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On the Maximal Inequalities for Partial Sums of Strong Mixing Random Variables with Applications[1]

Guo-dong Xing† **and Shan-chao Yang**‡

†Department of Mathematics, Hunan University of Science and Engineering,
Yongzhou, Hunan 425100, China
e-mail : xingguod@163.com
‡Department of Mathematics, Guangxi Normal University,
Guilin, Guangxi 541004, China

Abstract : Maximal inequalities for partial sums of strong mixing random variables are established. To show the applications of the inequalities obtained, we discuss the strong consistency of Gasser-Müller estimator of fixed design regression estimate and obtain the almost sure convergence rate $n^{-1/2}(\log\log n)^{1/\xi}\log^{3/2} n$ with any $0<\xi<2$, which closes to the optimal achievable convergence rate for independent random variables under an iterated logarithm.

Keywords : Convergence rate; Gasser-Müller estimator; Maximal inequality; Strong mixing.
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1 Introduction and Inequalities

Definition 1.1. *Assume that* $\{X_i : i \in \boldsymbol{Z}\}$ *is a real-valued random variable sequence on a probability space* $(\Omega\ \mathcal{B}\ P)$. *Let* $\Re_m^n$ *denote the* σ*-algebra generated by* $(X_i : m \le i \le n)$. *Set*

$$\alpha(n) = \sup_{m\ge 1} \sup_{A\in\Re_{-\infty}^m, B\in\Re_{m+n}^\infty} \{|P(AB) - P(A)P(B)|\}$$

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The sequence $\{X_i\}$ is said to be strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

Since Rosenblatt [1] introduced the strong mixing coefficient, one has been recognizing its importance more and more. One can refer to Chanda [2], Gorodeskii [3], Withers [4], Liang et al. [5], Liang and Uña-Álvarez [6] and Xing et al. [7] for further understanding.

In this paper, we'll prove the following maximal inequalities for strong mixing sequences.

Theorem 1.2. *Let $1 < r \le 2, \delta > 0$ and $\{X_i, i \ge 1\}$ be a strong mixing sequence of random variables with zero mean. Assume that*

$$\alpha(n) \le Cn^{-\theta} \quad \text{for some} \quad C > 0 \quad \text{and} \quad \theta > r(r+\delta)/(2\delta). \tag{1.1}$$

Then, for any $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon, r, \delta, \theta, C)$ such that

$$E \max_{1\le j\le n} |S_j|^r \le K \left\{ n^{\varepsilon} \sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2 \right)^{r/2} \right\}. \tag{1.2}$$

Remark 1.3.

(1) *Although the mixing coefficient condition of Theorem 1.2 in this paper is stronger than that of Theorem 1.2 of Xing et al. [7], the bound of the result (1.2) is smaller than that of Theorem 1.1 of Xing et al. [7].*

(2) *Since the result (1.2) holds without the assumption that $E|X_i|^{r+\delta} < \infty$, Theorem 1.1 improves Theorem 1 in Huang and Xing [8].*

(3) *Since the up-boundary of the inequality (1.2) contains the information of moment summations, it may be of much efficiency in exploring the asymptotical property of weighted sums.*

Theorem 1.4. *Let $\{X_i, i \ge 1\}$ be a strong mixing sequence of random variables with zero mean and $\alpha(i)$ satisfy*

$$\sum_{i=1}^{\infty} \alpha(i)^{(u-2)/u} < \infty \tag{1.3}$$

for some $u > 2$. Then, we have

$$E \max_{1\le j\le n} |S_j|^2 \le C \log^2(2n) \sum_{i=1}^{n} \|X_i\|_u^2. \tag{1.4}$$

To illustrate the applications of the inequalities above, we explore the strong consistency of Gasser-Müller estimator of fixed design regression estimate under strong mixing errors by Theorem 1.2 and obtain the almost sure convergence rate

$n^{-1/2}(\log\log n)^{1/\xi}\log^{3/2} n$ with any $0<\xi<2$ for strong mixing sequences by Theorem 1.2, which closes to the optimal achievable convergence rate for independent random variables under an iterated logarithm.

Throughout this paper, we always suppose that C denotes constant which only depends on some given numbers and may vary from one appearance to the next, $a_n = O(b_n)$ represents $a_n \le Cb_n$, $a_n \ll b_n$ means $a_n = O(b_n)$, $[x]$ denotes the integer part of x, $\|X\|_r = (E|X|^r)^{1/r}$ and $a\wedge b = \min\{a,b\}$. The paper is organized as following. Section 2 contains the applications of the maximal inequalities, section 3 provides the proofs of the maximal inequalities.

2 Applications

In this section, we'll show the applications of Theorem 1.2 and Theorem 1.4. Firstly, let us investigate the strong consistency of Gasser-Müller estimator of fixed design regression estimate. Let A be a compact set in R. Consider observations

$$Y_i = g(x_{ni}) + \varepsilon_i, \quad i = 1,2,...,n$$

where $x_{n1}, x_{n2}, ..., x_{nn} \in A$ are fixed points, g is a bounded real valued function on A and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are random errors with $E\varepsilon_i = 0, i = 1,2,...,n$. The general linear smooth estimate is defined by the formula

$$g_n(x) = \sum_{i=1}^{n} \omega_{ni}(x) Y_i, \quad x \in A \subset R$$

where weight functions $\omega_{ni}, i = 1,2,...,n$, depend on the fixed design points $x_{n1}, x_{n2}, ..., x_{nn}$ and the number of observations n. Assume

$$\omega_{ni}(x) = \frac{K\left(\frac{x-x_{ni}}{h_n}\right)}{\sum\limits_{j=1}^{n} K\left(\frac{x-x_{nj}}{h_n}\right)},$$

where $0 = x_{n0} \le x_{n1} \le \cdots \le x_{nn} = 1,\ 0 < h_n \to 0,\ K(\cdot)$ is a probability density function and $g(\cdot)$ is bounded and integrable in $[0,1]$. Denote Gasser-Müller estimator by

$$g_n(x) = \sum_{i=1}^{n} \omega_{ni}(x) Y_i. \tag{2.1}$$

By the proof of Theorem 2.3 in Xing et al. [7] and Theorem 1.2, we can obtain the following theorem.

Theorem 2.1. *Let $2 \ge r > p \ge 1$ and $\{\varepsilon_i\}$ be a strong mixing sequence of random variables. Assume*

(i) $E\varepsilon_i = 0, \sup_{i\ge 1} E|\varepsilon_i|^r < \infty.$ (2.2)

(ii) $\alpha(n) \leq Cn^{-\theta}$ *for some* $\theta > rp/(2(r-p))$. (2.3)

(iii) $K(u)$ *is continuous almost everywhere in* R, *nonincreasing in* $[0,\infty)$, *nondecreasing in* $(-\infty,0)$ *and* $\lim_{|u|\to\infty}|u|K(u)=0$. *There exists a majorcant* $H(u)$ *which is bounded, symmetric, nonincreasing in* $[0,\infty)$ *and integrable over* R, *such that* $K(u) \leq H(u)$ *for* $u \in R$.

(iv) *There exists two constants* C_1 *and* C_2 *such that* $\frac{C_1}{n} \leq x_{ni} - x_{n,i-1} \leq \frac{C_2}{n}$ *for* $i = 1, 2, ..., n$.

(v) $(nh_n)^{-1} = O(n^{-1/p})$. (2.4)

Then at every continuous point $x \in A$ *of the function* g, *we obtain*

$$g_n(x) \to g(x), a.s. \tag{2.5}$$

Next, we will investigate almost sure convergence rate for α-mixng sequeces by Theorem 1.4. The result is

Theorem 2.2. *Let* $\{X_i, i \geq 1\}$ *be a strong mixing sequence of random variables with* $EX_i = 0$, *the mixing coefficient* $\alpha(i)$ *satisfying*

$$\sum_{i=1}^{\infty} \alpha(i)^{\eta/(2+\eta)} < \infty \tag{2.6}$$

for some $\eta > 0$ *and* $\sup_{i\geq 1} E|X_i|^{v+\eta_1} < \infty$ *for some* $1 \leq v \leq 2$ *and* $\eta_1 = v\eta/2$. *Let* $S_n = \sum_{i=1}^{n} X_i$. *Then, we have, for any* $0 < \xi < 2$,

$$S_n/(n(\log\log n)^{2/\xi}\log^3 n)^{1/v} \to 0 \quad a.s. \tag{2.7}$$

Proof. Set $b_n = (n(\log\log n)^{2/\xi}\log^3 n)^{1/v}$, $X_{i1} = X_i I(|X_i| \leq b_n)$ and $S_{j1} = \sum_{i=1}^{j}(X_{i1} - EX_{i1})$. By subsequence method, it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |S_j| > \varepsilon b_n) < \infty \tag{2.8}$$

for any $\varepsilon > 0$. We first show that

$$b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} EX_{i1} \right| \to 0. \tag{2.9}$$

Since $E|X_i|I(|X_i| > b_n) \leq b_n^{1-v-\eta_1} E|X_i|^{v+\eta_1} I(|X_i| > b_n) \ll b_n^{1-v-\eta_1}$, we can get

$$\sum_{i=1}^{n} E|X_i|I(|X_i| > b_n) \ll nb_n^{1-v-\eta_1}.$$

By this and $EX_i = 0$, we have

$$\begin{aligned} b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_{i1}\right| &= b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_i I(|X_i| \le b_n)\right| \\ &= b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_i I(|X_i| > b_n)\right| \\ &\le b_n^{-1} \sum_{i=1}^{n} E|X_i| I(|X_i| > b_n) \\ &\le n b_n^{-v-\eta_1} \to 0. \end{aligned}$$

Hence, (2.9) holds. From (2.9), it follows that for sufficiently large n,

$$\begin{aligned} &P\left(\max_{1\le j\le n} |S_j| > \varepsilon b_n\right) \\ &= P\left(\max_{1\le j\le n} |S_j| > \varepsilon b_n,\ \exists |X_i| > b_n\right) + P\left(\max_{1\le j\le n} |S_j| > \varepsilon b_n,\ \forall |X_i| \le b_n\right) \\ &\le P\left(\max_{1\le i\le n} |X_i| > b_n\right) + P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} X_{i1}\right| > \varepsilon b_n\right) \\ &\le P\left(\max_{1\le i\le n} |X_i| > b_n\right) + P\left(\max_{1\le j\le n} |S_{j1}| > \varepsilon b_n - \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_{i1}\right|\right) \\ &\le \sum_{i=1}^{n} P\left(|X_i| > b_n\right) + P\left(\max_{1\le j\le n} |S_{j1}| > \varepsilon b_n/2\right). \end{aligned}$$

Thus, we need only to prove that

$$I := \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|X_i| > b_n\right) < \infty,$$

$$II := \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |S_{j1}| > \varepsilon b_n/2\right) < \infty. \tag{2.10}$$

By Markov inequality, it follows that

$$I = \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|X_i| > b_n\right) \le \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} b_n^{-v} E|X_i|^v \ll \sum_{n=1}^{\infty} b_n^{-v} < \infty.$$

By Theorem 1.4, we have

$$\begin{aligned} II &= \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}|S_{j1}| > \varepsilon b_n/2\right) \\ &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-2} E\max_{1\le j\le n}|S_{j1}|^2 \\ &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-2}\log^2(2n)\sum_{i=1}^{n}\|X_{i1}\|_{2+\eta}^2 \\ &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-2}\log^2 n\sum_{i=1}^{n}\left(E|X_i|^{2+\eta} I(|X_i|\le b_n)\right)^{2/(2+\eta)} \\ &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-2}\log^2 n\sum_{i=1}^{n}\left(b_n^{2+\eta} E\left(|X_i|^{2+\eta}/b_n^{2+\eta}\right) I(|X_i|\le b_n)\right)^{2/(2+\eta)} \end{aligned}$$

$$\begin{aligned} &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-2}\log^2 n\sum_{i=1}^{n}\left(b_n^{2+\eta} E\left(|X_i|^{v+\eta_1}/b_n^{v+\eta_1}\right) I(|X_i|\le b_n)\right)^{2/(2+\eta)} \\ &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-2}\log^2 n\sum_{i=1}^{n}\left(b_n^{2+\eta-v-\eta_1} E|X_i|^{v+\eta_1} I(|X_i|\le b_n)\right)^{2/(2+\eta)} \\ &\le C\sum_{n=2}^{\infty} b_n^{-v}\log^2 n \\ &< \infty. \end{aligned}$$

Now we complete the proof of Theorem 2.2. □

Remark 2.3. *For the case $v = 2$, we can obtain that the almost sure convergence rate of S_n/n is $n^{-1/2}(\log\log n)^{1/\xi}\log^{3/2} n$ with any $0 < \xi < 2$, which closes to the optimal rate obtained under the iterated logarithm for independent random variables.*

3 Proofs

Let $k = [(n/2)^{\lambda}]$ and $m = [(n/2)^{1-\lambda}]$, where $0 < \lambda < 1$ which will be given later on. Obviously,

$$n < 2(m+1)k,\ Cn^{\lambda} < k < 2n^{\lambda},\ m < 2n^{1-\lambda} \tag{3.1}$$

Fix n and redefine X_i as $X_i = X_i$ for $1 \le i \le n$ and $X_i = 0$ for $i > n$. For $l = 1, 2, ..., [\frac{j}{2k}] + 1 (1 \le j \le n)$, put

$$Y_l = \sum_{2(l-1)k+1}^{j\wedge(2l-1)k} X_i,\ Z_l = \sum_{(2l-1)k+1}^{j\wedge 2lk} X_i$$

and $S_{1,l} = \sum_{i=1}^{l} Y_i$, $S_{2,l} = \sum_{i=1}^{l} Z_i$.

Lemma 3.1.

$$\max_{1\le j\le n} |S_j|^r \le C\left\{\max_{1\le l\le m+1} |S_{1,l}|^r + \max_{1\le l\le m+1} |S_{2,l}|^r\right\} \tag{3.2}$$

Proof. By (3.1) and so-called C_r inequality, we immediately get (3.2). It is easy to observe that

$$\max_{1\le l\le m+1} |S_{1,l}|^r \le 2^{r-1}\left|\max_{1\le l\le m+1} S_{1,l}\right|^r + 2^{r-1}\left|\max_{1\le l\le m+1} (-S_{1,l})\right|^r. \tag{3.3}$$

□

Let M_l, N_l, $\widetilde{M}_l$, $\tilde{N}_l$ be as in Xing et al. [7]. Then, by the proof of Lemma 3.1 in Xing et al. [7], we have the following lemma.

Lemma 3.2. *If $\theta > r(r+\delta)/(2\delta)$, then for any $\tau > 0$, there exist positive constants $C_\tau = C(\tau, r, \delta, \theta) < \infty$ and $C_r = C(r) < \infty$ such that*

$$\sum_{l=1}^{m+1} E\left(Y_l M_l^{r-1}\right) \le C_\tau \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2\right)^{r/2} + \tau C_r E \max_{1\le l\le m+1} |S_{1,l}|^r, \tag{3.4}$$

$$\sum_{l=1}^{m+1} E\left(Y_l \widetilde{M}_l^{r-1}\right) \le C_\tau \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2\right)^{r/2} + \tau C_r E \max_{1\le l\le m+1} |S_{1,l}|^r. \tag{3.5}$$

Lemma 3.3. *If $\theta > r(r+\delta)/(2\delta)$, then*

$$E \max_{1\le l\le m+1} |S_{1,l}|^r \le C\left\{\sum_{l=1}^{m+1} E|Y_l|^r + \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2\right)^{r/2}\right\}, \tag{3.6}$$

$$E \max_{1\le l\le m+1} |S_{2,l}|^r \le C\left\{\sum_{l=1}^{m+1} E|Z_l|^r + \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2\right)^{r/2}\right\}. \tag{3.7}$$

Proof. From the proof of Lemma 3.2 in Xing et al. [7] and Lemma 3.2, we can get the desired results and so the details are omitted here. □

Proof of Theorem 1.2. It follows from Lemma 3.1 and Lemma 3.3,

$$E \max_{1\le j\le n} |S_j|^r \le C\left\{\sum_{l=1}^{2(m+1)} \left(E|Y_l|^r + E|Z_l|^r\right) + \left(\sum_{i=1}^{n} \|X_i\|_{r+\delta}^2\right)^{r/2}\right\}. \tag{3.8}$$

Using so-called Cr-inequality for $E|Y_l|^r$, $E|Z_l|^r$ mentioned above, and noting (3.3), we have

$$\begin{aligned} E\max_{1\le j\le n}|S_j|^r &\le C\left\{k^{r-1}\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\right)^{r/2}\right\} \\ &\le C\left\{n^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\right)^{r/2}\right\}. \end{aligned}$$

Applying the result to $E|Y_l|^r$, $E|Z_l|^r$ in (3.8),

$$\begin{aligned} E\max_{1\le j\le n}|S_j|^r &\le C\left\{k^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\right)^{r/2}\right\} \\ &\le C\left\{n^{\lambda^2(r-1)}\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\right)^{r/2}\right\}. \end{aligned}$$

Repeating t times in this way for $E|Y_l|^r$, $E|Z_l|^r$ in (3.8), we obtain

$$E\max_{1\le j\le n}|S_j|^r\le C\left\{n^{\lambda^t(r-1)}\sum_{i=1}^{n}E|X_i|^r+\left(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\right)^{r/2}\right\}$$

for integer $t\ge 1$. Since $0<\lambda<1, \lambda^t(r-1)<\varepsilon$ for some $t>1$. Hence (1.2) holds. The proof is completed.

In order to prove Theorem 1.4, we need

Lemma 3.4. (Stout [9]) *Let* $S_n=\sum_{i=1}^n X_i$. *If* $ES_k^2\le C\sum_{i=1}^k\|X_i\|_u^2$ *for some* $u>2$, *then*

$$E\max_{1\le j\le n}|S_j|^2\le C\log^2(2n)\sum_{i=1}^{n}\|X_i\|_u^2.$$

Proof of Theorem 1.4. By Theorem 7.3 in Roussas and Ioannidies [10] and the condition (1.3), we have

$$ES_n^2\le C\sum_{i=1}^{n}\|X_i\|_u^2,$$

which, together with Lemma 3.4, yields the desired result (1.4).

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