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On Nil-semicommutative Rings

Weixing Chen

School of Mathematics and Information Science,
Shandong Institute of Business and Technology,
Yantai 264005, China
e-mail : wxchen5888@163.com

Abstract : In this note a ring R is defined to be nil-semicommutative in case for any $a, b \in R$, ab is nilpotent implies that arb is nilpotent whenever $r \in R$. Examples of such rings include semicommutative rings, 2-primal rings, NI-rings etc.. It is proved that if I is an ideal of a ring R such that both I and R/I are nil-semicommutative then R is nil-semicommutative and that if R is a semicommutative ring satisfying the α-condition for an endomorphism α of R then the skew polynomial ring $R[x;\alpha]$ is nil-semicommutative. However the polynomial ring $R[x]$ over a nil-semicommutative ring R need not be nil-semicommutative. It is an open question whether a nil-semicommutative ring is an NI-ring, which has a close connection with the famous Koethe's conjecture.

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1 Introduction

Rings considered are associative with identity unless otherwise stated. For a ring R, we use the symbol $Nil(R)$ to denote the set of nilpotent elements in R. The prime radical, the Levitzki radical, the upper nil radical and the Jacobson radical of a ring R are denoted by $Nil_*(R)$, $Rad - L(R)$, $Nil^*(R)$ and $J(R)$ respectively. The symbol $T_n(R)$ stands for the ring of upper triangular matrices over a ring R, and $M_n(R)$ stands for the $n \times n$ matrix ring over R.

Recall that a ring is reduced if it has no nonzero nilpotent elements, a ring R is 2-primal if $Nil(R) = Nil_*(R)$, and R is an NI-ring if $Nil(R) = Nil^*(R)$. A ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is known that reduced $\Rightarrow$semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ NI, and no reversal holds (cf. [1]). Historically, some of the earliest results known to us about semicommutative rings is due to Shin [2]. Since then there are many papers to investigate semicommutative rings and their generalizations (see [1, 3, 4, 5, 6]). Liang et al. in [5] define a ring R to be weakly semicommutative if $ab = 0$ implies $aRb \subseteq Nil(R)$ for $a, b \in R$. This notion is a proper generalization of semicommutative rings. In this note we define a ring R to be nil-semicommutative if for any $a, b \in R$, $ab \in Nil(R)$ implies $aRb \subseteq Nil(R)$, which is another proper generalization of semicommutative rings. It is proved that if R is a ring and I an ideal of R such that I and R/I are both nil-semicommutative then R is nil-semicommutative. It follows that a ring R is nil-semicommutative if and only if so is $T_n(R)$. Moreover it is proved that if R is a semicommutative ring satisfying the α-condition for an endomorphism α of R then $R[x; \alpha]$ is nil-semicommutative, improving one of the main results of Liang et al. in [5]. However the polynomial ring $R[x]$ over a nil-semicommutative ring R need not be nil-semicommutative. Whether a nil-semicommutative ring is an NI-ring is an open question which has a close connection with the famous Koethe's conjecture.

2 Examples and Extensions

Definition 2.1. *A ring R is called nil-semicommutative if $a, b \in R$ satisfy $ab \in Nil(R)$, then $arb \subseteq Nil(R)$ for any $r \in R$. And an ideal I of a ring R is called nil-semicommutative if I satisfies the above condition as R.*

Obviously a ring R is nil-semicommutative if and only if for any $n \geq 2$ and $a_1, a_2, ..., a_n \in R$, whenever $a_1 a_2 \cdots a_n \in Nil(R)$ then $a_1 r_1 a_2 r_2 \cdots a_{n-1} r_{n-1} a_n \in Nil(R)$ where $r_1, r_2, ..., r_{n-1} \in R$. In particular, if $a \in Nil(R)$ and $r \in R$ then $ar, ra \in Nil(R)$. This means that aR and Ra are nil one-sided ideals for any $a \in Nil(R)$ in such a ring R. A nil semicommutative ring is weakly semicommutative by Definition 2.1, but the converse is not true as the following example shows.

Example 2.2. ([7, Example 1]) *There exists a weakly semicommutative ring R which is not nil-semicommutative, and R has a homomorphic image S which is Armendariz but not nil-semicommutative.*

Proof. Let F be a field, $F < X, Y >$ the free algebra on X, Y over F and I denote the ideal $(X^2)^2$ of $F < X, Y >$, where (X^2) is the ideal of $F < X, Y >$ generated by (X^2). Let $R = F < X, Y > /I$ and $x = X + I$. Then by the computation in [7, Example 1]), $Nil(R) = xRx + Rx^2R + Fx$, $Nil_*(R) = Rx^2R$ and $Nil_*(R)$ contains all nilpotent elements of index two. We claim that R is weakly semicommutative. Assume $a, b \in R$ with $ab = 0$. Then $(ba)^2 = 0$ and so $ba \in Nil_*(R)$. This means $bar \in Nil_*(R)$ and so $arb \in Nil(R)$ for any $r \in R$. Hence R is a weakly semicommutative ring. Since $yxx \in Nil(R)$ but $yxyx \notin Nil(R)$, R is not nil-semicommutative. To prove the second statement, let $S = F < X, Y > /(X^2)$, then $R/Nil^*(R) \cong S$ and $Nil(S) = xSx + Fx$ by the proof [7, Example 1]) where $x = X + (X^2)$. If a, b are two nonzero elements of S such that $ab = 0$, then $a \in Sx$ and $b \in xS$ by the proof of [7, Example 1]). Now we have $yxx = 0$ in S, but $yxyx$ is not nilpotent by the expression of $Nil(S)$. This implies that the homomorphism image S of R is not a nil-semicommutative ring. □

Recall that a ring R is Armendariz if $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for all i and j. Note that the ring S in Example 2.2 is an Armendariz ring by [8, Example 4.8]. This shows that an Armendariz ring need not be nil-semicommutative.

Theorem 2.3. *Let R be a ring and I be an ideal of R. If I and R/I are both nil-semicommutative, then R is a nil-semicommutative ring.*

Proof. For $a, b \in R$ with $ab \in Nil(R)$ and any $r \in R$, then there exists a positive integer k such that $(ab)^k = 0$. Write $\overline{R} = R/I$, $\bar{a} = a + I$, $\bar{b} = b + I$ and $\bar{r} = r + I$, then $\bar{a}\bar{b} \in Nil(\overline{R})$. This implies that $\bar{a}\bar{r}\bar{b} \in Nil(\overline{R})$ since $\overline{R}$ is nil-semicommutative. There exists a positive integer n such that $(arb)^n \in I$. Clearly, $rb(arb)^n ar, (arb)^n a, b(ab)^s (arb)^n$ are all in I for $s = 1, 2, ..., k-1$. Since $(ab)^k = 0$, $[(arb)^n a][b(ab)^{k-1}(arb)^n] = (arb)^n (ab)^k (arb)^n = 0$ in I. Inserting $rb(arb)^n ar$ between the two square brackets, then $(arb)^n a(rb(arb)^n ar)b(ab)^{k-1}(arb)^n$ is in $Nil(I)$ since I is nil-semicommutative. It follows that $(arb)^{2n+2}(ab)^{k-1}(arb)^n$ is in $Nil(I)$. Note that

$$(arb)^{2n+2}(ab)^{k-1}(arb)^n = \left[(arb)^{2n+2}a\right]\left[b(ab)^{k-2}(arb)^n\right] \in Nil(I).$$

Similar to the above argument, we have

$$\left[(arb)^{2n+2}a\right] rb(arb)^n ar \left[b(ab)^{k-2}(arb)^n\right] \in Nil(I).$$

This means $(arb)^{3n+4}(ab)^{k-2}(arb)^n \in Nil(I)$. Continuing this process, finally we have $(arb)^{(k+2)n+2k} \in Nil(I) \subseteq Nil(R)$. The proof is completed. □

Corollary 2.4. *Let R be a ring and I be an ideal contained in $Nil(R)$. Then R is nil-semicommutative if and only if R/I is nil-semicommutative.*

Proposition 2.5. *A ring R is nil-semicommutative if and only if $T_n(R)$ is nil-semicommutative if and only if $R[x]/(x^n)$ is nil-semicommutative for any $n \geq 2$ where (x^n) is the ideal generated by x^n in $R[x]$.*

Proof. Clearly a direct sum of finite many nil- semicommutative rings is nil-semicommutative. Since $T_n(R)/Nil^*(T_n(R)) \cong \bigoplus_{i=1}^{n} R/Nil^*(R)$, we get the first conclusion by Corollary 2.4. Also since $R[x]/(x^n)$ is isomorphic with the subring

$V_n(R)$ of $T_n(R)$ where $V_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ & a_0 & a_1 & \dots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & a_0 & a_1 \\ & & & & a_0 \end{pmatrix} \mid a_i \in R \right\}$

(see [9, Theorem 3.9] for the details). The proof is completed. □

Theorem 2.6. *There exists a nil-semicommutative ring R over which the polynomial ring $R[x]$ is not nil-semicommutative.*

Proof. First note for a nil-semicommutative ring R that $Nil(R) \subseteq J(R)$ holds. In fact for any $a \in Nil(R)$, aR is a nil right ideal of R and so $a \in aR \subseteq J(R)$. It is well known that Koethe's conjecture (whether every one-sided nil ideal of any associative ring is contained in a two-sided nil ideal of the ring) has a positive solution if and only if for every nil algebra S over any field, the polynomial algebra $S[x]$ is Jacobson radical (cf. [10]). Now suppose that for any nil-semicommutative ring R, the polynomial ring $R[x]$ is nil-semicommutative. Then for any nil algebra over a field K, the ring $R = K + S$ (as a sum of K-algebra) is nil-semicommutative by Corollary 2.4, since $Nil^*(R) = S$ and $R/S \cong K$. Since $R[x]$ is nil-semicommutative by the above assumption, $S[x]$ is nil-semicommutative as a subring (without 1) of $R[x]$. Hence we have $Nil(R[x]) \subseteq J(R[x])$ by the beginning argument. It is easy to see that $Nil(R) = Nil(S)$. We claim that $J(R[x]) = J(S[x])$. In fact, since $S[x]$ is an ideal of $R[x]$, $J(S[x]) = S[x] \bigcap J(R[x]) \subseteq J(R[x])$. On the other hand, $J(R[x]) = I[x]$ for some nil ideal I of R by [11, Theorem 1]. This implies $J(R[x]) \subseteq S[x]$. Hence $J(R[x])$ is a quasi-regular ideal of $S[x]$ and so $J(R[x]) \subseteq J(S[x])$. Thus we have $Nil(S[x]) \subseteq Nil(R[x]) \subseteq J(R[x]) = J(S[x])$. Now for any $f(x) \in S[x]$, write $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Because a_ix^i is contained in $Nil(S[x])$ for all i, it is in $J(S[x])$, and so $f(x) \subseteq J(S[x])$. Hence $S[x]$ is Jacobson radical. This means that Koethe's conjecture has a positive solution. For any nil-semicommutative ring R and any $a \in Nil(R)$, then the nil right ideal $aR \subseteq Nil^*(R)$ and so $a \in Nil^*(R)$. This implies that any nil-semicommutative ring R is an NI-ring. It yields that the class of nil-semicommutative rings coincides with that of NI-rings. But it is known that there is an NI-ring over which the polynomial ring is not an NI ring (see [4] for the details). This leads a contradiction. □

Proposition 2.7. *Let R be a nil-semicommutative ring. If R is an Armendariz ring, then $R[x]$ is a nil-semicommutative ring.*

Proof. Since R is an Armendariz ring, $Nil(R)$ is a subring (without 1) of R by [8, Corollary 3.3], and $R[x]$ is also Armendariz by [12, Theorem 1]. Hence $Nil(R)[x] = Nil(R[x])$ by [8, Proposition 2.7 and Theorem 5.3]. Assume $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) \in Nil(R[x])$, then $a_ib_j \in Nil(R)$ for all i and j by [12, Proposition 1] since R is Armendariz. Now for any $h(x) = \sum_{k=0}^{s} c_kx^k \in R[x]$, the coefficient of $f(x)h(x)g(x)$ has the form

$\sum a_i c_k b_j$. Since R is nil-semicommutative, $a_i b_j \in Nil(R)$ implies $a_i c_k b_j \in Nil(R)$. This means $\sum a_i c_k b_j \in Nil(R)$. It follows that $f(x)h(x)g(x) \in Nil(R)[x] = Nil(R[x])$. □

Proposition 2.8. *A nil-semicommutative ring R is directly finite.*

Proof. If R is not directly finite, then it must contain matrix units e_{ij} satisfying $0 \neq e_{ij}$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ $(i, j, k, l = 1, 2, ...)$ (see [13, p. 328]). Since $e_{11}e_{21} = 0$, but $e_{11}e_{12}e_{21} = e_{11}$ is a nonzero idempotent, this contradicts the assumption. The proof is completed. □

The proof of Proposition 2.8 implies that for any ring R, the matrix ring $M_n(R)$ is not nil-semicommutative whenever $n \geq 2$.

3 Semicommutative Rings Satisfying α-condition

Recall [5] that a ring R is said to satisfy the α-condition for an endomorphism α of R in case $ab = 0 \Leftrightarrow a\alpha(b) = 0$ where $a, b \in R$. Clearly R satisfies the α-condition if and only if for $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha^n(b) = 0$ where n is any nonnegative integer. More generally, we have the following fact.

Lemma 3.1. *Let R be a ring which satisfies the α-condition for an endomorphism α of R. Then $a_1a_2\cdots a_n = 0 \Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) = 0$ where $k_1, k_2, ..., k_n$ are arbitrary nonnegative integers and $a_1, a_2, ..., a_n$ are arbitrary elements in R.*

Proof. First observe that for a nonzero ring R the α-condition implies that α is a monomorphism and so α^k is a monomorphism for any nonnegative integer k. Using this and the fact that $a\alpha^n(b) = 0 \Leftrightarrow ab = 0$ where $a, b \in R$ and $n \geq 0$, we have the following equivalence.

$$\begin{aligned}
a_1a_2\cdots a_n = 0 &\Leftrightarrow \alpha^{k_1}(a_1a_2\cdots a_n) = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_1}(a_2\cdots a_n) = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)a_2\cdots a_n = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2\cdots a_n) = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\alpha^{k_2}(a_3\cdots a_k) = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)(a_3\cdots a_k) = 0 \\
&\Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\alpha^{k_3}(a_3\cdots a_k) = 0
\end{aligned}$$

Continuing this process, $a_1a_2\cdots a_n = 0 \Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) = 0$ holds eventually. □

Corollary 3.2. *Let R be a ring satisfying the α-condition for an endomorphism α of R. Then $a_1a_2\cdots a_n \in Nil(R) \Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) \in Nil(R)$ where $k_1, k_2, ..., k_n$ are arbitrary nonnegative integers and $a_1, a_2, ..., a_n$ are arbitrary elements in R. In particular, $ab \in Nil(R)$ if and only if $a\alpha(b) \in Nil(R)$ for any $a, b \in R$.*

According to [14], a ring R is called α-skew Armendariz for an endomorphism α of R if for any $f(x)=\sum_{i=0}^{m} a_i x^i, g(x)=\sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ whenever $f(x)g(x)=0$ then $a_i\alpha^i(b_j)=0$ for all i and j.

Lemma 3.3. *Let R be a ring and α be an endomorphism of R. Then R is α-skew Armendariz if and only if for any $k\geq 2$ and $f_s(x)=a_{s0}+a_{s1}x+\cdots+a_{sn}x^n=\sum_{i_s=0}^{n} a_{si_s}x^{i_s}\in R[x;\alpha](s=1,2,...,k)$ whenever $f_1(x)f_2(x)\cdots f_k(x)=0$, then $a_{1i_1}\alpha^{i_1}(a_{2i_2})\alpha^{i_1+i_2}(a_{3i_3})\cdots\alpha^{i_1+\cdots+i_{k-1}}(a_{ki_k})=0$ where $a_{1i_1}, a_{2i_2}, ..., a_{ki_k}$ are arbitrary coefficients of $f_1(x), f_2(x), ..., f_k(x)$, respectively.*

Proof. Clearly α induces an endomorphism of $R[x]$ via $a_0+a_1x+\cdots+a_nx^n\longmapsto \alpha(a_0)+\alpha(a_1)x+\cdots+\alpha(a_n)x^n$, still denoted by α for simplification. Now we prove the only if part of Lemma 3.3 by induction on k. It is true in the case of $k=2$ by the definition of an α-skew Armendariz ring. Assume that the conclusion is true for $k-1$. In the case of k, assume $f_1(x)f_2(x)\cdots f_k(x)=0$. Then $a_{1i_1}\alpha^{i_1}(f_2(x)f_3(x)\cdots f_k(x))=a_{1i_1}\alpha^{i_1}(f_2(x))\alpha^{i_1}(f_3(x))\cdots\alpha^{i_1}(f_k(x))=0$ for any coefficient a_{1i_1} of $f_1(x)$. Write $g_s(x)=\alpha^{i_1}(a_{s0})+\alpha^{i_1}(a_{s1})x+\cdots+\alpha^{i_1}(a_{sn})x^n=\sum_{i_s=0}^{n} b_{si_s}x^{i_s}$ where $b_{si_s}=\alpha^{i_1}(a_{si_s})$ $(s=2,...,k)$. It follows that $(a_{1i_1}g_2(x))g_3(x)\cdots g_k(x)=0$. By the inductive assumption, we have the equality $a_{1i_1}b_{2i_2}\alpha^{i_2}(b_{3i_3})\cdots\alpha^{i_2+\cdots+i_{k-1}}(b_{ki_k})=0$ where $b_{2i_2},\cdots,b_{ki_k}$ are arbitrary coefficients of $g_2(x),...,g_k(x)$, respectively. This means that $a_{1i_1}\alpha^{i_1}(a_{2i_2})\alpha^{i_1+i_2}(a_{3i_3})\cdots\alpha^{i_1+i_2+\cdots+i_{k-1}}(a_{ki_k})=0$. The if part of Lemma 3.3 is obvious. □

Following [15], an endomorphism α of a ring R is called rigid if $a\alpha(a)=0$ implies $a=0$ where $a\in R$, and in [14] a ring R is called α-rigid if there exists a rigid endomorphism α of R. It is known by [14, Corollary 4] that if R is α-rigid then R is α-skew Armendariz.

Lemma 3.4. *Let R be a semicommutative ring and α be an endomorphism of R. If R satisfies the α-condition, then $Nil(R[x;\alpha])=Nil(R)[x;\alpha]$.*

Proof. Since R is semicommutative, it is 2-primal and so $Nil(R)=Nil_*(R)$. The endomorphism α of R induces an endomorphism of $R/Nil(R)$, denoted by $\bar{\alpha}$, via $a+Nil(R)\mapsto\alpha(a)+Nil(R)$ where $a\in R$. Clearly $R/Nil(R)$ is reduced. Because R satisfies the α-condition, it is easy to check that $R/Nil(R)$ is $\bar{\alpha}$-rigid, and so $R/Nil(R)[x;\bar{\alpha}]$ is $\bar{\alpha}$-skew Armendariz. Also it is a routine task to check that there is a ring homomorphism between $R[x;\alpha]$ and $R/Nil(R)[x;\bar{\alpha}]$ via $a_0+a_1x+\cdots+a_nx^n\mapsto a_0+Nil(R)+(a_1+Nil(R))x+\cdots+(a_n+Nil(R))x^n$, and that $R[x;\alpha]/Nil(R)[x;\alpha]\cong R/Nil(R)[x;\bar{\alpha}]$. Now for any $f(x)=a_0+a_1x+\cdots+a_nx^n\in Nil(R[x;\alpha])$, then there exists a positive integer k such that $f(x)^k=0$ and so $\bar{f}(x)=\bar{a_0}+\bar{a_1}x+\cdots+\bar{a_n}x^n$ satisfies $\bar{f}(x)^k=\bar{0}$ in $R/Nil(R)[x;\bar{\alpha}]$. Hence $\bar{a}_i\bar{\alpha}^i(\bar{a}_i)\cdots\bar{\alpha}^{(k-1)i}(\bar{a}_i)=\bar{0}$ for all $i=0,1,...,n$ by Lemma 3.3. This means $\bar{a}_i^k=\bar{0}$ by Lemma 3.1. Hence $a_i\in Nil(R)$ for each i. On the other hand, $Nil(R)[x;\alpha]\subseteq Nil(R[x;\alpha])$ by [5, Lemma 3.4]. The proof is completed. □

Theorem 3.5. *Let R be a semicommutative ring. If R satisfies α-condition for an endomorphism α of R, then $R[x;\alpha]$ is a nil-semicommutative ring.*

Proof. Assume that $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) \in Nil(R[x;\alpha])$. Then there exists a positive integer k such that $(f(x)g(x))^k = 0$, and so $(\bar{f}(x)\bar{g}(x))^k = \bar{0}$ in $R/Nil(R)[x;\bar{\alpha}]$. By the proof of Lemma 3.4, $R/Nil(R)[x;\bar{\alpha}]$ is $\bar{\alpha}$-skew Armendariz. So $(\bar{f}(x)\bar{g}(x))^k = \bar{0}$ implies that $\bar{a}_i\bar{\alpha}^i(\bar{b}_j)\bar{\alpha}^{i+j}(\bar{a}_i)\bar{\alpha}^{i+j+i}(\bar{b}_j) \cdots \bar{\alpha}^{(k-1)i+(k-1)j}(\bar{a}_i)\bar{\alpha}^{ki+(k-1)j}(\bar{b}_j) = \bar{0}$ for all $i = 0,1,...,n$ and $j = 0,1,...,m$ by Lemma 3.3. Hence $(\bar{a}_i\bar{b}_j)^k = \bar{0}$ by Lemma 3.1. This means that $a_ib_j \in Nil(R)$. Now for any $h(x) = c_0 + c_1x + \cdots + c_px^p \in R[x;\alpha]$, we have $a_ic_lb_j \in Nil(R)$ where $l = 0,1,...,p$. Note that each coefficient of $f(x)h(x)g(x)$ has the form $\sum a_i\alpha^i(c_l)\alpha^{i+l}(b_j)$, which is in $Nil(R)$ by Corollary 3.2. Hence $f(x)h(x)g(x) \in Nil(R)[x;\alpha] = Nil(R[x;\alpha])$. □

Corollary 3.6. ([5, Theorem 3.1]) *Let R be a semicommutative ring and α be an endomorphism of R. If R satisfies the α-condition, then $R[x;\alpha]$ is a weakly semicommutative ring.*

Proposition 3.7. *Let R be a semicommutative ring satisfying the α-condition for an endomorphism α. If R is α-skew Armendariz, then $R[x;\alpha]$ is semicommutative.*

Proof. Suppose that $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$ in $R[x;\alpha]$. Then $a_i\alpha^i(b_j) = 0$ for all i and j since R is α-skew Armendariz. This means $a_ib_j = 0$ for all i and j by Lemma 3.1. Now for any $h(x) = c_0 + c_1x + \cdots + c_px^p \in R[x;\alpha]$, we have $a_ic_lb_j = 0$ for all $l = 0,1,...,p$ since R is semicommutative. It follows that $a_i\alpha^i(c_l)\alpha^{i+l}(b_j) = 0$ by Lemma 3.1. Hence $f(x)h(x)g(x) = 0$ in $R[x;\alpha]$ and so $R[x;\alpha]$ is semicommutative. □

4 Final Remarks

Nil-semicommutative rings have many common properties with NI-rings (cf. [4]). However it is difficult to answer the question whether a nil-semicommutative ring is an NI-ring. A negative answer will lead to a negative solution to Koethe's conjecture, since we can show that if Koethe's conjecture has a positive solution then a nil-semicommutative ring R is an NI-ring. In fact for any $a \in Nil(R)$, then Ra is a nil left ideal of R and so $Ra \subseteq Nil^*(R)$, similarly $aR \subseteq Nil^*(R)$. Hence for any $a, b \in Nil(R)$, we have $a, b \in Nil^*(R)$, and so $a - b \in Nil^*(R) \subseteq Nil(R)$. This gives $Nil(R) = Nil^*(R)$. In particular, the question has a positive answer if a ring R has bounded index of nilpotency. In this case, $Ra, aR \subseteq L - rad(R)$ for any $a \in Nil(R)$ (cf. [16, p. 111]). Hence we have $Nil(R) = L - rad(R)$.

Proposition 4.1. *Let R be a ring. If $R[x]$ is nil-semicommutative, then R is an NI-ring.*

Proof. By [11, Theorem 1] for any ring R, $J(R[x]) = I[x]$ holds where I is a nil ideal of R. Hence we have $J(R[x]) \subseteq Nil(R)[x]$. Since $R[x]$ is nil-semicommutative, we have $Nil(R[x]) \subseteq J(R[x])$. Now for any $a, b \in Nil(R)$, then $a, b \in Nil(R[x]) \subseteq J(R[x])$. Hence $a - b \in J(R[x]) \subseteq Nil(R)[x]$. This means $a - b \in Nil(R)$. Since R is nil-semicommutative as a subring of $R[x]$, $ab \in Nil(R)$ and so $Nil(R)$ is a subring (without 1) of R. It follows that $Nil(R)$ is an ideal of R and $Nil(R) = Nil^*(R)$. □

Recall that a ring R is nil-Armendariz if $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) \in Nil(R)[x]$ then $a_i b_j \in Nil(R)$ for all i and j.

Proposition 4.2. *Let R be a nil-semicommutative ring. If $Nil(R)[x] = Nil(R[x])$, then $R[x]$ is an NI-ring.*

Proof. Let $a, b \in Nil(R)$. Then $a - bx \in Nil(R)[x]$ ans so $a - b \in Nil(R)$. This means R is an NI-ring by the proof of Proposition 4.1. Since R is an NI-ring, it is nil-Armendariz. So $Nil(R)[x] = Nil(R[x])$ implies $Nil(R[x])$ is a subring (without 1) of $R[x]$ by [8, Proposition 2.3]. Now for any $f(x) = \sum_{i=0}^{m} a_i x^i \in Nil(R[x])$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, then $a_i b_j \in Nil(R)$. Hence $f(x)g(x) \in Nil(R[x])$. Similarly, $g(x)f(x) \in Nil(R[x])$. Hence $Nil(R[x])$ is an ideal of $R[x]$. □

We conclude this note by posing the following question.

Question 4.3. Is there a nil-semicommutative ring R which is not a NI-ring?

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