim_{n\to\infty} |b_{n,0}| \\
&\quad = \lim_{n\to\infty} \left| \frac{v_0 v_1 \cdots v_{n-1}}{(u_0-\alpha)(u_1-\alpha)\cdots(u_n-\alpha)} - \frac{u_0 v_2 \cdots v_{n-1}}{(u_0-\alpha)(u_2-\alpha)\cdots(u_n-\alpha)} - \cdots \right. \\
&\qquad \left. + \frac{u_0 u_2 \cdots u_{n-2}}{(u_0-\alpha)\cdots(u_n-\alpha)} \right| \\
&\quad \le \frac{1}{|\sqrt{V^2 - 4U(U-\alpha)}|} \left(\lim_{n\to\infty} |w_1|^n + \lim_{n\to\infty} |w_2|^n \right) = 0.
\end{aligned}$$

Thus, $\lim_{n\to\infty} |b_{n,0}| = 0$. Similarly, for n is odd, $\lim_{n\to\infty} |b_{n,0}| = 0$.
Again, we can show that $\lim_{n\to\infty} |b_{n,k}| = 0$ for all $k = 1, 2, 3, \ldots$. Thus,

$$(\Delta_{uv}^2 - \alpha I)^{-1} \in B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| \frac{2(U-\alpha)}{V + \sqrt{V^2 - 4U(U-\alpha)}} \right| > 1. \quad (3.1)$$

Next we will show that domain of the operator $(\Delta_{uv}^2 - \alpha I)^{-1}$ is dense in c_0. This statement holds if and only if range of the operator $(\Delta_{uv}^2 - \alpha I)$ is dense in c_0. Since $(\Delta_{uv}^2 - \alpha I)^{-1} \in (c_0, c_0)$, which implies that range of the operator $(\Delta_{uv}^2 - \alpha I)$ is dense in c_0. Hence we have

$$\sigma(\Delta_{uv}^2, c_0) \subseteq \left\{ \alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{V + \sqrt{V^2 - 4U(U-\alpha)}} \right| \le 1 \right\}. \quad (3.2)$$

Part (II): We now prove the reverse inequality, i.e.,

$$\left\{ \alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{V + \sqrt{V^2 - 4U(U-\alpha)}} \right| \le 1 \right\} \subseteq \sigma(\Delta_{uv}^2, c_0). \quad (3.3)$$

First we prove the inclusion (3.3) under the assumption that $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$, i.e., we want to show that one of the conditions of Definitions 2.1 fails. Let $\alpha \in \mathcal{S}$. Clearly, $(\Delta_{uv}^2 - \alpha I)$ is a triangle and hence $(\Delta_{uv}^2 - \alpha I)^{-1}$ exists. So, condition (R1) is satisfied but condition (R2) fails as can be seen below:
First, let $V^2 = 4U(U-\alpha)$, then $a_n = (\frac{2n}{V})[\frac{V}{2(U-\alpha)}]^n$, which gives

$$\lim_{n\to\infty} |b_{n,0}| = \lim_{n\to\infty} \left| \frac{2n}{V} \right| \left| \frac{V}{2(U-\alpha)} \right|^n = \infty,$$

since $|\frac{V}{2(U-\alpha)}| \ge 1$. So, we may assume that $V^2 \neq 4U(U-\alpha)$. Suppose $\alpha \in \mathbb{C}$

with $\left|\frac{2(U-\alpha)}{V+\sqrt{V^2-4U(U-\alpha)}}\right| < 1$. Then $|w_1| > 1$ and $|w_1| > |w_2|$ always, consequently,

$$\begin{aligned}\lim_{n\to\infty}|b_{n,0}| &= \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|}\lim_{n\to\infty}\Big|(w_1)^n-(w_2)^n\Big| \\ &\geq \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|}\lim_{n\to\infty}\Big\{|w_1|^n-|w_2|^n\Big\} \\ &= \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|}\lim_{n\to\infty}|w_1|^n\Big\{1-\Big(\frac{|w_2|}{|w_1|}\Big)^n\Big\}\to\infty,\end{aligned}$$

which gives $\lim_{n\to\infty}|b_{n,k}| \neq 0$ for each k. Hence

$$(\Delta^2_{uv}-\alpha I)^{-1}\notin B(c_0) \text{ for } \alpha\in\mathbb{C} \text{ with } \left|\frac{2(U-\ \alpha)}{V\ +\ \sqrt{V^2-\ 4U(U-\ \alpha)}}\right|\ <\ 1. \quad (3.4)$$

Next, we consider $\alpha \in \mathbb{C}$ with $\left|\frac{2(U-\alpha)}{V+\sqrt{V^2-4U(U-\alpha)}}\right| = 1$. Then $|w_1| = 1$ and $|w_2| < |w_1| = 1$,

$$\begin{aligned}\lim_{n\to\infty}|b_{n,0}| &\geq \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|}\lim_{n\to\infty}\Big\{|w_1|^n-|w_2|^n\Big\} \\ &= \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|}\lim_{n\to\infty}\Big\{1-|w_2|^n\Big\} \\ &= \frac{1}{|\sqrt{V^2-\ 4U(U-\ \alpha)}|},\end{aligned}$$

this tells $\lim_{n\to\infty}|b_{n,0}| \neq 0$. Thus,

$$(\Delta^2_{uv}-\alpha I)^{-1}\notin B(c_0) \text{ for } \alpha\in\mathbb{C} \text{ with } \left|\frac{2(U-\ \alpha)}{V\ +\ \sqrt{V^2-\ 4U(U-\ \alpha)}}\right|\ =\ 1. \quad (3.5)$$

Finally, we prove the inclusion (3.3) under the assumption that $\alpha = U$ and $\alpha = u_k$ for each $k \in \mathbb{N}_0$. We have

$$(\Delta^2_{uv}-\alpha I)x = \begin{pmatrix} (u_0-\alpha)x_0 \\ -v_0x_0+(u_1-\alpha)x_1 \\ u_0x_0-v_1x_1+(u_2-\alpha)x_2 \\ u_1x_1-v_2x_2+(u_3-\alpha)x_3 \\ \vdots \end{pmatrix}.$$

Case (i): If (u_k) is a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$, then

$$(\Delta^2_{uv}-UI)x = 0 \ \ \Rightarrow x_0 = 0, x_1 = 0, x_2 = 0, \ldots.$$

This shows that the operator $(\Delta^2_{uv} - UI)$ is one to one, but $R(\Delta^2_{uv} - UI)$ is not dense in c_0. So, condition (R3) fails. Hence $U \in \sigma(\Delta^2_{uv}, c_0)$.

Case (ii): If (u_k) is a strictly decreasing sequence, then for fixed k, $k \geq 0$,

$$(\Delta^2_{uv} - u_k I)x = 0 \quad \Rightarrow \quad x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0, x_{k+1} = \frac{v_k}{u_{k+1} - u_k} x_k,$$

$$x_{k+2} = \frac{v_{k+1}x_{k+1} - u_k x_k}{u_{k+2} - u_k} = \left\{ \frac{v_k v_{k+1} + u_k(u_k - u_{k+1})}{(u_{k+2} - u_k)(u_{k+1} - u_k)} \right\} x_k,$$

are non-zero since $x_k \neq 0$ and we have chosen u_k to be a strictly decreasing sequence. Similarly it can be shown that, for $n \geq k+3$, x_n is non-zero by using the expression

$$x_{n+1} = \frac{v_n x_n - u_{n-1}x_{n-1}}{(u_{n+1} - u_k)}.$$

Hence we get non-zero solution of $(\Delta^2_{uv} - u_k I)x = 0$. This shows that $(\Delta^2_{uv} - u_k I)$ is not injective. So, condition $(R1)$ fails. Hence $u_k \in \sigma(\Delta^2_{uv}, c_0)$ for all $k \in \mathbb{N}_0$. Hence we have

$$\left\{ \alpha \in \mathbb{C} : \left| \frac{2(U - \alpha)}{V + \sqrt{V^2 - 4U(U - \alpha)}} \right| \leq 1 \right\} \subseteq \sigma(\Delta^2_{uv}, c_0). \tag{3.6}$$

From inclusions 3.2 and 3.6, we get

$$\sigma(\Delta^2_{uv}, c_0) = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(U - \alpha)}{V + \sqrt{V^2 - 4U(U - \alpha)}} \right| \leq 1 \right\}.$$

This completes the proof. □

Theorem 3.3. *Point spectrum of the operator Δ^2_{uv} on sequence space c_0 is*

$$\sigma_p(\Delta^2_{uv}, c_0) = \begin{cases} \emptyset, & \text{if } (u_k) \text{ is a constant sequence,} \\ \{u_0, u_1, \ldots\}, & \text{if } (u_k) \text{ is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.

Case (i): Suppose (u_k) is a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. Consider $\Delta^2_{uv}x = \alpha x$ for $x \in c_0$ and $x \neq \theta$, which gives

$$\left.\begin{aligned} u_0 x_0 &= \alpha x_0 \\ -v_0 x_0 + u_1 x_1 &= \alpha x_1 \\ u_0 x_0 - v_1 x_1 + u_2 x_2 &= \alpha x_2 \\ u_1 x_1 - v_2 x_2 + u_3 x_3 &= \alpha x_3 \\ &\vdots \\ u_{k-2}x_{k-2} - v_{k-1}x_{k-1} + u_k x_k &= \alpha x_k \\ &\vdots \end{aligned}\right\} \tag{3.7}$$

Let (x_t) be the first non-zero entry of the sequence $x = (x_n)$. So equation $Ux_{t-2} - v_{t-1}x_{t-1} + Ux_t = \alpha x_t$, implies $\alpha = U$, and from the equation $Ux_{t-1} -$

$v_t x_t + U x_{t+1} = \alpha x_{t+1}$, we get $x_t = 0$, which is a contradiction to our assumption. Therefore

$$\sigma_p(\Delta^2_{uv}, c_0) = \emptyset.$$

Case (ii): Suppose (u_k) is a strictly decreasing sequence. Consider $\Delta^2_{uv}x = \alpha x$ for $x \in c_0$ and $x \neq \theta$, which gives system of equations (3.7).
If $\alpha = u_0$, then

$$x_1 = \frac{v_0}{(u_1 - u_0)}x_0,\ x_2 = \frac{v_0 v_1 + u_0(u_0 - u_1)}{(u_1 - u_0)(u_2 - u_0)}x_0,$$

are non-zero, since u_k is a strictly decreasing sequence and by taking $x_0 \neq 0$. Similarly, it can be shown that, for $n \geq 3$, x_n is non-zero by using the expression

$$x_{n+1} = \frac{v_n x_n - u_{n-1}x_{n-1}}{(u_{n+1} - u_0)}, \text{ for all } n \geq 2.$$

Hence we get non-zero solution of $(\Delta^2_{uv} - u_0 I)x = 0$.
If $\alpha = u_k$, for all $k \geq 1$, then solving system of equations, we get $x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0$, $x_{k+1} = \frac{v_k}{u_{k+1}-u_k}x_k$,

$$x_{k+2} = \frac{v_{k+1}x_{k+1} - u_k x_k}{u_{k+2} - u_k} = \left\{\frac{v_k v_{k+1} + u_k(u_k - u_{k+1})}{(u_{k+2} - u_k)(u_{k+1} - u_k)}\right\}x_k,$$

is non-zero, since u_k is strictly decreasing sequence and by taking $x_k \neq 0$. Similarly, it can be shown that, for $n \geq k+3$, x_n is non-zero by using the expression

$$x_{n+1} = \frac{v_n x_n - u_{n-1}x_{n-1}}{(u_{n+1} - u_k)}, \text{ for all } n \geq k+2.$$

Hence we get non-zero solution of $(\Delta^2_{uv} - u_k I)x = 0$. This shows that $(\Delta^2_{uv} - u_k I)$ is not injective. So, condition $(R1)$ fails. we get non-zero solution of $(\Delta^2_{uv} - \alpha I)x = 0$. Thus,

$$\sigma_p(\Delta^2_{uv}, c_0) = \{u_0, u_1, u_2, \ldots\}.$$

This completes the proof. □

4 Point Spectrum of the Adjoint Operator Δ^{2*}_{uv} of Δ^2_{uv} on Dual Sequence Space c_0^*

Let $T : X \to X$ be a bounded linear operator having matrix representation A and the dual space of X is denoted by X^*. Again, let T^* be its adjoint operator on X^*. Then the matrix representation of T^* is the transpose of the matrix A.

Theorem 4.1. *Point spectrum of the adjoint operator Δ^{2*}_{uv} over c_0^* is*

$$\sigma_p(\Delta^{2*}_{uv}, c_0^*) = \left\{\alpha \in \mathbb{C} : \frac{2|(U - \alpha)|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1\right\}.$$

Proof. If $\Delta_{uv}^{2*} f = \alpha f$ for $0 \neq f \in c_0^* \cong l_1$, where

$$\Delta_{uv}^{2*} = \begin{pmatrix} u_0 & -v_0 & u_0 & 0 & 0 & \dots \\ 0 & u_1 & -v_1 & u_1 & 0 & \dots \\ 0 & 0 & u_2 & -v_2 & u_2 & \dots \\ 0 & 0 & 0 & u_3 & -v_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}.$$

Consider the system of linear equations

$$\left.\begin{array}{r} u_0 f_0 - v_0 f_1 + u_0 f_2 = \alpha f_0 \\ u_1 f_1 - v_1 f_2 + u_1 f_3 = \alpha f_1 \\ u_2 f_2 - v_2 f_3 + u_2 f_4 = \alpha f_2 \\ \vdots \end{array}\right\} \tag{4.1}$$

Solving the system of linear equations (4.1) in terms of f_0 and f_1, we obtain for $k \geq 2$,

$$f_k = (-1)^k (b_{k-1,0} f_1 - b_{k-1,1} f_0) \frac{(\alpha - u_0)(\alpha - u_1) \cdots (\alpha - u_{k-1})}{u_0 u_1 \cdots u_{k-2}},$$

where $b_{k-1,0}$ and $b_{k-1,1}$ are defined as in last section. For $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$ and $u = (u_k)$ is a constant or strictly decreasing positive real sequence, we get for $n = 0, 1, \dots, k-2$

$$\begin{aligned} \left|\frac{\alpha - u_n}{u_n}\right| &= \left|\frac{\alpha}{u_n} - 1\right| \\ &= \left(\left(\frac{\alpha_1}{u_n} - 1\right)^2 + \left(\frac{\alpha_2}{u_n}\right)^2\right)^{\frac{1}{2}} \\ &\leq \left(\left(\frac{\alpha_1}{U} - 1\right)^2 + \left(\frac{\alpha_2}{U}\right)^2\right)^{\frac{1}{2}} \\ &= \left|\frac{\alpha - U}{U}\right|. \end{aligned}$$

Then

$$|f_k| \leq |b_{k-1,0} f_1 - b_{k-1,1} f_0| \left|\frac{\alpha - U}{U}\right|^{k-1} |\alpha - U|.$$

Taking limit on both sides and choosing $f_0 = 1$ and $f_1 = \frac{1}{w_1}$, we obtain

$$\begin{aligned}
\lim_{k\to\infty}|f_k| &\leq \lim_{k\to\infty}|a_k f_1 - a_{k-1}f_0|\left|\frac{\alpha-U}{U}\right|^{k-1}|\alpha-U| \\
&= \lim_{k\to\infty}\frac{|(w_1^k-w_2^k)f_1-(w_1^{k-1}-w_2^{k-1})f_0|}{|\sqrt{V^2-4U(U-\alpha)}|}\left|\frac{U-\alpha}{U}\right|^{k-1}|U-\alpha| \\
&= \lim_{k\to\infty}\frac{|w_2|^{k-1}|w_1-w_2|}{|w_1||\sqrt{V^2-4U(U-\alpha)}|}\left|\frac{U-\alpha}{U}\right|^{k-1}|U-\alpha| \\
&= \lim_{k\to\infty}\frac{|w_2|^{k-1}}{|w_1|}\left|\frac{U-\alpha}{U}\right|^{k-1}. \qquad (4.2)
\end{aligned}$$

We have the relation

$$\frac{U-\alpha}{U} = \frac{2(U-\alpha)}{V+\sqrt{V^2-4U(U-\alpha)}} \times \frac{2(U-\alpha)}{V-\sqrt{V^2-4U(U-\alpha)}} = \frac{1}{w_1 w_2}. \qquad (4.3)$$

Then using (4.3) in (4.2), we obtain

$$\lim_{k\to\infty}|f_k| \leq \lim_{k\to\infty}\frac{|w_2|^{k-1}}{|w_1|}\frac{1}{|w_1w_2|^{k-1}} = \lim_{k\to\infty}\left|\frac{1}{w_1}\right|^k.$$

If $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} < 1$, then $\frac{1}{|w_1|} < 1$. By Cauchy root test $\lim_{k\to\infty}|f_k|^{\frac{1}{k}} \leq \frac{1}{|w_1|} < 1$. Hence $\sum_{k=0}^{\infty}|f_k|$ converges.

Conversely, we have to show that $\sum_{k=0}^{\infty}|f_k|$ converges implies $\frac{1}{|w_1|} < 1$. This is equivalent to show that $\frac{1}{|w_1|} \geq 1$ implies $\sum_{k=0}^{\infty}|f_k|$ diverges. To prove this, we write another representation of f_k in terms of f_{k-1} and f_{k-2} from the system of equation (4.1) as

$$f_k = \Big(\frac{\alpha-u_{k-2}}{u_{k-2}}\Big)f_{k-2} + \Big(\frac{v_{k-2}}{u_{k-2}}\Big)f_{k-1} \text{ for all } k \geq 2.$$

Hence, dividing both sides by f_k, we obtain

$$\Big(\frac{\alpha-u_{k-2}}{u_{k-2}}\Big)\frac{f_{k-2}}{f_{k-1}}\frac{f_{k-1}}{f_k} + \Big(\frac{v_{k-2}}{u_{k-2}}\Big)\frac{f_{k-1}}{f_k} = 1.$$

Taking limit both sides and denoting $\lim_{k\to\infty}\frac{f_{k-1}}{f_k} = L$, we obtain a quadratic equation,

$$\left(\frac{\alpha-U}{U}\right)L^2 + \left(\frac{V}{U}\right)L - 1 = 0.$$

Solving the above equation, we get two roots as

$$L_1 = \frac{V+\sqrt{V^2-4U(U-\alpha)}}{2(U-\alpha)} \text{ and } L_2 = \frac{V-\sqrt{V^2-4U(U-\alpha)}}{2(U-\alpha)}.$$

Now we describe some cases:

(i) If $1 < \frac{1}{|w_1|} < \frac{1}{|w_2|}$, then

$$\begin{aligned}\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| &= \left|\frac{1}{L_1}\right| = \frac{1}{|w_1|} > 1. \\ &= \left|\frac{1}{L_2}\right| = \frac{1}{|w_2|} > 1.\end{aligned}$$

Hence by ratio test, series $\sum_{k=0}^{\infty} |f_k|$ diverges.

(ii) If $1 < \frac{1}{|w_1|} = \frac{1}{|w_2|}$, then

$$\begin{aligned}\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| &= \left|\frac{1}{L_1}\right| = \frac{1}{|w_1|} > 1. \\ &= \left|\frac{1}{L_2}\right| = \frac{1}{|w_2|} > 1.\end{aligned}$$

Hence by ratio test, series $\sum_{k=0}^{\infty} |f_k|$ diverges.

(iii) If $1 = \frac{1}{|w_1|} < \frac{1}{|w_2|}$

$$\begin{aligned}\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| &= \left|\frac{1}{L_1}\right| = \frac{1}{|w_1|} = 1. \\ &= \left|\frac{1}{L_2}\right| = \frac{1}{|w_2|} > 1.\end{aligned}$$

Thus ratio test fails for one case. In this case, we take $f = (f_k)$ in such a way that it is an increasing sequence of positive real numbers and $\lim_{k\to\infty} |\frac{f_k}{f_{k-1}}| = 1$. Clearly, $f = (f_k)$ is a divergent sequence and consequently series $\sum_{k=0}^{\infty} |f_k|$ diverges.

(iv) If $1 = \frac{1}{|w_1|} = \frac{1}{|w_2|}$. Here also ratio test fails for both the expression for L. Thus by choosing as same way in (iii), we able to prove that series $\sum_{k=0}^{\infty} |f_k|$ diverges. This completes the proof. □

5 Residual and Continuous Spectrum of the Operator Δ_{uv}^2 on Sequence Space c_0

Define two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ as,

$$\mathcal{S}_1 = \left\{\alpha \in \mathbb{C} : \frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1\right\},$$

and

$$\mathcal{S}_2 = \left\{\alpha \in \mathbb{C} : \frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} = 1\right\}.$$

Theorem 5.1. *Residual spectrum $\sigma_r(\Delta^2_{uv}, c_0)$ of the operator Δ^2_{uv} over c_0 is*

$$\sigma_r(\Delta^2_{uv}, c_0) = \begin{cases} \mathcal{S}_1, & if\ (u_k)\ is\ a\ constant\ sequence, \\ \mathcal{S}_1 \setminus \{u_0, u_1, \dots\}, & if\ (u_k)\ is\ a\ strictly\ decreasing\ sequence. \end{cases}$$

Proof. The proof of the theorem is divided into two cases.

Case (i): Let (u_k) be a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. For $\alpha \in \mathbb{C}$ with $2|U - \alpha| < |V + \sqrt{V^2 - 4U(U - \alpha)}|$, the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle except $\alpha = U$ and consequently $(\Delta^2_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta^2_{uv} - \alpha I)$ is one to one for $\alpha = U$ and hence has an inverse.

By Theorem 4.1, the operator $(\Delta^2_{uv} - \alpha I)^* = \Delta^{2*}_{uv} - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ with $\frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1$. Hence by Lemma 2.6, range of the operator $(\Delta^2_{uv} - \alpha I)$ is not dense in c_0. Thus,

$$\sigma_r(\Delta^2_{uv}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1\right\}.$$

Case (ii): Let (u_k) be a strictly decreasing sequence. For $\alpha \in \mathbb{C}$ such that

$$\frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1,$$

the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle except for $\alpha = u_k$ for all $k \in \mathbb{N}_0$ and consequently the operator $(\Delta^2_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta^2_{uv} - \alpha I)$ is not one to one for $\alpha = u_k$ for all $k \in \mathbb{N}_0$. So, $(\Delta^2_{uv} - \alpha I)^{-1}$ does not exist.

On the basis of argument as given in Case (i), it is easy to verify that the range of the operator $(\Delta^2_{uv} - \alpha I)$ is not dense in c_0. Thus,

$$\sigma_r(\Delta^2_{uv}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} < 1\right\} \setminus \{u_0, u_1, u_2, \dots\}.$$

□

Theorem 5.2. *Continuous spectrum $\sigma_c(\Delta^2_{uv}, c_0)$ of operator Δ^2_{uv} over c_0 is*

$$\sigma_c(\Delta^2_{uv}, c_0) = \begin{cases} \mathcal{S}_2, & if\ (u_k)\ is\ a\ constant\ sequence, \\ \mathcal{S}_2 \setminus \{u_0, u_1, \dots\}, & if\ (u_k)\ is\ a\ strictly\ decreasing\ sequence. \end{cases}$$

Proof. The proof of this theorem is divided into two cases.

Case (i): Let (u_k) be a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. For $\alpha \in \mathbb{C}$ with $\frac{2|U - \alpha|}{|V + \sqrt{V^2 - 4U(U - \alpha)}|} = 1$, the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle because $\alpha \neq U$ and has an inverse. The operator $(\Delta^2_{uv} - \alpha I)^{-1}$ is discontinuous by statement (3.5). Therefore, the operator $(\Delta^2_{uv} - \alpha I)$ has an unbounded inverse.

By Theorem 4.1, the operator $(\Delta^2_{uv} - \alpha I)^*$ is one to one for $\alpha \in \mathbb{C}$ with $\frac{2|U-\alpha|}{|V+\sqrt{V^2-4U(U-\alpha)}|} = 1$. Hence by Lemma 2.6, range of the operator $(\Delta^2_{uv} - \alpha I)$ is dense in c_0. Thus,

$$\sigma_c(\Delta^2_{uv}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|U-\alpha|}{|V+\sqrt{V^2-4U(U-\alpha)}|} = 1\right\}.$$

Case (ii): Let (u_k) be a strictly decreasing sequence. For $\alpha \in \mathbb{C}$ such that $\frac{2|U-\alpha|}{|V+\sqrt{V^2-4U(U-\alpha)}|} = 1$, the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle except for $\alpha = u_k$ for all $k \in \mathbb{N}_0$ and consequently the operator $(\Delta^2_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta^2_{uv} - \alpha I)$ is not one to one for $\alpha = u_k$ for all $k \in \mathbb{N}_0$. So, $(\Delta^2_{uv} - \alpha I)^{-1}$ does not exist.

On the basis of argument as given in Case (i), it is easy to verify that the range of the operator $(\Delta^2_{uv} - \alpha I)$ is dense in c_0. Thus,

$$\sigma_c(\Delta^2_{uv}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|U-\alpha|}{|V+\sqrt{V^2-4U(U-\alpha)}|} = 1\right\} \setminus \{u_0, u_1, u_2, \dots\}.$$

□

6 Fine Spectrum of the Operator Δ^2_{uv} on Sequence Space c_0

Theorem 6.1. *If α satisfies $\frac{2|U-\alpha|}{|V+\sqrt{V^2-4U(U-\alpha)}|} > 1$, then $(\Delta^2_{uv} - \alpha I) \in A_1$.*

Proof. It is required to show that the operator $(\Delta^2_{uv} - \alpha I)$ is bijective and has an inverse for $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|V+\sqrt{V^2-4U(U-\alpha)}|} > 1$. Since $\alpha \neq U$, therefore the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle. Hence it has an inverse. The operator $(\Delta^2_{uv} - \alpha I)^{-1}$ is continuous for $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|V+\sqrt{V^2-4U(U-\alpha)}|} > 1$ by statement (3.1). Also the equation

$$(\Delta^2_{uv} - \alpha I)x = y \text{ gives } x = (\Delta^2_{uv} - \alpha I)^{-1}y, i.e., x_n = ((\Delta^2_{uv} - \alpha I)^{-1}y_n), n \in \mathbb{N}_0.$$

Thus , for every $y \in c_0$, we can find $x \in c_0$ such that

$$(\Delta^2_{uv} - \alpha I)x = y, \text{ since } (\Delta^2_{uv} - \alpha I)^{-1} \in (c_0, c_0).$$

This shows that the operator $(\Delta^2_{uv} - \alpha I)$ is onto and hence $(\Delta^2_{uv} - \alpha I) \in A_1$. This completes the proof. □

Theorem 6.2. *Let (u_k) be a constant sequence, say $u_k = U$ and $\alpha = U$. Then $\alpha \in C_1\sigma(\Delta^2_{uv}, c_0)$.*

Proof. We have

$$\sigma_p(\Delta_{uv}^{2*}, c_0^*) = \left\{\alpha \in \mathbb{C} : 2|(U - \alpha)| < |V + \sqrt{V^2 - 4U(U - \alpha)}|\right\}.$$

For $\alpha = U$, the operator $(\Delta_{uv}^2 - \alpha I)^*$ is not one to one. By Lemma 2.6, $R(\Delta_{uv}^2 - \alpha I)$ is not dense in c_0. Again by Theorem 3.3, since $\alpha = U$ does not belong to the set $\sigma_p(\Delta_{uv}^2, c_0)$, therefore the operator $(\Delta_{uv}^2 - \alpha I)$ has an inverse.

Next, we show that the operator $(\Delta_{uv}^2 - \alpha I)^{-1}$ is continuous. By Lemma 2.7, it is enough to show that $(\Delta_{uv}^2 - \alpha I)^*$ is onto, i.e., for given $y = (y_n) \in c_0^*$, we can find $x = (x_n) \in c_0^*$ such that $(\Delta_{uv}^2 - \alpha I)^* x = y$. Now, $(\Delta_{uv}^2 - UI)^* x = y$, i.e.,

$$\begin{aligned} -v_0x_1 + Ux_2 &= y_0 \\ -v_1x_2 + Ux_3 &= y_1 \\ &\vdots \\ -v_{i-1}x_i + Ux_{i+1} &= y_{i-1} \\ &\vdots \end{aligned}$$

Thus, $-v_{n-1}x_n + Ux_{n+1} = y_{n-1}$ for all $n \geq 1$ which implies $\sum_{n=0}^{\infty} |x_n| < \infty$, since $y \in l_1$. This shows that the operator $(\Delta_{uv}^2 - \alpha I)^*$ is onto and hence $\alpha \in C_1\sigma(\Delta_{uv}^2, c_0)$. □

Theorem 6.3. *Let (u_k) be a constant sequence, say $u_k = U$ and $\alpha \neq U$, $\alpha \in \sigma_r(\Delta_{uv}^2, c_0)$. Then $\alpha \in C_2\sigma(\Delta_{uv}^2, c_0)$.*

Proof. Since $\alpha \neq U$, therefore the operator $(\Delta_{uv}^2 - \alpha I)$ is a triangle. Hence it has an inverse. For $U \neq \alpha \in \mathbb{C}$ with $2|(U - \alpha)| < |V + \sqrt{V^2 - 4U(U - \alpha)}|$, the operator $(\Delta_{uv}^2 - \alpha I)^{-1}$ is discontinuous by statement (3.4). Thus, $(\Delta_{uv}^2 - \alpha I)$ is injective and $(\Delta_{uv}^2 - \alpha I)^{-1}$ is discontinuous.

Again by Theorem 4.1, the operator $(\Delta_{uv}^2 - \alpha I)^*$ is not one to one for $\alpha \in \mathbb{C}$ with $2|(U - \alpha)| < |V + \sqrt{V^2 - 4U(U - \alpha)}|$. But Lemma 2.6 yields the fact that range of the $(\Delta_{uv}^2 - \alpha I)$ is not dense in c_0 and $\alpha \in C_2\sigma(\Delta_{uv}^2, c_0)$. □

Theorem 6.4. *Let (u_k) be constant sequence. If $|U| < |v_k|$ for each k, then $U \in C_1\sigma(\Delta_{uv}^2, c_0)$. If $|U| \geq |v_k|$ for each k, then $U \in C_2\sigma(\Delta_{uv}^2, c_0)$.*

Proof. If $\alpha = U$, then by Theorem 5.1 $(\Delta_{uv}^2 - \alpha I)$ is in state C_1 or C_2. A left inverse of Δ_{uv}^2 is

$$B = (\Delta_{uv}^2 - UI)^{-1} = \begin{pmatrix} 0 & \left(\frac{-1}{v_0}\right) & 0 & 0 & \cdots \\ 0 & \left(\frac{-U}{v_0v_1}\right) & \left(\frac{-1}{v_1}\right) & 0 & \cdots \\ 0 & \left(\frac{-U^2}{v_0v_1v_2}\right) & \left(\frac{-U}{v_1v_2}\right) & \left(\frac{-1}{v_2}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix B is in $B(c_0)$ for $|U| < |v_k|$ and is not in $B(c_0)$ for $|U| \geq |v_k|$, $k \in \mathbb{N}_0$. That is $(\Delta^2_{uv} - UI)$ has a continuous inverse for $|U| < |v_k|$, $k \in \mathbb{N}_0$ but it does not have a continuous inverse for $|U| \geq |v_k|$, $k \in \mathbb{N}_0$. Therefore, $U \in C_1\sigma(\Delta^2_{uv}, c_0)$ for $|U| < |v_k|$, $k \in \mathbb{N}_0$, and $U \in C_2\sigma(\Delta^2_{uv}, c_0)$ for $|U| \geq |v_k|$, $k \in \mathbb{N}_0$. This completes the proof. □

Theorem 6.5. *Let (u_k) be a strictly decreasing sequence and $\alpha \in \sigma_r(\Delta^2_{uv}, c_0)$. Then $\alpha \in C_2\sigma(\Delta^2_{uv}, c_0)$.*

Proof. We have,

$$\sigma_r(\Delta^2_{uv}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|(U - \alpha)|}{|V + \sqrt{V^2 - 4U(U - \alpha)|}} < 1\right\} \setminus \{u_0, u_1, u_2, \dots\}.$$

Since $\alpha \neq u_k$ for all k, therefore the operator $(\Delta^2_{uv} - \alpha I)$ is a triangle. Hence it has an inverse. For $u_k \neq \alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|V + \sqrt{V^2 - 4U(U-\alpha)|}} < 1$, the inverse of the operator $(\Delta^2_{uv} - \alpha I)$ is discontinuous by statement 3.4. Thus $(\Delta^2_{uv} - \alpha I)$ injective and $(\Delta^2_{uv} - \alpha I)^{-1}$ is discontinuous.

On the basis of argument as given in Theorem 6.3, it is easy verify that the range of the operator $(\Delta^2_{uv} - \alpha I)$ is not dense in c_0 and hence $\alpha \in C_2\sigma(\Delta^2_{uv}, c_0)$. □

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