



Variational Iteration Method for Solving Eighth-Order Boundary Value Problems

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Abstract : The variational iteration method (VIM), which is a powerful tool, is applied to numerical solution of eighth-order boundary value problems. The VIM usually gives a solution in the form of a rapidly convergent series of a correction functional. The correction functional is constructed by using generalized Lagrange multipliers and the calculus of variations. Analytical results are given for several examples to illustrate the implementation and efficiency of the method. A comparison of the results obtained by the present method with results obtained by the modified decomposition method and the homotopy perturbation method reveals that the present method is very effective and convenient.

Keywords : Variational Iteration Method; Numerical Method.

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1 Introduction

Consider the general eighth-order boundary value problems of the type:

$$y^{(viii)}(x) = f(x, y', y'', y^{(3)}, y^{(iv)}, y^{(v)}, y^{(vi)}, y^{(vii)}), \quad a \leq x \leq b \quad (1.1)$$

with conditions

$$\begin{aligned} y(a) &= A_1, \quad y'(a) = A_2, \quad y''(a) = A_3, \quad y^{(3)}(a) = A_4 \\ y^{(iv)}(a) &= A_5, \quad y^{(v)}(a) = A_6, \quad y(b) = B_1, \quad y'(b) = B_2, \end{aligned}$$

where f is a differentiable function as required for $a \leq x \leq b$ and A_i , $i = 1, 2, \dots, 6$ and B_i , $i = 1, 2$ are real constants. Eighth-order boundary value problems are known to arise in the mathematics, physic and engineering sciences [18]. Several

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numerical methods including spectral Galerkin and collocation [16], sixth B-spline, decomposition [19], spline collocation approximation [20], Chow-Yorke algorithm [21], and homotopy perturbation [22] have been developed for solving the problem [1]. The variational iteration method (VIM) was first proposed by Professor Ji-Huan He for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations [2]. The idea of VIM is to construct a correction functional using a general Lagrange multiplier which we can then identify by variational theory. The multiplier in the functional should be chosen such that its correction solution is superior to an initial approximation (a trial function) and is the best within the flexibility of the trial function. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions. The method gives rapidly convergent successive approximations to the exact solution if such a solution exists. VIM has successfully been applied to many problems. For example, Wazwaz [3] used VIM to solve linear and nonlinear Schrodinger equations. Dehghan and Shakeri [4] applied VIM to solve the Cauchy reaction-diffusion problem. Jinbo and Jiang [5] used VIM to solve an inverse parabolic equation. Ramos [6] applied VIM to solve nonlinear differential equations. Das [7] used VIM to obtain the solution of a fractional diffusion equation. Assas [8] applied VIM to solve coupled-KdV equations. Inc [9] used VIM to solve space- and time-fractional Burgers equations with initial conditions. Javidi and Jalilian [10] applied VIM to obtain wave solution of Boussinesq equation. Wazwaz [11] applied VIM to some linear and nonlinear systems of PDEs. Batiha et al. [12] used VIM to solve systems of PDEs. In this paper, we apply the variational iteration method to solve eighth-order boundary value problems. By using a suitable transformation, the variational iteration method can be used to show that eighth-order boundary value problems are equivalent to a system of integral equations. This technique has been developed by Noor and Mohyud-Din [13]. We apply this technique to solve eighth-order boundary value problems. We compare the results from the variational iteration method with exact solutions.

2 Variational iteration method

We consider the following differential equation

$$Lu(x) + Nu(x) = g(x) \quad (2.1)$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is the forcing term. In the variational iteration method, a correctional functional is constructed as follows [23]

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + Nu_n(s) - g(s))ds, \quad (2.2)$$

where λ is a Lagrange multiplier [25], which can be identified optimally via the variational theory by using the stationary conditions, the subscript n denotes n^{th}

-order approximation, u_n is considered as a restricted variation [24], i.e., $\delta\tilde{u} = 0$. The principles of the variational iteration method and examples of its application for numerical solution of various kinds of differential equations are given in [?]. We consider the following system of differential equations:

$$x_i' = f_i(t, x_i), \quad i = 1, 2, 3, \dots, n \quad (2.3)$$

subject to the conditions:

$$x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n$$

Following the variational iteration method for solving the system of differential equations, we rewrite system (2.3) in the following form: subject to the boundary conditions: $x_i(0) = c_i$, $i = 1, 2, 3, \dots, n$ where g_i is defined as in (2.1). The correctional functional for the system of differential equations (2.3) can be approximated as

$$\begin{aligned} x_1^{(k+1)}(t) &= x_1^{(0)}(t) + \int_0^t \lambda_1 \left(x_1^{(k)}(s), f_1(\tilde{x}_1^{(k)}(s), \dots, \tilde{x}_n^{(k)}(s)) - g_1(s) \right) ds, \\ x_2^{(k+1)}(t) &= x_2^{(0)}(t) + \int_0^t \lambda_2 \left(x_1^{(k)}(s), f_2(\tilde{x}_1^{(k)}(s), \dots, \tilde{x}_n^{(k)}(s)) - g_2(s) \right) ds, \\ &\vdots \\ x_n^{(k+1)}(t) &= x_n^{(0)}(t) + \int_0^t \lambda_n \left(x_n^{(k)}(s), f_n(\tilde{x}_1^{(k)}(s), \dots, \tilde{x}_n^{(k)}(s)) - g_n(s) \right) ds, \end{aligned} \quad (2.4)$$

where $\lambda_i = \pm 1$, $i = 1, 2, 3, \dots, n$ are Lagrange multipliers, and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_n$ denote the restricted variations. The approximation can be completely determined. If the series of approximate solutions converges to the exact solution, we can obtain the exact solution as:

$$x_i(t) = \lim_{k \rightarrow \infty} x_i^{(k)}(t), \quad i = 1, 2, 3, \dots, n$$

A finite number of terms from this series will give an approximate solution.

3 Applications

In this section, we apply the VIM to obtain approximate solutions for some eight-order linear and nonlinear initial-value and boundary-value problems.

Example 3.1. Consider the following eighth-order linear problem

$$y^{(viii)}(x) = y(x), \quad 0 \leq x \leq 1 \quad (3.1)$$

with the following initial conditions

$$y^{(i)}(0) = 1, \quad i = 1, 2, 3, \dots, 7.$$

The exact solution of this problem is

$$y(x) = e^x. \quad (3.2)$$

Using the transformation, we can rewrite the eight-order initial-value problem as a system of first-order differential equations as follows. We let

$$\begin{aligned} \frac{dy}{dx} &= q(x), \quad \frac{dq}{dx} = f(x), \quad \frac{df}{dx} = s(x), \quad \frac{ds}{dx} = t(x), \\ \frac{dt}{dx} &= z(x), \quad \frac{dz}{dx} = w(x), \quad \frac{dw}{dx} = k(x), \quad \frac{dk}{dx} = y(x), \end{aligned}$$

with $y_0(0) = q_0(0) = f_0(0) = s_0(0) = t_0(0) = z_0(0) = w_0(0) = k_0(0) = 1$. Using the VIM, we can rewrite the above system of differential equations as a system of integral equations with Lagrange multipliers $\lambda_i = 1, i = 1, 2, 3, \dots, n$

$$\begin{aligned} y^{m+1}(x) &= 1 + \int_0^x q^m(s)ds, & q^{m+1}(x) &= 1 + \int_0^x f^m(s)ds, \\ f^{m+1}(x) &= 1 + \int_0^x s^m(s)ds, & s^{m+1}(x) &= 1 + \int_0^x t^m(s)ds, \\ t^{m+1}(x) &= 1 + \int_0^x z^m(s)ds, & z^{m+1}(x) &= 1 + \int_0^x w^m(s)ds, \\ w^{m+1}(x) &= 1 + \int_0^x k^m(s)ds, & k^{m+1}(x) &= 1 + \int_0^x y^m(s)ds, \end{aligned}$$

with $y_0(x) = q_0(x) = f_0(x) = s_0(x) = t_0(x) = z_0(x) = w_0(x) = k_0(x) = 1$

Consequently, the VIM obtains the following approximations

$$\begin{aligned} y_1(x) &= 1 + x, \\ y_2(x) &= 1 + x + \frac{x^2}{2!} \\ y_3(x) &= 1 + x + \frac{x^3}{3!} \\ y_4(x) &= 1 + x + \frac{x^3}{3!} + \frac{x^4}{4!} \\ &\vdots \\ y_m(x) &= 1 + x + \frac{x^2}{3!} + \frac{x^4}{4!} + \dots + \frac{x^m}{m!} \end{aligned}$$

The approximation can be completely determined as

$$y(x) = \lim_{m \rightarrow \infty} y_m(x) = \lim_{m \rightarrow \infty} \left(1 + x + \frac{x^2}{3!} + \frac{x^4}{4!} + \dots + \frac{x^m}{m!} \right) = e^x$$

which is the same as the exact solution of (3.2).

Example 3.2. Consider the eighth-order nonlinear boundary-value problem [19,22]

$$y^{(viii)}(x) = 8e^x + y(x), \quad 0 \leq x \leq 1$$

with the initial and boundary conditions

$$\begin{aligned} y(0) &= 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y^{(iii)}(0) = -2 \\ y^{(iv)}(0) &= -3, \quad y^{(v)}(0) = -4, \quad y^{(vi)}(1) = -e, \quad y^{(vii)}(1) = -2e, \end{aligned}$$

The exact solution is

$$y(x) = (1-x)e^x, \quad i = 1, 2, 3, \dots, 7. \quad (3.3)$$

Using the transformation, the problem can be written as the system of first-order differential equations

$$\begin{aligned} \frac{dy}{dx} &= q(x), \quad \frac{dq}{dx} = f(x), \quad \frac{df}{dx} = s(x), \quad \frac{ds}{dx} = t(x), \\ \frac{dt}{dx} &= z(x), \quad \frac{dz}{dx} = w(x), \quad \frac{dw}{dx} = k(x), \quad \frac{dk}{dx} = y(x). \end{aligned}$$

Using the VIM, we can transform the above system of differential equations into the system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1, 2, 3, \dots, n$

$$\begin{aligned} y^{m+1}(x) &= 1 + \int_0^x q^m(s) ds, \\ q^{m+1}(x) &= \int_0^x f^m(s) ds, \\ f^{m+1}(x) &= -1 + \int_0^x s^m(s) ds, \\ s^{m+1}(x) &= -2 + \int_0^x t^m(s) ds, \\ t^{m+1}(x) &= -3 + \int_0^x z^m(s) ds, \\ z^{m+1}(x) &= -4 + \int_0^x w^m(s) ds, \\ w^{m+1}(x) &= A + \int_0^x k^m(s) ds, \\ k^{m+1}(x) &= B + \int_0^x (-8e^x + y^m(s)) ds, \end{aligned} \quad (3.4)$$

with $y_0(x) = 1$, $q_0(x) = 0$, $f_0(x) = -1$, $s_0(x) = -2$, $t_0(x) = -3$, $z_0(x) = -4$, $w_0(x) = A$, $k_0(x) = B$

Consequently, the VIM obtains the following approximations

$$\begin{aligned}
 y_1(x) &= 1, \\
 y_2(x) &= 1 - \frac{x^2}{2} \\
 y_3(x) &= 1 - \frac{x^2}{2!} - \frac{x^3}{3!} \\
 y_4(x) &= 1 + x + \frac{x^3}{3!} + \frac{x^4}{4!} \\
 &\vdots \\
 y_8(x) &= -8e^x + \frac{x^8}{40320} + \frac{x^7}{630} + B\frac{x^7}{5040} + \frac{x^6}{90} + A\frac{x^6}{720} \\
 &\quad + \frac{x^5}{30} + 5\frac{x^4}{24} + \frac{7}{2}x^2 + 8x + 9
 \end{aligned}$$

Using the condition $y^{(vi)}(1) = -e$, $y^{(vii)}(1) = -2e$, to obtain the constants A and B , we have

$$A = -5.0074, \quad B = -5.9710$$

Then the approximate solution is given as

$$\begin{aligned}
 y_8(x) &= -8e^x + \frac{x^8}{40320} + \frac{x^7}{630} - 5.0074\frac{x^7}{5040} + \frac{x^6}{90} \\
 &\quad - 5.9710\frac{x^6}{720} + \frac{x^5}{30} + 5\frac{x^4}{24} + \frac{7}{2}x^2 + 8x + 9
 \end{aligned}$$

which as shown in Table 1 is a good approximation to the exact solution given in (3.3).

x	Exact solution	Error VIM, N=8
0.25	0.9630190628	3.8922e-10
0.50	0.8243606355	1.1571e-7
0.75	0.5292500042	1.0479e-6
1.00	0	4.2188e-6

Table 1: Comparison of VIM solution of Example 3.2 with exact solution

4 Conclusion

The variational iteration method has been applied to obtain numerical solutions of a linear and a nonlinear eighth-order boundary-value problem. The results

show that the method gives rapidly converging series solutions in both cases. The method is easy to apply and can easily be applied to similar problems that arise in physical and engineering sciences problems.

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