Thai Journal of Mathematics
Volume 9 (2011) Number 1 : 29–37

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

On the Extended Null Scrolls in Minkowski 3-space

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Abstract : In this paper, we defined an extended null scroll in Minkowski 3-space $\mathbb{R}^3_1$ which is obtained by a null line moving with (proper) null frame along a null curve. We proved the well-known theorem due to Bonnet in the 3-dimensional Euclidean space for an extended null scroll. We calculated the geodesic and normal curvature of a curve on the extended null scroll.

Keywords : Minkowski 3-space; Null Scrolls; Cartan Frame.
2010 Mathematics Subject Classification : 53A35; 53B30.

1 Introduction

The Classification of surfaces and curves in Euclidean or Non-Euclidean spaces has been of particular interest for geometers. Many interesting results in Non-Euclidean spaces have neen obtained by many mathematicians. This subject have been studied by many researcher [1, 2, 3, 4, 5].

In this study, we have done a study about null scrolls in Minkowski 3-sapce. Let $\mathbb{R}^3_1$ be a Minkowski 3-space with the natural Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

in terms of natural coordinates. The vector product operation of $\mathbb{R}^3_1$ is defined by

$$X \Lambda Y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

for $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$.

The norm of $\overrightarrow{v} \in \mathbb{R}^3_1$ is denoted by $\|\overrightarrow{v}\|$ and defined as

$$\|\overrightarrow{v}\| = \sqrt{|\langle \overrightarrow{v}, \overrightarrow{v} \rangle|}.$$

Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, recall that a vector $\overrightarrow{v} \in \mathbb{R}^3_1$ can have one of three causal characters: it can be spacelike if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle > 0$ or $\overrightarrow{v} = 0$, timelike if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle < 0$ and null (lightlike) if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle = 0$ for $\overrightarrow{v} \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}^3_1$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(t)$ are respectively spacelike, timelike or null (lightlike) [6].

Definition 1.1. ([7]) *A surface in a Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentz metric, i.e., the normal on the surface is a spacelike vector.*

Lemma 1.2. ([8]) *In the Minkowski 3-space $\mathbb{R}^3_1$, the following properties are satisfied:*

(i) Two timelike vectors are never orthogonal.

(ii) Two null vectors are orthogonal if and only if they are linearly dependent.

(iii) A timelike vector is never orthogonal to a null (lightlike) vector.

A basis $F = \left\{ \overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z} \right\}$ of $\mathbb{R}^3_1$ is called a (proper) null frame if it satisfies the following conditions:

$$\left\langle \overrightarrow{X}, \overrightarrow{X} \right\rangle = \left\langle \overrightarrow{Y}, \overrightarrow{Y} \right\rangle = 0, \quad \left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle = -1,$$

$$\left\langle \overrightarrow{X}, \overrightarrow{Z} \right\rangle = \left\langle \overrightarrow{Y}, \overrightarrow{Z} \right\rangle = 0, \quad \left\langle \overrightarrow{Z}, \overrightarrow{Z} \right\rangle = 1,$$

$\det(X, Y, Z) = 1$, [9].

Let $\alpha = \alpha(t)$ be a null curve in $\mathbb{R}^3_1$, namely, a smooth curve whose tangent vectors $\alpha'(t)$, $\forall t \in I$ are null. For a given smooth positive function $k_0 = k_0(t)$ let us put $X = X(t) = k_0^{-1}\alpha'$. Then X is a null vector field along α. Moreover, there exists a null vector field $Y = Y(t)$ along α satisfying $\langle X, Y \rangle = -1$. Here if we put $Z = X \Lambda Y$ then we can obtain a (proper) null frame field $F = \{X, Y, Z\}$ along α. In this case the pair (α, F) is said to be a (proper) framed null curve. A framed null curve (α, F) satisfies the following, so called the Frenet equations:

$$\begin{aligned} X'(t) &= k_1(t)X(t) + k_2(t)Z(t), \\ Y'(t) &= -k_1(t)Y(t) + k_3(t)Z(t), \\ Z'(t) &= k_3(t)X(t) + k_2(t)Y(t), \end{aligned} \tag{1.1}$$

where $k_i = k_i(t)$, $i = 1, 2, 3$ are smooth functions defined by

$$k_1 = -\langle X', Y\rangle, \quad k_2 = \langle X', Z\rangle, \quad k_3 = \langle Y', Z\rangle.$$

The function k_i is called an $i - th$ curvature of the framed null curve. It follows from the fundamental theorem of ordinary differential equations that a framed null curve $(\alpha, F) = (\alpha(t), F(t))$ is uniquely determined by the functions $k_0(> 0)$, k_1, k_2, k_3 and the initial condition. The functions k_2 and k_3 are called the curvature and torsion of α, respectively.

A framed null curve (α, F) with $k_0 = 1$ and $k_1 = 0$ is called a Cartan framed null curve and the frame field F is called a Cartan frame.

We call the vector fields X, Y, Z a tangent vector field, a binormal vector field and a (principal) normal vector field of α, respectively.

Note that α is called null geodesic if $k_2 = 0$ [9, 10].

Lemma 1.3. ([11]) *Assume that $\alpha(t)$ is a null curve in $\mathbb{R}^3_1$ and $\{X, Y, Z\}$ be its (proper) null frame field. Then*

$$X\Lambda Y = Z, \quad Y\Lambda Z = -Y, \quad X\Lambda Z = X.$$

2 Null Scrolls in $\mathbb{R}^3_1$

Let $(\alpha, F) = (\alpha(t), F(t))$ be a null curve with frame $F = \{X, Y, Z\}$. A ruled surface is a surface swept out by a straight line Y moving along a curve α. The various positions of the generating line Y are called the rulings of the surface. Such a surface, thus has a parametrization in ruled form as follows:

$$\begin{gathered}\Psi : U \longrightarrow \mathbb{R}^3_1, \\ \Psi(t, v) = \alpha(t) + vY(t); \quad \forall (t, v) \in U\end{gathered}$$

is called a null scroll and denoted by M. We call α to be the base curve and Y to be the director curve. If the tangent plane is constant along each ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. One can see that M is a timelike surface. Furthermore, for a Cartan framed null curve α with Cartan frame $F = \{X, Y, Z\}$ the ruled surface is called a B-scroll [11].

Now consider a ruled surface in $\mathbb{R}^3_1$ generated by a null generator $\overrightarrow{L}(t)$ moving with (proper) null frame of a null curve $\alpha = \overrightarrow{\alpha}(t)$, i.e.,

$$\overrightarrow{L}(t) = \ell_1(t)X(t) + \ell_2(t)Y(t) + \ell_3(t)Z(t), \tag{2.1}$$

where the components $\ell_i = \ell_i(t)$, $\ell_2 \neq 0$ $(i = 1, 2, 3)$ are scalar functions of the parameter of the null curve $\alpha = \overrightarrow{\alpha}(t)$. Thus if $\overrightarrow{L}$ moves with (proper) null frame, the constructed ruled surface is given by the following parametrization

$$\Psi(t, v) = \overrightarrow{\alpha}(t) + v\overrightarrow{L}(t), \quad (t, v) \in U \subset \mathbb{R}^2,$$

$$< \overrightarrow{L}(t), \overrightarrow{L}(t) >= \ell_3^2(t) - 2\ell_1(t)\ell_2(t) = 0. \tag{2.2}$$

This ruled surface is called an extended null scroll.

From (2.1) and using (1.1), we obtain

$$\overrightarrow{L'}(t) = (\ell_1' + \ell_1 k_1 + \ell_3 k_3)\overrightarrow{X} + (\ell_2' - \ell_2 k_1 + \ell_3 k_2)\overrightarrow{Y} + (\ell_3' + \ell_1 k_2 + \ell_2 k_3)\overrightarrow{Z}. \tag{2.3}$$

It obvious that the vector $\overrightarrow{L'}(t)$ is a spacelike vector or null vector and in the second case it is linearly dependent with the generator.

The assumption $\overrightarrow{L'}(t) \neq 0$, is usually expressed by saying that the ruled surface M is a noncylindrical.

From (2.2), one can obtain the first fundamental quantities of the extension

$$g_{11} = -2k_0 v(\ell_2' - \ell_2 k_1 + \ell_3 k_2) + v^2 \left\| L' \right\|^2, \quad g_{12} = -k_0 \ell_2, \quad g_{22} = 0. \tag{2.4}$$

Thus the induced metric on the extended null scroll is a Lorentz metric. Therefore the extended null scroll is a timelike ruled surface.

The unit normal vector field $\overrightarrow{n} = \overrightarrow{n}(t,v)$ on the extended null scroll in $\mathbb{R}_1^3$ is

$$\overrightarrow{n} = \frac{\alpha'(t) \wedge \overrightarrow{L}(t) + v\overrightarrow{L'}(t) \wedge \overrightarrow{L}(t)}{\left\| \alpha'(t) \wedge \overrightarrow{L}(t) + v\overrightarrow{L'}(t) \wedge \overrightarrow{L}(t) \right\|}. \tag{2.5}$$

Thus, from (2.5) the unit normal vector to the surface M at the point (t,0) is

$$\overrightarrow{n}(t,0) = \frac{\ell_3 X + \ell_2 Z}{|\ell_2|}, \quad \ell_2 \neq 0. \tag{2.6}$$

Definition 2.1. ([1]) *If there exists a common perpendicular to two preceding rulings in the skew surface, the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction.*

Using (1.1) and (2.3), it is easy to see that the parametrization of the striction curve on an extended null scroll (2.2) is given by

$$\overrightarrow{\beta}(t) = \overrightarrow{\alpha}(t) + \frac{k_0(\ell_2' - \ell_2 k_1 + \ell_3 k_2)}{\left\| \overrightarrow{L'} \right\|^2} \overrightarrow{L}(t) \tag{2.7}$$

From (2.6), it follows that the base curve of the extended null scroll (2.2) to be a geodesic curve ($\overrightarrow{n}(t,0) = \overrightarrow{Z}$) if $\ell_2 \neq 0$ and $\ell_3 = 0$.

Theorem 2.2. *Let M be an extended null scroll. Then the curve of striction is a timelike curve in an extended null scroll M.*

Proof. If we use the equation (2.6), we can show easily that the tangent vector field of the curve of striction is a timelike vector field. □

Thus we have the Bonnet's theorem for the extended null scroll which is formulated as follows:

Theorem 2.3. *If a timelike curve on an extended null scroll in $\mathbb{R}^3_1$ has two of the following properties, it has the third also*

(i) it is a geodesic ($k_2 = 0$) and $k_1 = 0$,

(ii) it cut the rulings at a constant angle ($\ell_2 = const.$),

(iii) it is a striction curve.

Now we examine the extended null scroll for which the striction curve is the base curve, i.e.,

$$M^s : \Psi^s(t, v) = \overrightarrow{\alpha}(t) + v\overrightarrow{L}(t)$$

$$\ell_3^2 - 2\ell_1\ell_2 = 0, \quad \ell_2' - \ell_2 k_1 + \ell_3 k_2 = 0. \tag{2.8}$$

Corollary 2.4. *Let the curve of striction be the base curve of an extended null scroll M. Then the vector $\overrightarrow{L'}(t)$ is a spacelike vector.*

Proof. From (2.3), using (2.8), it can be seen easily. □

Let us consider an extended null scroll M^s. Then since $\left\langle \overrightarrow{L}(t), \overrightarrow{L'}(t) \right\rangle = 0$ and $\left\langle \overrightarrow{\alpha'}, \overrightarrow{L'}(t) \right\rangle = 0$, we can write

$$\overrightarrow{\alpha'}(t) \wedge \overrightarrow{L}(t) = \lambda \overrightarrow{L'}(t),$$

where

$$\lambda = \lambda(t) = -\frac{\det(\overrightarrow{\alpha'}(t), \overrightarrow{L}(t), \overrightarrow{L'}(t))}{\left\|\overrightarrow{L'}(t)\right\|^2}. \tag{2.9}$$

The function $\lambda = \lambda(t)$ is called the distribution parameter of the extended null scroll M^s. In more explicitly using (1.1), (2.1) and (2.3), we get

$$\lambda(t) = -\frac{k_0\ell_2(\ell_1 k_2 + \ell_2 k_3 + \ell_3')}{\left\|\overrightarrow{L'}(t)\right\|^2}. \tag{2.10}$$

The normal vector field on M^s takes the form

$$\overrightarrow{N} = \lambda\overrightarrow{L'}(t) + v\overrightarrow{L'}(t) \wedge \overrightarrow{L}(t). \tag{2.11}$$

Thus, from (2.10) we have

$$\ell_2^2 = \frac{\lambda^2}{k_0^2}\left\|\overrightarrow{L'}(t)\right\|^2. \tag{2.12}$$

Hence, we have

Corollary 2.5. *The singular points on the extended null scroll* M^s *are the points for which* $\lambda = 0$. *Since* $\left\|\overrightarrow{L'}\right\| \neq 0$, *i.e.,* $\ell_1 k_2 + \ell_2 k_3 + \ell_3' \neq 0$, *the singular points are given* $\ell_2 = 0$.

Let M^s be an extended null scroll for which the striction curve is the base curve in $\mathbb{R}^3_1$. The unit normal vector to the extended null scroll M^s at (t, v) is given from (2.11) and (2.12) by

$$\overrightarrow{n}(t,v) = k_0 \overrightarrow{\ell} + k_0 \frac{v}{\lambda} \overrightarrow{\ell} \wedge \overrightarrow{L},$$

where $\overrightarrow{\ell} = \frac{\overrightarrow{L'}(t)}{\left\|\overrightarrow{L'}(t)\right\|}$.

For a regular patch on M^s ($\lambda \neq 0$), it is easy to see that the normal along the striction curve on M^s is given by

$$\overrightarrow{n}_0(t,0) = k_0 \overrightarrow{\ell}.$$

Since $\overrightarrow{n}_0$ is a unit spacelike vector and $\overrightarrow{n}$ is unit spacelike vector, thus if θ is the angle of rotation from the normal $\overrightarrow{n}_0$ to the normal $\overrightarrow{n}$ we get

$$\sin\theta = \|\overrightarrow{n}_0 \wedge \overrightarrow{n}\| = \left\| k_0 \overrightarrow{\ell}(t) + v\frac{k_0}{\lambda} \overrightarrow{\ell}(t) \wedge \overrightarrow{L}(t) \wedge \overrightarrow{\ell}(t) \right\|.$$

Routine calculation, one can obtain $\theta = 0$. Thus we have without loose of generality following theorem which is similar to Chasles theorem for the extended null scroll in $\mathbb{R}^3_1$.

Theorem 2.6. *For the extended null scroll* M^s *in* $\mathbb{R}^3_1$, *the normal vector* $\overrightarrow{n}$ *at a point of a ruling and the normal vector* $\overrightarrow{n}_0$ *at the striction point of this ruling are parallel.*

3 Extended Null Scrolls with Constant Parameter of Distribution

From (2.10), one can see that an extended null scroll M^s with a constant distribution parameter satisfies the following differential equation:

$$-k_0\ell_2(\ell_1\ell_2 + \ell_2 k_3 + \ell_3' = c\left\langle \overrightarrow{L'(t)}, \overrightarrow{L'(t)} \right\rangle,$$

where c is constant. If we consider (2.8), we get easily,

$$\begin{aligned} \ell_2 &= \int (k_1\ell_2 - k_2\ell_3)dt + c_1 \\ \ell_3 &= -\int \Big(\frac{c\left\langle \overrightarrow{L'(t)}, \overrightarrow{L'(t)} \right\rangle}{k_0\ell_2} + \frac{\ell_3^2}{2} + k_3\ell_2\Big)dt + c_2 \qquad (3.1) \\ \ell_3^2 - 2\ell_1\ell_2 &= 0. \end{aligned}$$

The first equation of (3.1) is an integral equation for the unknown $\ell_1 = \ell_1(t)$. Therefore if $\ell_2 = \ell_2(t)$ and $\ell_3 = \ell_3(t)$ are given we get $\ell_1 = \ell_1(t)$.

Theorem 3.1. *The range of existence of a one parametric extended null scrolls $\{M^s\}$ with constant parameter of distribution comprises within two arbitrary functions of one variable.*

Developable null scrolls are special class of the ruled surfaces which is described $k_1 = k_3 = 0$ [11].

Definition 3.2. ([11]) *Let M^s be an extended null scroll in $\mathbb{R}^3_1$. If there exists a curve which makes constant angle with one of the rulings, this curve is called a pseudo-orthogonal trajectory of M^s.*

Theorem 3.3. *Let M be an extended null scroll in $\mathbb{R}^3_1$. The shortest distance between two ruling is measured only on the curve of striction which is one of the pseudo-orthogonal trajectories.*

Now we calculate the geodesic curvature and normal curvature of an extended null scroll M^s in $\mathbb{R}^3_1$.

Let $\overrightarrow{\gamma} = \overrightarrow{\gamma}(t)$ be a curve on the extended null scroll M^s. Then it can be represented in the form

$$\overrightarrow{\gamma}(t) = \overrightarrow{\alpha}(t) + v(t)\overrightarrow{L}(t).$$

Using (1.1), it is easy to see that the unit tangent vector along the curve $\overrightarrow{\gamma} = \overrightarrow{\gamma}(t)$ is

$$\overrightarrow{\gamma}'_0(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{\xi_1 X + \xi_2 Y + \xi_3 Z}{\sqrt{|\xi_3^2 - 2\xi_1\xi_2|}},$$

where,

$$\begin{aligned}
\xi_1 &= k_0 v'\ell_1 + v\eta_1 \\
\xi_2 &= \eta_2 \\
\xi_3 &= v'\ell_3 + v\eta_3 \\
\eta_1 &= \ell_1' + \ell_1 k_1 + \ell_3 k_3 \\
\eta_2 &= v'\ell_2 \\
\eta_3 &= \ell_3' + \ell_1 k_2 + \ell_2 k_3.
\end{aligned}$$

Therefore, one can see that

$$\begin{aligned}
\overrightarrow{\gamma}''_0 &= \frac{1}{R^{\frac{1}{2}}}[(\xi_1' + \xi_1 k_1 + \xi_3 k_3)X + (\xi_2' - \xi_2 k_1 + \xi_3 k_2)Y + (\xi_3' + \xi_1 k_2 + \xi_2 k_3)Z] \\
&\quad - \frac{R'}{2R^{\frac{3}{2}}}(\xi_1 X + \xi_2 Y + \xi_3 Z),
\end{aligned}$$

where $R = \left|\xi_3^2 - 2\xi_1\xi_2\right|$.

Using (2.5), the unit normal vector field on the extended null scroll M^s along the curve $(v = v(t))$ is

$$\overrightarrow{n}(t, v(t)) = \frac{[k_0\ell_3 + v(\ell_3\eta_1 - \ell_1\eta_3)]X + (v\ell_2\eta_3)Y + (k_0\ell_2 + v\ell_2\eta_1)Z}{\sqrt{(k_0\ell_2 + v\ell_2\eta_1)^2 - 2v\ell_2\eta_3[k_0\ell_3 + v(\ell_3\eta_1 - \ell_1\eta_3)]}}.$$

The geodesic curvature of the curve $\overrightarrow{\gamma} = \overrightarrow{\gamma}(t)$ is given by

$$k_g = \frac{1}{2\left\|\overrightarrow{n}(t, v(t))\right\| R^{\frac{3}{2}}} \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \begin{matrix} 2R(\xi_1' + \xi_1 k_1 \\ +\xi_3 k_3) - R'\xi_1 \end{matrix} & \begin{matrix} 2R(\xi_2' - \xi_2 k_1 \\ +\xi_3 k_2) - R'\xi_2 \end{matrix} & \begin{matrix} 2R(\xi_3' + \xi_1 k_2 \\ +\xi_2 k_3) - R'\xi_3 \end{matrix} \\ \begin{matrix} k_0\ell_3 \\ +v(\ell_3\eta_1 - \ell_1\eta_3) \end{matrix} & v\ell_2\eta_3 & k_0\ell_2 + v\ell_2\eta_1 \end{vmatrix}. \tag{3.2}$$

Then, the normal curvature of a curve $\overrightarrow{\gamma_0} = \overrightarrow{\gamma_0}(t)$ is given by

$$\begin{aligned} k_n \;=\; & \frac{1}{2\left\|\overrightarrow{n}(t, v(t))\right\| R^{\frac{3}{2}}}\{-[2R(\xi_1' + \xi_1 k_1 + \xi_3 k_3) - R'\xi_1](v\ell_2\eta_3) \\ & -[2R(\xi_2' - \xi_2 k_1 + \xi_3 k_2) - R'\xi_2](k_0\ell_3 + v(\ell_3\eta_1 - \ell_1\eta_3)) \\ & +[2R(\xi_3' + \xi_1 k_2 + \xi_2 k_3) - R'\xi_3](k_0\ell_2 + v\ell_2\eta_1)\}. \end{aligned}$$

Corollary 3.4. *From (3.2) and (3.3) the geodesic curvature k_g and the normal curvature k_n of a base curve on the extended null scroll M^s in $\mathbb{R}^3_1$ are not define.*

The results in the study are confirmed by the following example:

Example 3.5. *$\varphi(t, v) = (t + v, \cos t + v\cos t, \sin t + v\sin t)$ is a null scroll where $\alpha(t) = (t, \cos t, \sin t)$ is a null base curve and $\overrightarrow{L}(t) = (1, \cos t, \sin t)$ is a null generator.The striction curve is $\overline{\alpha}(t) = \alpha(t) - \overrightarrow{L}(t)$. The distribution parameter is $\lambda = -1$ and it is nondevelopable null scroll.*

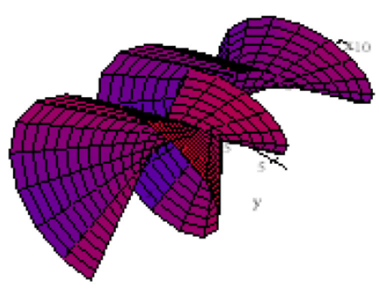

Fig1.

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(Received 15 April 2009)
(Accepted 8 November 2010)