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# A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces

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**Abstract :** We prove the existence of fixed points of a generalized weakly contractive map in $T$-orbitally complete metric spaces. This result generalizes the results of Babu and Alemayehu [1] and Rhoades [2] when the space under consideration is bounded.

**Keywords :** Weakly contractive maps; Generalized weakly contractive maps; Fixed point; $T$-orbitally complete metric spaces.
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## 1 Introduction

In 1997, Alber and Guerre-Delabriere [3] introduced the concept of weakly contractive maps in Hilbert spaces and proved the existence of fixed points. Rhoades [2] extended this concept to Banach spaces and established the existence of fixed points.

Throughout this paper, $(X, d)$ is a metric space which we denote simply by $X$ and $T{:}X \to X$ a selfmap of $X$. We denote $R^+ = [0, \infty)$, $N$, the set of all natural numbers and $R$, the set of all real numbers. We write

$\Phi = \{\varphi : [0,\infty) \to [0,\infty)/$(i) $\varphi$ is continuous (ii) $\varphi$ is non-decreasing (iii) $\varphi(t) > 0$ for $t > 0$ and (iv) $\varphi(0) = 0\}$.

A selfmap $T : X \to X$ is said to be a *weakly contractive map* if there exists a $\varphi \in \Phi$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \text{ for all } x, y \in X. \tag{1}$$

Here we observe that every contraction map $T$ on $X$ with contraction constant $k$ is a weakly contractive map with $\phi(t) = (1-k)t,\ t \geq 0$. But its converse need not be true.

**Theorem 1.1.** (Rhoades [2]) *Let (X, d) be a complete metric space and T a weakly contractive map. Then T has a unique fixed point in X.*

A selfmap $T : X \to$X is said to be a *generalized weakly contractive map* if there exists a $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)) \text{ for all } x, y \in X, \tag{2}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$.

**Remark 1.2.** (Babu and Alemayehu [1]) *Every weakly contractive map defined on a bounded metric space with a positive diameter is a generalized weakly contractive map, but it's converse need not be true.*

**Theorem 1.3.** (Babu and Alemayehu [1]) *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmap. If $T$ is a generalized weakly contractive map on $X$, then $T$ has a unique fixed point in $X$.*

If $X$ is a complete metric space which is bounded, then by Remark 1.2, Theorem 1.1 follows as a corollary to Theorem 1.3. In fact in this case, Theorem 1.3 is a generalization of Theorem 1.1 (Example 3.2 of Babu and Alemayehu, [1]).

For $x_0 \in X, O(x_0) = \{T^n x_0/n = 0, 1, 2, ...\}$ is called the *orbit of* $x_0$, where $T^0 = I$, $I$ the identity map of $X$.

A metric space $X$ is said to be *T-orbitally complete* if every Cauchy sequence which is contained in $O(x)$ for all $x$ in $X$ converges to a point of $X$.

Here we note that every complete metric space is $T$- orbitally complete for any $T$, but a $T$- orbitally complete metric space need not be a complete metric space. For more details, we refer Turkoglu, Ozer and Fisher [4].

In this paper, we prove the existence of fixed points of a generalized weakly contractive map in $T$-orbitally complete metric spaces. Our theorems generalize the results of Babu and Alemayehu [1] and Rhoades [2].

We use the following lemma to prove our main result, whose proof is well known. But for completeness sake we present it's proof.

**Lemma 1.4.** *Suppose $(X, d)$ is a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon, d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ and* *(i)* $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$;

*(ii)* $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$;

*(iii)* $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$.

*Proof.* If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $m(k) > n(k) > k$ satisfying

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{1.4.1}$$

We choose $m(k)$, the least positive integer satisfying (1.4.1). Then we have

$$m(k) > n(k) > k \text{ with } d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{1.4.2}$$

We now prove (i).

(i) By using the triangle inequality, we have

$$\begin{aligned}\epsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)+1}) \\ &\quad + d(x_{n(k)+1}, x_{n(k)}).\end{aligned}$$

By taking limit inferior as $k \to \infty$, we get

$$\begin{aligned}\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{m(k)-1}) &+ \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) \\ &+ \liminf_{k\to\infty} d(x_{n(k)+1}, x_{n(k)}).\end{aligned}$$

Now, on using $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$, we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}). \tag{1.4.3}$$

Now,

$$\begin{aligned}d(x_{m(k)-1}, x_{n(k)+1}) &\leq d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) \\ &< \epsilon + d(x_{n(k)}, x_{n(k)+1}).\end{aligned}$$

Now taking limit superior as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) \leq \epsilon + \limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon.$$

Therefore,

$$\limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) \leq \epsilon. \tag{1.4.4}$$

From (1.4.3) and (1.4.4), we get

$$\liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1}) = \epsilon,$$

so that

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) \text{ exists and } \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon.$$

Hence (i) holds.

(ii) We have, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, and hence

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}). \tag{1.4.5}$$

Now,

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + \epsilon. \end{aligned}$$

This implies,

$$\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)-1}) + \epsilon,$$

so that

$$\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \epsilon. \tag{1.4.6}$$

From (1.4.5) and (1.4.6), we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \epsilon.$$

Therefore, $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$. Hence (ii) holds.

(iii) We have, $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$. Hence

$$\limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon. \tag{1.4.7}$$

Now,

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}).$$

Hence

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{m(k)-1}) + \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}).$$

Thus by using the property $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ we have

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}). \tag{1.4.8}$$

From (1.4.7) and (1.4.8), we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) \leq \limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon.$$

Hence, $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$. Hence (iii) holds. This completes the proof of the lemma. □

## 2 Main Result

**Theorem 2.1.** *Let $(X, d)$ be a $T$-orbitally complete metric space. Assume that for some $x_0 \in X$, there exists a $\varphi_{x_0} \in \Phi$ such that*

$$d(Tx, Ty) \leq M(x, y) - \varphi_{x_0}(M(x, y)) \quad \textit{for all } x, y \in \overline{O(x_0)}, \tag{2.1.1}$$

*where* $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$. *Then the sequence $\{T^n x_0\}$ is Cauchy in $X$, $\lim_{n\to\infty} T^n x_0 = z$, $z \in X$ and $z$ is a fixed point of $T$. Further, $z$ is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of $T$.*

*Proof.* If $M(x, y) = 0$ for some $x, y \in \overline{O(x_0)}$, then we are through. Suppose $M(x, y) \neq 0$ for all $x, y \in \overline{O(x_0)}$. We define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for $n$ = 0, 1, 2,... . If $x_n = x_{n+1}$ for some $n$, then the conclusion of the theorem follows trivially. So, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n = 0$, 1, 2,... . Let $\alpha_n = d(x_n, x_{n+1})$ for $n$ =0, 1, 2,... . Note that $\alpha_n > 0$ for all $n = 0$, 1, 2,... . Now by using (2.1.1), we have

$$\begin{aligned}
\alpha_{n+1} = d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq M(x_n, x_{n+1}) - \varphi_{x_0}(M(x_n, x_{n+1})) \qquad (2.1.2) \\
&< M(x_n, x_{n+1}) \\
&= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \tfrac{1}{2}[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \tfrac{1}{2}[d(x_n, x_{n+2})]\} \\
&\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]\} \\
&= \max\{\alpha_n, \alpha_{n+1}, \tfrac{1}{2}(\alpha_n + \alpha_{n+1})\} \\
&= \max\{\alpha_n, \alpha_{n+1}\}. \qquad (2.1.3)
\end{aligned}$$

Hence it follows that $\alpha_{n+1} < \alpha_n$ for all $n = 0$, 1, 2,... . So, $\{\alpha_n\}$ is a strictly decreasing sequence of real numbers. Let $\lim_{n\to\infty}\alpha_n = \alpha$, $\alpha \in$ R and $\alpha \geq 0$. Now, from (2.1.3), we have

$$\alpha_{n+1} < M(x_n, x_{n+1}) \leq \alpha_n \quad \text{for all } \; n = 0, 1, 2, ....$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} M(x_n, x_{n+1}) = \alpha$. Let $\alpha > 0$. From (2.1.2), we have

$$\alpha_{n+1} \leq M(x_n, x_{n+1}) - \varphi_{x_0}(M(x_n, x_{n+1})) \quad \text{for each } \; n = 0, 1, 2, ....$$

On taking limits as $n \to \infty$, we get $\alpha \leq \alpha - \varphi_{x_0}(\alpha)$, a contradiction. So $\alpha = 0$.

We now show that the sequence $\{x_n\} \subset O(x_0)$ is Cauchy. Otherwise, there exists an $\epsilon>0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ and from Lemma 1.4 we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon,\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$$

and

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon \tag{2.1.5}$$

Hence,

$$\begin{aligned}
\epsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}) \\
&= d(Tx_{m(k)-1}, Tx_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}) \\
&\leq M(x_{m(k)-1}, x_{n(k)}) - \varphi_{x_0}(M(x_{m(k)-1}, x_{n(k)})) \\
&\quad +d(x_{n(k)+1}, x_{n(k)}) \\
&= \max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), \\
&\quad \frac{1}{2}[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})]\} \\
&\quad -\varphi_{x_0}(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \\
&\quad d(x_{n(k)}, x_{n(k)+1}), \frac{1}{2}[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})]\}) \\
&\quad +d(x_{n(k)+1}, x_{n(k)}) \\
&< \max\{\epsilon, d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), \\
&\quad \frac{1}{2}[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})]\} \\
&\quad -\varphi_{x_0}(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \\
&\quad d(x_{n(k)}, x_{n(k)+1}), \frac{1}{2}[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})]\}) \\
&\quad +d(x_{n(k)+1}, x_{n(k)}).
\end{aligned}$$

Now on taking limits as $k \to \infty$ and using (2.1.5), we have

$$\epsilon \leq \max\{\epsilon, 0, 0, \epsilon\} - \varphi_{x_0}(\max\{\epsilon, 0, 0, \epsilon\}) = \epsilon - \varphi_{x_0}(\epsilon) < \epsilon,$$

a contradiction. Therefore, $\{x_n\} \subset O(x_0)$ is Cauchy. Since $X$ is $T$-orbitally complete, $\lim_{n\to\infty} x_n = z$ (say), $z \in X$.

We now show that $Tz = z$. Suppose $Tz \neq z$. We consider,

$$\begin{aligned}
d(Tx_n, Tz) &\leq M(x_n, z) - \varphi_{x_0}(M(x_n, z)) \\
&= \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \frac{1}{2}[d(x_n, Tz) + d(z, x_{n+1})]\} \\
&\quad -\varphi_{x_0}(\max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \\
&\quad \frac{1}{2}[d(x_n, Tz) + d(z, x_{n+1})]\}).
\end{aligned}$$

On letting $n \to \infty$, it follows that $d(z, Tz) \le d(z, Tz) - \varphi_{x_0}(d(z, Tz))$, a contradiction. Hence $Tz = z$. Uniqueness of $z$ follows trivially from the inequality (2.1.1). ☐

**Corollary 2.2.** *Let $(X, d)$ be a $T$-orbitally complete bounded metric space. Assume that for some $x_0 \in X$, there exists $\varphi_{x_0} \in \Phi$ such that*

$$d(Tx, Ty) \le d(x, y) - \varphi_{x_0}((d(x, y)) \quad \textit{for all } x, y \in \overline{O(x_0)}. \tag{2.2.1}$$

*Then the sequence $\{T^n x_0\}$ is Cauchy in $X$, $\lim_{n\to\infty} T^n x_0 = z, z \in X$ and $z$ is a fixed point of $T$. Further, $z$ is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of $T$.*

*Proof.* By Remark 1.2 we have that the inequality (2.2.1) implies the inequality (2.1.1). Thus the conclusion of the corollary follows from Theorem 2.1. ☐

**Remark 2.3.**

*(i) If $X$ is bounded then Theorem 1.1 follows as a corollary to Corollary 2.2.*

*(ii) Theorem 1.3 follows as a corollary to Theorem 2.1.*

*(iii) The following example shows that Corollary 2.2 is a generalization of Theorem 1.1 and Theorem 2.1 is a generalization of Theorem 1.3, when the space $X$ is bounded.*

**Example 2.4.** *Let $X = [0, 1]$ with the usual metric. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{2}, & if \quad x \in [0,\ \frac{1}{2}] \\ x, & if \quad x \in (\frac{1}{2},\ 1]. \end{cases}$$

*Let $x_0 = \frac{1}{2}$, then $O(x_0) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ and $\overline{O(x_0)} = O(x_0) \cup \{0\}$. $X$ is $T$-orbitally complete and satisfies the inequality (2.2.1) with $\varphi_{x_0} : [0, \infty) \to [0, \infty)$ defined by*

$$\varphi_{x_0}(t) = \begin{cases} \frac{t}{2}, & if \quad 0 \le t \le 1 \\ \\ \frac{1}{2}, & if \quad t \ge 1. \end{cases}$$

*Since the inequality (1) fails to hold for any $x, y \in (\frac{1}{2}, 1]$ with $x \ne y$ and for any $\varphi \in \Phi$, $T$ is not a weakly contractive map on $X$.*

*Also, we observe that $T$ is not a generalized weakly contractive map on $X$, for let $x, y \in (\frac{1}{2}, 1]$ with $x \ne y$. Then $M(x, y) = |x - y|$, so that for any $\varphi \in \Phi, T$ fails to satisfy (2).*

*Thus $T$ is neither a weakly contractive map nor a generalized weakly contractive map on $X$, and hence Theorem 1.1 and Theorem 1.3 are not applicable. But, $T$ satisfies the inequality (2.2.1). Since $\overline{O(x_0)}$ is bounded, by Remark 1.2, the inequality (2.1.1) holds so that $T$ satisfies all the hypotheses of Theorem 2.1; and 0 is the unique fixed point of $T$ in $\overline{O(x_0)}$.*

**Remark 2.5.** *Under the hypotheses of Theorem* 2.1*, $T$ may have more than one fixed point in $X$. It is illustrated in the following example.*

**Example 2.6.** *Let $X = R$ with the usual metric. We define $T : X \to X$ by $T(x) = [x]$. For any $x_0 \in X, O(x_0) = \{x_0, [x_0], [x_0], ...\}$ and $\overline{O(x_0)} = O(x_0)$. Then $T$ satisfies all the conditions of Theorem* 2.1 *with $\varphi_{x_0}(t) = t, t \geq 0$ and $T$ has a unique fixed point $[x_0]$ in $\overline{O(x_0)}$, but $T$ has more than one fixed point in $X$. In fact, $T$ has infinitely many fixed points in $X$.*

The following example shows that if the inequality (2.1.1) holds in $O(x_0)$ instead of $\overline{O(x_0)}$ for some $x_0$ in $X$, then $T$ may not have a fixed point.

**Example 2.7.** *Let $X = \{0, 1\} \cup \{\frac{1}{2^n} : n \in N\} \cup \{1 + \frac{1}{2^{n+1}} : n \in N\}$ with the usual metric. We define $T$: $X \to X$ by $T0 = 1, T1 = 0, T(\frac{1}{2^n}) = \frac{1}{2^{n+1}}$ and $T(1 + \frac{1}{2^n}) = 1 + \frac{1}{2^{n+1}}$. Let $x_0 = 1 + \frac{1}{2}, O(x_0) = \{1 + \frac{1}{2}, 1 + \frac{1}{2^2}, 1 + \frac{1}{2^3}, .....\}$ and $\overline{O(x_0)} = O(x_0) \cup \{1\}$. Then the inequality* (2.1.1) *holds for all $x, y$ in $O(x_0)$ with $\varphi_{x_0}(t) = \frac{t}{2}, t \geq 0$. But the inequality* (2.1.1) *fails to hold in $\overline{O(x_0)}$ for any $\varphi \in \Phi$. For, at $x = 1$ and $y = 1 + \frac{1}{2}$, we have $d(Tx, Ty) = 1 + \frac{1}{2^2} = \frac{5}{4}$, and*

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\} \\ &= \max\left\{\frac{1}{2}, 1, \frac{1}{2^2}, \frac{1}{2}\left[\frac{1}{2^2} + 1 + \frac{1}{2}\right]\right\} \\ &= \max\left\{\frac{1}{2}, 1, \frac{1}{4}, \frac{7}{8}\right\} = 1. \end{aligned}$$

*Therefore, $d(Tx, Ty) = \frac{5}{4} \not\leq 1 - \varphi(1) = M(x, y) - \varphi((M(x, y))$ for any $\varphi \in \Phi$. Further, for $x = 0$ and $y = 1$, $d(Tx, Ty) = 1 \not\leq 1 - \varphi(1) = M(x, y) - \varphi((M(x, y))$ for any $\varphi \in \Phi$. Thus the inequality* (2.1.1) *fails to hold on $\overline{O(x_0)}$ for $x_0 = 1 + \frac{1}{2}$ and $T$ has no fixed points in $X$.*

The following is an example in support of Theorem 2.1.

**Example 2.8.** *Let $X = \{0, 1, 2\} \cup \{\sum_{i=0}^{n} \frac{1}{2^i} : n \in N\}$ with the usual metric. We define $T$: $X \to X$ by $T0 = 1$, $T1 = 0$, $T2 = 2$ and $T\ (\sum_{i=0}^{n} \frac{1}{2^i}) = \sum_{i=0}^{n+1} \frac{1}{2^i}$. Let $x_0 = 1 + \frac{1}{2}$, then $O(x_0) = \{\sum_{i=0}^{n} \frac{1}{2^i} : n \in N\}$ and $\overline{O(x_0)} = O(x_0) \cup \{2\}$. We define $\varphi_{x_0} : R^+ \to R^+$ by $\varphi_{x_0}(t) = \frac{t}{2}$, $t \geq 0$. Let $x, y \in \overline{O(x_0)}$.*

*Case* (i)*: $x = \sum_{i=0}^{n} \frac{1}{2^i}$, $y = \sum_{i=0}^{m} \frac{1}{2^i}$ and $n > m$. In this case, $|Tx - Ty| = \frac{1}{2}\sum_{i=m+1}^{n} \frac{1}{2^i}$, $|x - y| = \sum_{i=m+1}^{n} \frac{1}{2^i}$, $|x - Tx| = \frac{1}{2^{n+1}}$, $|y - Ty| = \frac{1}{2^{m+1}}$, $|x - Ty| = \sum_{i=m+2}^{n} \frac{1}{2^i}$, $|y - Tx| = \sum_{i=m+1}^{n+1} \frac{1}{2^i}$. Therefore,*

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right]\right\} \\ &= \sum_{i=m+1}^{n+1} \frac{1}{2^i} \end{aligned}$$

*and* $M(x,y) - \varphi_{x_0}((M(x,y)) = \frac{1}{2}\sum_{i=m+1}^{n+1}\frac{1}{2^i}$, *so that* $|Tx - Ty| < M(x,y) - \varphi_{x_0}((M(x,y))$ *holds.*

*Case* (ii)*:* $x = 2, y = \sum_{i=0}^{n}\frac{1}{2^i}$. *Now,* $|Tx-Ty| = |2-\sum_{i=0}^{n+1}\frac{1}{2^i}| = \frac{1}{2^{n+1}}$, $|x-y| = |2-\sum_{i=0}^{n}\frac{1}{2^i}| = \frac{1}{2^n}$, $|x-Tx| = 0$, $|y-Ty| = \frac{1}{2^{n+1}}$, $|y-Tx| = |\sum_{i=0}^{n}\frac{1}{2^i} - 2| = \frac{1}{2^n}$. $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\} = \frac{1}{2^n}$ *and* $M(x,y) - \varphi_{x_0}((M(x,y)) = \frac{1}{2^n} - \frac{1}{2}\frac{1}{2^n} = \frac{1}{2^{n+1}}$. *Hence,* $|Tx - Ty| = M(x,y) - \varphi_{x_0}((M(x,y))$ *holds.*

*Thus, from case (i) and case (ii), we have* $T$ *is a weakly contractive map on* $\overline{O(x_0)}$ *with* $\varphi_{x_0}(t) = \frac{t}{2}$, $t \geq 0$. *Therefore,* $T$ *satisfies all the conditions of Theorem* 2.1. *Further, the sequence* $\{T^n x_0\}$ *converges to* 2 *and* 2 *is the unique fixed point of* $T$ *in* $\overline{O(x_0)}$.

The following is an example in support of Corollary 2.2, in abstract spaces.

**Example 2.9.** *Let* $C_0 = \{y = \{y_n\}_{n=0}^{\infty} / y_n \to 0$ *as* $n \to \infty\}$ *with the metric* $d$, *defined by* $d(x,y) = \|x-y\| = \sup\{|x_n - y_n|/n \in N\}$, *where* $x = \{x_n\}_{n=0}^{\infty}$ *and* $y = \{y_n\}_{n=0}^{\infty}$ *in* $C_0$. *With this metric* $d$ *on* $C_0, C_0$ *is a complete metric space. Let* $x_0 = (0,0,0,...), x_1 = (1,\frac{1}{2},\frac{1}{3},...), x_2 = (0,\frac{1}{2},\frac{1}{3},...), x_3 = (0,0,\frac{1}{3},\frac{1}{4},...),..., x_n = (0,0,0,...,\frac{1}{n},\frac{1}{n+1},...),....$ *Now, let* $X = \{x_0, x_1, x_2, ..., x_n, ...\}$. *We define* $T$ *on* $X$ *by* $Tx_0 = x_0$ *and* $Tx_n = x_{n+1}$ *for* $n = 1,2,....$ *Then* $O(x_1) = \{x_1, x_2, x_3, ...\}$ *and* $\overline{O(x_1)} = O(x_1) \cup \{x_0\}$. *Since the sequence* $O(x_1)$ *converges to* $x_0$, *every Cauchy sequence in* $O(x_1)$ *converges to* $x_0$. *Hence* $X$ *is* $T$*-orbitally complete. We define* $\varphi_{x_1} : R^+ \to R^+$ *by* $\varphi_{x_1}(t) = \frac{t^2}{1+t}, t \geq 0$. *Then* $\varphi_{x_1} \in \Phi$. *Let* $x_n = (0,0,0,...,\frac{1}{n},\frac{1}{n+1},...)$ *and* $x_{n+k} = (0,0,0,...,\frac{1}{n+k},\frac{1}{n+k+1},...)$ *be in* $\overline{O(x_1)}$. *Now,* $d(Tx_n, Tx_{n+k}) = d(x_{n+1}, x_{n+k+1}) = \|x_{n+1} - x_{n+k+1}\| = \frac{1}{n+1}$ *and* $d(Tx_0, Tx_n) = d(x_0, x_{n+1}) = \|x_0 - x_{n+1}\| = \frac{1}{n+1}$. *Now,* $d(x_n, x_{n+k}) = \|x_n - x_{n+k}\| = \frac{1}{n}$ *and* $d(x_0, x_n) = \|x_0 - x_n\| = \frac{1}{n}$. *Since,* $\frac{1}{n+1} = \frac{1}{n} - [\frac{1}{n} - \frac{1}{n+1}] = \frac{1}{n} - [\frac{1}{n(n+1)}] = \frac{1}{n} - \frac{(1/n)^2}{1+\frac{1}{n}}$, *we have* $d(Tx_n, Tx_{n+k}) = d(x_n, x_{n+k}) - \varphi_{x_1}(d(x_n, x_{n+k}))$. *Thus,* $T$ *satisfies the inequality* (2.2.1) *and all the hypotheses of Corollary* 2.2 *with* $T^n x_1 \to x_0$ *as* $n \to \infty$ *and* $T$ *has a unique fixed point* $x_0$ *in* $\overline{O(x_1)}$. *Here we observe that* $T$ *is not a contraction. For, if* $T$ *is a contraction, then there exists a* $q \in (0,1)$, *such that* $d(Tx_n, Tx_{n+k}) \leq q.d(x_n, x_{n+k})$. *This implies* $\frac{1}{n+1} \leq q.\frac{1}{n}$, *i.e.,* $\frac{n}{n+1} \leq q$. *Now as* $n \to \infty$, *we have* $\lim_{n\to\infty}\frac{n}{n+1} \leq q$, *i.e.,* $1 \leq q$, *a contradiction. Thus, Corollary* 2.2 *is a generalization of Banach contraction principle, when the space under consideration is bounded.*

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