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More Accurate Approximations for the Gamma Function

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Abstract : A series transformation idea inspired by a formula of R.W. Gosper and some asymptotic expansions for the central binomial coefficients leads us to new accurate approximations for the Gamma function.

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1 Introduction

The Gamma function plays an important role in several fields of mathematics such as probability theory or combinatorics. One often has to evaluate the function for large positive values. One way to aim this is to use asymptotic approximations. It is well known that for large values of x the Gamma function has the asymptotic series of the form [1, 2, 3]

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{2\pi x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\cdots\right). \tag{1.1}$$

Equation (1.1) is called Stirling's formula however, Laplace was the first who derived it by his approximation method for special integrals. Another famous result is the Stirling series [1, 2, 3]

$$\log\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \cdots . \quad (1.2)$$

A main advantage of this latter series is that it has only odd powers of the variable. For the past almost three hundred years several authors established fascinating new asymptotic formulas to improve the accuracy of (1.1). For example, Karatsuba [4] showed that a formula of Ramanujan can turn into an asymptotic expansion:

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{\pi}\sqrt[6]{8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\cdots}. \quad (1.3)$$

Mortici [5] proved in his more recent paper the following expansion similar to Karatsuba's:

$$n! = \Gamma(n+1) \sim n^n e^{-n}\sqrt{\pi}\sqrt{2n+\frac{1}{3}+\frac{1}{36n}-\frac{31}{3240n^2}-\frac{139}{77760n^3}+\cdots}. \quad (1.4)$$

For further developments in this topic, see, for example, [5, 6, 7, 8, 9, 10, 11, 12].

We develop some new variants of (1.1) in this paper and show that these formulas are numerically more efficient than much of the early ones in many cases. The first few values of the newly introduced coefficients and sequences can be found in Appendix A.

2 New asymptotic expansions

The motivating examples are the following two asymptotic expansions [3, p. 12]:

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}-\frac{21}{32768n^4}+\cdots\right), \quad (2.1)$$

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\left(1-\frac{1}{64\left(n+\frac{1}{4}\right)^2}+\frac{21}{8192\left(n+\frac{1}{4}\right)^4}-\cdots\right). \quad (2.2)$$

The first one is the standard asymptotic series of the central binomial coefficients. If one expands them into a series in powers of $1/(n+1/4)$, the asymptotic series contains only even powers. This remarkable result suggests that there might have been an asymptotic expansion similar to (1.1) that is, it contains only even powers of the shifted variable. The formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(1+\mathcal{O}\left(\frac{1}{x^2}\right)\right), \quad (2.3)$$

known as Gosper's approximation [13] can be a good starting point. Our aim is to elaborate the asymptotic series part in Gosper's formula. It seems from (2.2) that another series in terms of $1/(x+1/6)$ would be the right choice. It can be shown that a series like that contains even and odd powers. If we insist to have even powers only we are lead to the form

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi\left(x+\frac{1}{6}\right)} \left(g_0 + \frac{g_1}{(x+v_1)^2} + \frac{g_2}{(x+v_2)^4} + \cdots\right), \qquad (2.4)$$

where the sequences $\{g_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 1}$ has to be determined. One of our main result is

Theorem 2.1. *The Gamma function has an asymptotic series expansion of the form*

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi\left(x+\frac{1}{6}\right)} \left(g_0 + \sum_{n\geq 1} \frac{g_n}{(x+v_n)^{2n}}\right), \qquad (2.5)$$

as $x \to \infty$, where the sequences $\{g_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 1}$ can be found from the recurrence

$$g_0 = a_0 = 1, \ \sum_{j=0}^{n} \binom{-1/2}{j} \frac{a_{n-j}}{6^j} = \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{-2j}{n-2j} g_j v_j^{n-2j}, \ \ n \geq 1. \qquad (2.6)$$

Here the a_n coefficients are those appearing in (1.1)*, i.e.,*

$$\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} \sim \sum_{n\geq 0} \frac{a_n}{x^n}. \qquad (2.7)$$

Proof. As $x \to \infty$ we have

$$\frac{\Gamma(x+1)\, e^x}{x^x \sqrt{2\pi\left(x+\frac{1}{6}\right)}} = \frac{\Gamma(x+1)\, e^x}{x^x\sqrt{2\pi x}} \left(1+\frac{1}{6x}\right)^{-1/2} \sim \left(1+\frac{1}{6x}\right)^{-1/2} \sum_{n\geq 0} \frac{a_n}{x^n}.$$

From the binomial formula we find

$$\left(1+\frac{1}{6x}\right)^{-1/2} \sim \sum_{n\geq 0} \binom{-1/2}{n} \frac{1}{6^n x^n},$$

as $x \to \infty$. Thus we obtain the asymptotic expansion

$$\frac{\Gamma(x+1)\, e^x}{x^x \sqrt{2\pi\left(x+\frac{1}{6}\right)}} \sim \sum_{n\geq 0} \left(\sum_{j=0}^{n} \binom{-1/2}{j} \frac{a_{n-j}}{6^j}\right) \frac{1}{x^n}. \qquad (2.8)$$

Suppose the expansion of the form

$$\sum_{n\geq 0} \left(\sum_{j=0}^{n} \binom{-1/2}{j} \frac{a_{n-j}}{6^j}\right) \frac{1}{x^n} \sim g_0 + \sum_{n\geq 1} \frac{g_n}{(x+v_n)^{2n}}. \qquad (2.9)$$

Now we expand the sum on the right-hand side in powers of $1/x$:

$$\sum_{n\geq 1}\frac{g_n}{(x+v_n)^{2n}}=\sum_{n\geq 1}\frac{g_n}{x^{2n}}\left(1+\frac{v_n}{x}\right)^{-2n}=\sum_{n\geq 1}\frac{g_n}{x^{2n}}\sum_{l\geq 0}\binom{-2n}{l}\frac{v_n^l}{x^l}$$
$$=\sum_{n\geq 1}\sum_{l\geq 0}\binom{-2n}{l}\frac{g_n v_n^l}{x^{2n+l}}=\sum_{n\geq 1}\left(\sum_{j=1}^{\lfloor n/2\rfloor}\binom{-2j}{n-2j}g_j v_j^{n-2j}\right)\frac{1}{x^n}.$$

Clearly, from formula (2.9) we see that $g_0=a_0=1$. What remains is to show that the system

$$\sum_{j=0}^{n}\binom{-1/2}{j}\frac{a_{n-j}}{6^j}=\sum_{j=1}^{\lfloor n/2\rfloor}\binom{-2j}{n-2j}g_j v_j^{n-2j} \tag{2.10}$$

has (unique) solutions $\{g_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$. Depending on the parity of n, we have

$$g_k=\sum_{j=0}^{2k}\binom{-1/2}{j}\frac{a_{2k-j}}{6^j}-\sum_{j=1}^{k-1}\binom{-2j}{2k-2j}g_j v_j^{2k-2j} \tag{2.11}$$

and

$$-2kg_kv_k=\sum_{j=0}^{2k+1}\binom{-1/2}{j}\frac{a_{2k-j+1}}{6^j}-\sum_{j=1}^{k-1}\binom{-2j}{2k-2j+1}g_j v_j^{2k-2j+1}, \tag{2.12}$$

when $k\geq 1$. First, expression (2.11) gives $g_1=1/144$, and if we already found the terms $g_1,v_1,\ldots,g_k$, then from formula (2.12) the value of v_k follows. Now, $g_1,v_1,\ldots,g_k,v_k$ determine g_{k+1} by (2.11). Thus, by induction, system (2.10) defines the sequences $\{g_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ uniquely. □

The first few values of the sequences $\{g_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 1}$ can be found in Table 3. We have obtained an expansion in even powers however, the shift sequence $\{v_n\}_{n\geq 1}$ has different terms whereas (2.2) has constant ($=1/4$) shift in all terms. Numerical evaluation of the first few v_n (see Table 4) leads us to the

Conjecture 2.2. $\lim_{n\to\infty} v_n=1/4$.

This conjecture suggests a new asymptotic series to the Gamma function in terms of $1/(x+1/4)$. Our second result is

Theorem 2.3. *The Gamma function has an asymptotic series expansion of the form*

$$\Gamma(x+1)\sim x^x e^{-x}\sqrt{2\pi\left(x+\frac{1}{6}\right)}\sum_{n\geq 0}\frac{G_n}{\left(x+\frac{1}{4}\right)^n}, \tag{2.13}$$

as $x \to \infty$, where the G_n coefficients are given by

$$G_n = \sum_{j=0}^{n} \binom{-1/2}{j} \frac{a_{n-j}}{6^j} - \sum_{j=0}^{n-1} \binom{-j}{n-j} \frac{G_j}{4^{n-j}}. \tag{2.14}$$

The a_n coefficients are from (1.1).

Proof. Similarly to the previous proof we expand our series in terms of $1/x$:

$$\sum_{n\geq 0} \frac{G_n}{\left(x+\frac{1}{4}\right)^n} = \sum_{n\geq 0} \frac{G_n}{x^n} \sum_{l\geq 0} \binom{-n}{l} \frac{1}{4^l} \frac{1}{x^l} = \sum_{n\geq 0} \sum_{l\geq 0} \binom{-n}{l} \frac{G_n}{4^l} \frac{1}{x^{n+l}}$$
$$= \sum_{n\geq 0} \left\{ \sum_{j=0}^{n} \binom{-j}{n-j} \frac{G_j}{4^{n-j}} \right\} \frac{1}{x^n}.$$

According to (2.8) and the uniqueness of asymptotic series the proof of (2.14) is complete. □

3 Numerical comparisons

We will compare in this paragraph the numerical performance of some asymptotic formulas to the Gamma function with our new formulas for large values. We compare the following approximation formulas for $\Gamma(x+1)$.

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \cdots\right) \quad \text{(Stirling)}, \tag{3.1}$$

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right) \quad \text{(Laplace)}, \tag{3.2}$$

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \sqrt[6]{\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - \cdots\right)} \quad \text{(Ramanujan)}, \tag{3.3}$$

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \sqrt{\left(1 + \frac{1}{6x} + \frac{1}{72x^2} - \frac{31}{6480x^3} - \cdots\right)} \quad \text{(Mortici)}, \tag{3.4}$$

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi\left(x+\frac{1}{6}\right)} \left(1 + \frac{1}{144\left(x+\frac{1}{4}\right)^2} - \frac{1}{12960\left(x+\frac{1}{4}\right)^3} - \cdots\right) \text{(New)}. \tag{3.5}$$

Table 1 displays the number of exact decimal digits (edd) of the formulas for some values of x. Exact decimal digits are defined as follows:

$$\text{edd}(x) = -\log_{10}\left|1 - \frac{\text{approximation}(x)}{\Gamma(x+1)}\right|. \tag{3.6}$$

In the table below the (i)-th entry $(i = 1, 2, \ldots)$ in a line starting with "name" is the edd of the given approximation using the series up to the i-th order term. The "$-$" sign indicates that the approximation is smaller and the "$+$" sign (not displayed) indicates that the approximation is larger than the true value. Note that in the case of Stirling's formula the first n terms of the asymptotic series give the $2n$th order approximation.

Conclusion. It is seen that when we use odd order approximations, Laplace's formula is the most accurate. In the case of even orders Ramanujan's approximation is better than Stirling's, Laplace's and the one by Mortici, but our new formula gives better approximations even than that of Ramanujan's.

Formula	x	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Stirling	100		8.6		-13.1		17.2		-21.1
Laplace	100	**-6.5**	8.6	**11.7**	-13.1	**16.2**	17.2	**-20.4**	-21.1
Ramanujan	100	-5.7	-9.2	11.0	-13.3	-15.4	17.3	19.5	-21.1
Mortici	100	-6.2	8.6	11.4	-13.1	-15.9	17.2	-20.0	-21.1
New	100	-6.2	**10.1**	10.9	**-14.9**	-15.2	**19.4**	19.2	**-23.0**
Stirling	1000		11.6		-18.1		24.2		-30.1
Laplace	1000	**-8.5**	11.6	**15.6**	-18.1	**22.2**	24.2	**-28.3**	-30.1
Ramanujan	1000	-7.7	-12.2	15.0	-18.3	-21.4	24.3	27.5	-30.1
Mortici	1000	-8.2	11.6	15.4	-18.1	-21.9	24.2	-28.0	-30.1
New	1000	-8.2	**13.1**	14.9	**-19.7**	-21.2	**26.9**	27.2	**-33.5**
Stirling	10000		14.6		-23.1		31.2		-39.1
Laplace	10000	**-10.5**	14.6	**19.6**	-23.1	**28.2**	31.2	**-36.3**	-39.1
Ramanujan	10000	-9.7	-15.2	19.0	-23.3	-27.4	31.3	35.5	-39.1
Mortici	10000	-10.2	14.6	19.3	-23.1	-27.9	31.2	-36.0	-39.1
New	10000	-10.2	**16.1**	18.9	**-24.7**	-27.2	**33.7**	35.2	**-42.1**

Table 1: The number of exact decimal digits of the asymptotic series for some values of x.

Expression (2.5) is a slightly different than the previous ones, thus we consider it's edds in a separate table. The notations are the same except the fact that this series contains only even order terms.

Formula	x	(2)	(4)	(6)	(8)	(10)
Special	100	10.9	-15.2	19.2	-22.9	26.5
Special	1000	14.9	-21.2	27.2	-32.9	38.5
Special	10000	18.9	-27.2	35.2	-42.9	-50.5

Table 2: The number of exact decimal digits of the special asymptotic series (2.5) for some values of x.

A Tables of coefficients

g_0	1		
g_1	$\frac{1}{144}$	v_1	$\frac{23}{90}$
g_2	$-\frac{3857}{3110400}$	v_2	$\frac{1792627}{7289730}$
g_3	$\frac{20932906335329}{34283052002304000}$	v_3	$\frac{570984637359867601981}{2288928529497568067550}$

Table 3: The first few exact values of the sequences $\{g_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 1}$.

g_0	1.000000000000000		
g_1	0.006944444444444	v_1	0.255555555555555
g_2	−0.001240033436214	v_2	0.245911302613402
g_3	0.000610590513759	v_3	0.249454987345193
g_4	−0.000655407405149	v_4	0.249839892410196
g_5	0.001199164540953	v_5	0.249958497082160

Table 4: The first few numerical values of the sequences $\{g_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 1}$.

G_0	1	G_5	$-\frac{53}{2612736}$	G_{10}	$\frac{360182239526821}{300361133850624000}$
G_1	0	G_6	$\frac{5741173}{9405849600}$	G_{11}	$\frac{104939254406053}{210853515963138048000}$
G_2	$\frac{1}{144}$	G_7	$\frac{37529}{18811699200}$	G_{12}	$-\frac{508096766056991140541}{1518145314934593945600000}$
G_3	$-\frac{1}{12960}$	G_8	$-\frac{710165119}{1083553873920}$	G_{13}	$-\frac{70637580369737593}{1518145314934593945600000}$
G_4	$-\frac{257}{207360}$	G_9	$-\frac{3376971533}{4022693756928000}$	G_{14}	$\frac{28937569055247344296446 7}{218612925350581528166400000}$

Table 5: The first few G_n coefficients.

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