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On the Union of Graded Prime Submodules

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Abstract : Let G be a group with identity e. Let R be a G-graded commutative ring, and let M be a graded R-module. In this paper, we investigate finite and infinite union of graded submodules of a graded R-module M. Also, we give a number of results concerning the union of graded prime submodules.

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1 Introduction

Let G be a group with identity e. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g\in G} R_g$ such that $1 \in R_e$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$; $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be homogeneous element. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. We write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$. A graded ideal I of R is said to be graded prime ideal if $I \neq R$;

and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A graded ideal I of R is said to be graded maximal if $I \neq R$ and there is no graded ideal J of R such that $I \nsubseteq J \nsubseteq R$. A graded ring R is called graded local if it has a unique graded maximal ideal. A proper graded submodule N of a graded R-module M is called graded prime if $rm \in N$, then $m \in N$ or $r \in (N : M)$, where $r \in h(R), m \in h(M)$. A graded module M over a G-graded ring R is called to be graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$ where $x_{g_i} \in h(M)$. A graded R-module M is called graded cyclic if $M = Rx_g$ where $x_g \in h(M)$. A graded module M over a G-graded ring R is called to be graded multiplication if for each graded submodule N of M; $N = IM$ for some graded ideal I of R. One can easily show that if N is graded submodule of a graded multiplication module M, then $N = (N : M)M$ (see [3]). Similar to non graded case, a graded multiplication module has a graded maximal ideal. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (degs)^{-1}(degr)\}$. Let M be a graded module over a graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (degs)^{-1}(degm)\}$. Consider the graded homomorphism $\eta : M \longrightarrow S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non graded case, one can prove that $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N : M) = \emptyset$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the graded localization of R. This ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, R_P^g-module $S^{-1}M$ is denoted by M_P^g. For graded submodules N and K of M, if $N_P^g = K_P^g$ for every graded prime (graded maximal) ideal P of R, then $N = K$.

If K is a graded submodule of $S^{-1}R$-module $S^{-1}M$, then $K \cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. Moreover, similar to the non graded case one can prove that $S^{-1}(K \cap M) = K$. In this paper, we study unions of graded submodules of a graded R-module M. For example, we show that a graded multiplication module is a *ugp*-module.

2 The Union of Graded Prime Submodules

Let $N_1, N_2, ..., N_n$ be graded submodules of a graded R-module M, we call a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ efficient if N is not contained in the union of any $n-1$ of the graded submodules $N_1, N_2, ..., N_n$. We say that $N = N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient union, if non of the N_k may be excluded.

Similar to non graded case, if N, N_1, N_2 are graded submodules of a graded R-module M such that $N \subseteq N_1 \cup N_2$, then $N \subseteq N_1$ or $N \subseteq N_2$ (see [5]). Hence a covering of a graded submodule by two graded submodules is never efficient.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold:*

- *(i) If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals.*
- *(ii) If N is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.*
- *(iii) If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.*
- *(iv) Let N_λ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Lemma 2.2. *Let R be a G-graded ring and M a graded R-module and N a graded submodule of M. Let $N = N_1 \cup N_2 \cup \cdots \cup N_n$ be a efficient union of graded submodules of M, for $n > 1$. Then $\bigcap_{j \neq k} N_j = \bigcap_{j=1}^{n} N_j$ for all $1 \leq k \leq n$.*

Proof. It is straightforward. □

Lemma 2.3. *Let R be a G-graded ring and M a graded R-module and N a graded submodule of M. Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ be a efficient covering concisely of graded submodules of M, for $n > 1$. If $(N \bigcap N_j : N) \nsubseteq (N \cap N_k : N)$ for all $1 \leq k \leq n$, then no N_j, for $j \in \{1, 2, ..., n\}$, is a graded prime submodule.*

Proof. Clearly, by hypothesis, $N = (N \cup N_1) \cdots (N \cup N_n)$ is an efficient union. Moreover, by Lemma 2.2, $\bigcap_{j \neq k}(N \bigcap N_j) = \bigcap_{j=1}^{n}(N \bigcap N_j) \subseteq N \bigcap N_k$. Let N_k be a graded prime submodule of M. Now we show that $N \bigcap N_k$ is a graded prime submodule of N. If $r_g n_h \in N \bigcap N_k$ and $n_h \notin N \bigcap N_k$ where $r_g \in h(R)$ and $n_h \in h(N)$, then $r_g n_h \in N_k$ and $n_h \notin N_k$ and so $r_g M \subseteq N$. It follows that $N \bigcap N_k$ is a graded prime submodule and so $(N \bigcap N_k : N)$ is a graded prime ideal of R by [2, Proposition 2.7]. Since $(N \bigcap N_j : N) \nsubseteq (N \cap N_k : N)$ whenever $j \neq k$, we get that $(N \bigcap N_1 : N) \cdots (N \bigcap N_{k-1} : N)(N \bigcap N_{k+1} : N) \cdots (N \bigcap N_n : N) \nsubseteq (N \bigcap N_k : N)$ by [4, Proposition 1.4]. Therefore there exist $r \in [(N \bigcap N_1 : N) \cdots (N \bigcap N_{k-1} : N)(N \bigcap N_{k+1} : N) \cdots (N \bigcap N_n : N)] - (N \bigcap N_k : N)$ and so there exists $n \in N$ such that $rn \notin N \bigcap N_k$, but every $j \neq k$, $rn \in N \bigcap N_j$ which contracts to be $\bigcap_{j \neq k}(N \bigcap N_j) = \bigcap_{j=1}^{n}(N \bigcap N_j) \subseteq N \bigcap N_k$. Therefore, no N_k is a graded prime submodule. □

Theorem 2.4. *Let M be a graded R-module. Let $N_1, N_2, ..., N_n$ be graded submodules of M, and N a graded submodule of M such that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. Assume that at most two of the $N_k^{,}s$ are not graded prime and $(N \bigcap N_j : N) \nsubseteq (N \cap N_k : N)$ whenever, $j \neq k$. Then $N \subseteq N_i$ for some i.*

Proof. We may assume that the covering is efficient without loss of generality. Then $n \neq 2$. By Lemma 2.2, $n \leq 2$. Hence $n = 1$, and so $N \subseteq N_i$ for some i. □

Definition 2.5.

(i) *Let M be a graded R-module and N a graded submodule of M. N is called ug-submodule of M provided N contained in a finite union of graded submodules must be contained one of those graded submodules. M is called ug-module if every graded submodule of M is a ug-submodule.*

(ii) *Let M be a graded R-module and N a graded submodule of M. N is called ugp-submodule of M provided N contained in a finite union of graded prime submodules must be contained one of those graded prime submodules. M is called ugp-module if every graded submodule of M is a ugp-submodule.*

(iii) *Let M be a graded R-module and N a graded submodule of M. N is called ugm-submodule of M provided N contained in a finite union of graded maximal submodules must be contained one of those graded submodules. M is called ugm-module if every graded maximal submodule of M is a ugm-submodule.*

Theorem 2.6. *Let M be a graded finitely generated R-module. Then M is ugm-module if and only if every graded submodule N in M such that $N \subseteq \bigcup_{i=1}^{n} P_i$ where $P_i^{,}s$ are graded prime submodules implies that $N + P_i \neq M$ for some i.*

Proof. Let M be a graded finitely generated *ugm*-module. Suppose that N be a graded submodule of M such that $N \subseteq \bigcup_{i=1}^{n} P_i$ where $P_i^{,}s$ are graded prime submodules of M. By [1, Lemma 2.7] for each P_i, choose a graded maximal submodule M_i containing P_i. Then $N \subseteq \bigcup_{i=1}^{n} M_i$ and so $N \subseteq M_i$ by hypothesis. Since $P_i \subseteq M_i$, we have $N + P_i \subseteq M_i \neq M$.

Conversely, let N be a graded submodule of M such that $N \subseteq \bigcup_{i=1}^{n} M_i$ where $M_i^{,}s$ are graded maximal submodules of M. Then $N + M_i \neq M$ for some i by hypothesis. Therefore, since $M_i \subseteq N + M_i \subsetneq M$, then $N + M_i = M_i$, so $N \subseteq M_i$ for some i. The proof is completed. □

Proposition 2.7. *Let R be a G-graded ring and M a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R such that $S \bigcap p = \phi$, for every graded prime ideal p of R.*

(i) *M is a ugp-module if and only if $S^{-1}M$ is a ugp-module.*

(ii) *M is a ugm-module if and only if $S^{-1}M$ is a ugm-module.*

Proof. Let M be *ugp*-module. Let $K \subseteq Q_1 \bigcup Q_2 \bigcup \cdots \bigcup Q_n$ where K is a graded submodule of $S^{-1}M$ and $Q_1, Q_2, ..., Q_n$ are graded prime submodules of $S^{-1}M$. So $K = S^{-1}N$ and $Q_1 = S^{-1}P_1$, ...,$Q_n = S^{-1}P_n$, where N is a graded submodule of M, $P_1, P_2, ..., P_n$ are graded prime submodules of M, then $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$, because if $x \in N$, then $x = \sum_{g \in G} x_g$ where $x_g \in N \bigcap M_g$. So for any $g \in G$; $x_g \in N$. Hence for any $g \in G$; $x_g/1 \in S^{-1}N$, so $x_g/1 \in S^{-1}(P_1 \bigcup \cdots \bigcup P_n)$, hence $x_g/1 = p/s$ for some $p \in P_1 \bigcup \cdots \bigcup P_n$ and $s \in S$. So there exists $1 \leq k \leq n$ such that $p \in P_k$. Therefore, $tsx_g = pt \in P_k$ for some $t \in S$. Thus $x_g \in P_k$ since $ts \notin$

$(P_k : M)$ and P_k is a graded prime submodule of M, so $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$, since M is *ugp*-module; $N \subseteq P_i$ for some i. Then $S^{-1}N \subseteq S^{-1}P_i$, as needed.

Conversely, let $S^{-1}M$ be a *ugp*-module. Let $N \subseteq P_1 \bigcup \cdots \bigcup P_n$ where N is a graded submodule of M and $P_i^{,}s$ are graded prime submodules of M. Hence $S^{-1}N \subseteq S^{-1}(P_1 \bigcup \cdots \bigcup P_n) \subseteq S^{-1}P_1 \bigcup \cdots \bigcup S^{-1}P_n$. So $S^{-1}N \subseteq S^{-1}P_i$ for some i since $S^{-1}M$ is a *ugp*-module. So similar to the above proof, $N \subseteq P_i$. Therefore M is *ugp*-module.

(ii) Similar to (i). □

Theorem 2.8. *Every graded multiplication module is a ugp-module.*

Proof. Let M be a graded multiplication module. Let N be a graded submodule of M such that $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$ where at least $n-2$ of $P_1, P_2, ..., P_n$ are graded prime submodules. We may assume that the covering is efficient. Then $(P_j : M) \nsubseteq (P_k : M)$ whenever, $j \neq k$. Otherwise $(P_j : M) \subseteq (P_k : M)$, then $P_j = (P_j : M)M \subseteq (P_k : M)M = P_k$, a contradiction. Hence $N \subseteq P_k$ for some k. This result implies that M is a *ugp*-module. □

Definition 2.9. *By a chain of graded prime submodules of a graded R-module M we mean a finite strictly increasing sequence $P_1 \subsetneq \cdots \subsetneq P_n$; the graded dimension of this chain is n. We define the graded dimension of M to be the supremum of the lengths of all chains of graded prime submodules in M.*

Let M be a graded module over a G-graded ring R. Now consider the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in h(R)\}$. If R is a graded integral domain, then $T(M)$ is a graded submodule of M (see [1]). If $T(M) = 0$, then M is called graded torsion free and if $T(M) = M$, then M is called graded torsion.

Theorem 2.10. *Let M be a graded finitely generated R-module. Let M be a graded torsion free module with dimension 1, then M is a ugp-module if and only if M is a ugm-module.*

Proof. Let M be a *ugp*-module. Since every *ugp*-module is a *ugm*-module, so M is a *ugm*-module.

Conversely, let M be a *ugm*-module. Since M is a graded torsion free, 0 is a graded prime submodule by [1, Proposition 2.6]. Let N be a non-zero graded submodule of M such that $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$ where $P_i^{,}s$ are graded prime submodule of M. We may assume that $P_i \neq 0$ for all $i \in \{1, 2, ..., n\}$. By Theorem 2.6, $N + P_i \neq M$ for some i. There exists a graded maximal submodule of M such that $N + P_i \subseteq M_i$ by [1, Lemma 2.7]. Since the graded dimension of M is 1; $P_i = M_i$. Consequently, $N \subseteq P_i$. □

Definition 2.11. *Let R be a G-graded ring and M a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R. A non empty subset S^* of $h(M)$ is said to be graded S-closed if $se \in S^*$ for every $s \in S$ and $e \in S^*$.*

Theorem 2.12. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of graded ring R and S^* be a graded S-closed of a graded R-module of M. Let N be a graded submodule of M which is graded maximal in $M - S^*$. If the graded ideal $(N : M)$ is graded maximal in $R - S$, then N is a graded prime submodule of M.*

Proof. Assume that $r_g \notin (N : M)$ and $m_h \notin N$ for some $r_g \in h(R)$ and $m_h \in h(M)$ but $r_g m_h \in N$. Then there exist $s^* \in M$ and $r^* \in R$ such that $s^* \in (N + Rm_h) \bigcap S^*$ and $r^* \in ((N : M) + Rr_g) \bigcap S$. Therefore $r^* s^* \in (N : M) + Rr_g)(N + Rm_h) = (N : M)N + (N : M)Rm_h + Rr_g N + Rr_g Rm_h \subseteq N$. So $r^* s^* \in N \cap S^*$. This is a contradiction with $N \bigcap S^* = \emptyset$ □

Lemma 2.13. *Let M be a graded multiplication module. Let $P_i (i \in I)$ be a collection graded prime submodules of M with $(P_i : M) = p_i$ for any i and $M - S^* = \bigcup_{i \in I} P_i$ where $R - S = \bigcup_{i \in I} p_i$. If N is a graded maximal submodule in $M - S^*$, then N is a graded prime submodule of M.*

Proof. Let N be graded maximal submodule in $M - S^*$. Then the ideal $(N : M)$ is graded maximal in $R - S$. Otherwise, if $(N : M) \subsetneq T \subseteq \bigcup_i p_i$ where T is a graded ideal of R, then $N = (N : M)M \subsetneq TM \subseteq \bigcup_{i \in I} P_i$ which contradicts the presume that N is graded maximal in $M - S^*$. Then N is graded prime submodule of M by Theorem 2.12. □

Definition 2.14. *Let M be a graded R-module and N a graded submodule of M. N is called graded compactly packed by graded prime submodules if whenever N is contained in the union of a family of graded prime submodules of M, N is contained in one of the graded prime submodules of the family. M is called graded compactly packed by graded prime submodules if every graded submodule of M is graded compactly packed by graded prime submodules.*

Theorem 2.15. *Let M be a graded multiplication module. Then M is graded compactly packed by graded prime submodules if and only if every graded prime submodule of M is a graded compactly packed by graded prime submodules.*

Proof. It is clear that if M is graded compactly packed by graded prime submodules, then every graded prime submodule of M is graded compactly packed.

Conversely, suppose that every graded prime submodule of M be graded compactly packed by graded prime submodules. Let $N = \bigcup_{i=1}^{n} P_i$ where N is a graded submodule of M and $P_i^{,}s$ are graded prime submodules of M. Let L be a graded maximal submodule of M such that $N \subseteq L \subseteq \bigcup_{i=1}^{n} P_i$, then L is graded prime submodule of M by Lemma 2.13. Then $N \subseteq L \subseteq P_i$ for some i by hypothesis. □

Let N be a graded submodule of a graded R-module M. The graded radical of M, denoted by $gr - rad(N)$, is defined to be intersection of the graded prime submodules of M if such exist, and M otherwise (see [2]).

Theorem 2.16. *Let M be a graded multiplication module. Then M is graded compactly packed by graded prime submodules if and only if every graded prime submodule of M is graded radical of a graded cyclic submodule.*

Proof. Let M be graded compactly packed by graded prime submodules. Suppose that N be a graded prime submodule of M and not a graded radical of a graded cyclic submodule of M. Let $n = \sum_{g \in G} n_g \in N$. Since for each $n_g \in N$; $N \neq gr - rad(Rn_g)$ and since $gr - rad(N)$, is the intersection of all the graded prime submodules of M which contains $n_g (g \in G)$, there is a graded prime submodule P_{n_g} such that $n_g \in P_{n_g}$ but $N \nsubseteq P_{n_g}$. It is clear that $N \subseteq \bigcup_{n_g} P_{n_g}$, a contradiction. On the other hand, suppose that every graded prime submodule of M is the graded radical of a graded cyclic submodule of M. Let $N = \bigcup_{i \in I} P_i$ where N and $P_i (i \in I)$ are graded prime submodules of M and $N = gr - rad(Rn_g)$ for some $n_g \in h(M)$. Then $n_g \in \bigcup_{i \in I} P_i$ and so $n_g \in P_i$ for some $i \in I$. Therefore $N = gr - rad(Rn_g) \subseteq P_i$. Then M is graded compactly packed by graded prime submodules by Theorem 2.15. □

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References

[1] S. Ebrahimi Atani, F. Farzalipour, Notes On the graded prime submodules, Int. Math. Forum 1 (38) (2006) 1871–1880.

[2] S. Ebrahimi Atani, F. Farzalipour, On graded secondary modules, Turk. J. Math. 31 (2007) 371–378.

[3] S. Ebrahimi Atani, F. Farzalipour, On graded multiplication modules, (Submitted).

[4] M. Refaei, K. Alzobi, On graded primary ideals, Turkish. J. Math. 28 (2004) 217–229.

[5] R.Y. Sharp, Steps in Commutative Algebra, Cambridge University Press, 1990.

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