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# Weak and Strong Convergence of an Implicit Iteration Process for a Finite Family of Asymptotically Quasi-Nonexpansive Mappings

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**Abstract :** In this paper, a new implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings is introduced and studied. We prove that the implicit iteration sequence for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the family in a uniformly convex Banach space, requiring one member $T$ in the family which is either semi-compact or satisfies condition $(\overline{C})$. More precisely, weak convergence theorems are established for the implicit iteration process in a uniformly convex Banach space which satisfies Opial's condition. Our results generalize and extend the recent ones announced by Thianwan and Suantai [S. Thianwan and S. Suantai, Weak and strong convergence of an implicit iteration process for a finite family of nonexpansive Mappings, Scientiae Mathematicae Joponicae 66 (2007), 221–229], Sun [Z.H. Sun, Strong convergence of an implicit iteration for a finite family of asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl. 286 (2003), 351–358], and other authors.

**Keywords :** Implicit iteration process; Weak and strong convergence; Finite family of asymptotically quasi-nonexpansive mappings; Opial's condition; Common fixed point.

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## 1 Introduction

Let $X$ be a normed space and $C$ a nonempty subset of $X$, $T$ a self-mapping on $C$. $T$ is said to be nonexpansive on $C$ if for all $x, y \in C$ the following inequality holds: $\|Tx - Ty\| \leq \|x - y\|$. $T$ is called asymtotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq$

$(1+u_n)\|x-y\|$ for all $x, y \in C$ and $n \geq 1$. $T$ is said to be uniformly $L$-Lipschitzian if $\|T^n x - T^n y\| \leq L\|x-y\|$ for all $x, y \in C$ and $n \geq 1$, where $L$ is a positive constant. Denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$. $T$ is called asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that for all $x \in C$, the following inequality holds: $\|T^n x - q\| \leq (1 + u_n)\|x - q\|$ for all $q \in F(T)$ and $n \geq 1$.

It is clear from this definition that every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive.

In 1967, Diaz and Metcalf [7] gave the concept of quasi-nonexpansive mappings. In 1972, Goebel and Kirk [8] introduced the notion of asymptotically nonexpansive mappings. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Browder ([1], [2]), Goebel and Kirk [8], Liu [9], Wittmann [18], Reich [11], Shoji and Takahashi [14], Chang et al. [3] in the settings of Hilbert spaces and uniformly convex Banach spaces.

In 2001, Xu and Ori [19] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in J\}$ (here $J = \{1, 2, \ldots, N\}$) with $\{\alpha_n\}$ is a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 =& \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1, \\
x_2 =& \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2, \\
&\vdots \\
x_N =& \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N, \\
x_{N+1} =& \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_{N+1} x_{N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n \quad \forall n \geqslant 1, \tag{1.1}$$

where $T_n = T_{n(mod\ N)}$ (here the $mod\ N$ function takes values in $J$). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings in a Hilbert space.

In [20], Zhou and Chang studied the weak and strong convergence of this implicit iteration process to a common fixed point for a finite family of asymptotically nonexpansive mappings and a finite family of nonexpansive mappings. In [5], Chidume and Shahzad proved that Xu and Ori's iteration process converges strongly to a common fixed point for a finite family of nonexpansive mappings if one of the mappings is semi-compact. In 2003, Sun [16] defined an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings $\{T_i : i \in J\}$ with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$, as follows:

$$
\begin{aligned}
x_1 =& \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\
x_2 =& \alpha_2 x_1 + (1-\alpha_2) T_2 x_2, \\
& \vdots \\
x_N =& \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\
x_{N+1} =& \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\
& \vdots \\
x_{2N} =& \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_N, \\
x_{2N+1} =& \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\
& \vdots
\end{aligned}
$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n \quad \forall n \geqslant 1, \tag{1.2}$$

where $n = (k-1)N + i, i \in J$. More precisely, Sun proved the following results.

**Theorem 1.1** ([16], Theorem 3.1, p.354)**.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $\{T_i : i \in J\}$ be $N$ asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that $x_0 \in C$ and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (1.2) converges strongly to a common fixed point in $F$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

**Theorem 1.2** ([16], Theorem 3.3, p.355)**.** *Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$. Let $\{T_i : i \in J\}$ be $N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that there exists one semi-compact member $T$ in $\{T_i : i \in J\}$ and that $x_0 \in C$ and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (1.2) converges strongly to a common fixed point in $F$.*

In 2006, Rhoades and Soltuz [12] noted that the existence of $(I - tT_i^p)^{-1}$ for all $t \in (0,1)$, $i = 1, 2, \ldots N$, and all $p \geq 1$ should be assumed in order to have the iteration (1.2) well-defined. In 2007, Shahzad and Zegeye [13] introduced and studied the class of generalized asymptotically quasi-nonexpansive mappings and they proved the new strong convergence of the modified implicit iteration process (1.2) to a common fixed point of finite family of generalized asymptotically quasi-nonexpansive mappings.

Recently, Thianwan and Suantai [17] introduced a new implicit iteration process for a finite family of nonexpansive mappings. Let $X$ be a normed linear space and

$C$ a nonempty convex subset of $X$. Let $\{T_i : i \in J\}$ be a finite family of nonexpansive self-mappings of $C$ and suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \dots N$. A new implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$, $x_0 \in C$:

$$\begin{aligned}
x_1 =& \alpha_1 x_0 + \beta_1 T_1 x_0 + (1 - \alpha_1 - \beta_1) T_1 x_1, \\
x_2 =& \alpha_2 x_1 + \beta_2 T_2 x_1 + (1 - \alpha_2 - \beta_2) T_2 x_2, \\
&\vdots \\
x_N =& \alpha_N x_{N-1} + \beta_N T_N x_{N-1} + (1 - \alpha_N - \beta_N) T_N x_N, \\
x_{N+1} =& \alpha_{N+1} x_N + \beta_{N+1} T_{N+1} x_N + (1 - \alpha_{N+1} - \beta_{N+1}) T_{N+1} x_{N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1 - \alpha_n - \beta_n) T_n x_n, \ \forall n \geq 1, \tag{1.3}$$

where $T_n = T_{n(mod\, N)}$.

We note that Xu and Ori's iteration is a special case of (1.3). If $\beta_n \equiv 0$, then the implicit iterative scheme (1.3) reduces to Xu and Ori's iteration (1.1). Thianwan and Suantai [17] established the following theorems:

**Theorem 1.3** ([17], Theorem 2.2, p.225)**.** *Let $X$ be a uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ nonexpansive self-mappings of $C$ with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that $\{T_i : i \in J\}$ satisfies condition $(B)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.3). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

**Theorem 1.4** ([17], Theorem 2.3, p.226 )**.** *Let $X$ be a uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ nonexpansive self-mappings of $C$ with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_i : i \in J\}$ is semi-compact. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.3). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

**Theorem 1.5** ([17], Theorem 2.6, p.227 )**.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition and let $C$ be a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ nonexpansive self-mappings of $C$ with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is* *in* $[0,1]$ *for all* $n \geq 1$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by (1.3). Then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i : i \in J\}$.

Inspired and motivated by these facts, we will extend the process (1.3) to a process for a finite family of asymptotically quasi-nonexpansive mappings with $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 =& \alpha_1 x_0 + \beta_1 T_1 x_0 + (1-\alpha_1-\beta_1)T_1 x_1,\\
x_2 =& \alpha_2 x_1 + \beta_2 T_2 x_1 + (1-\alpha_2-\beta_2)T_2 x_2,\\
&\vdots\\
x_N =& \alpha_N x_{N-1} + \beta_N T_N x_{N-1} + (1-\alpha_N-\beta_N)T_N x_N,\\
x_{N+1} =& \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_N + (1-\alpha_{N+1}-\beta_{N+1})T_1^2 x_{N+1},\\
&\vdots\\
x_{2N} =& \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N-1} + (1-\alpha_{2N}-\beta_{2N})T_N^2 x_{2N},\\
x_{2N+1} =& \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + (1-\alpha_{2N+1}-\beta_{2N+1})T_1^3 x_{2N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1-\alpha_n-\beta_n)T_i^k x_n, \ \ \forall n \geq 1, \tag{1.4}$$

where $n = (k-1)N + i, \ T_n = T_{n(mod\, N)} = T_i, \ i \in J.$

Throughout this paper, we shall assume that $(I - tT_i^p)^{-1}$ exists for all $t \in (0,1)$, $i \in J$ and all $p \geq 1$. We always suppose that the sequence $\{x_n\}$ generated by process (1.4) exists. The purpose of this paper is to study the implicit iteration process (1.4) in the general setting of a uniformly convex Banach spaces and prove the strong convergence of the process to a common fixed point, requiring one member $T$ in the family $\{T_i : i \in J\}$ which is either semi-compact or satisfies condition $(\overline{C})$. More precisely, we prove weak convergence of the implicit iteration process in a uniformly convex Banach space $X$ which satisfies Opial's condition. The results presented in this paper generalize and extend the corresponding ones announced by Thianwan and Suantai [17], Sun [16], and other authors.

Now, we recall some well known concepts and results.

Let $X$ be a Banach space with dimension $X \geq 2$. The modulus of $X$ is the function $\delta_X : (0,2] \to [0,1]$ defined by $\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\}$. Banach space $X$ is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A mapping $T : C \to C$ is called *demi-closed* with respect to $y \in X$ if for each sequence $\{x_n\}$ in $C$ and each $x \in X$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$. A Banach space $X$ is said to satisfy *Opial's condition* [10] if for any sequence $\{x_n\}$ in $X$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in C$ with $x \neq y$. We recall that a mapping $T : C \to C$ is called semi-compact(or hemicompact) if any sequence $\{x_n\}$ in $C$ satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence. A family $\{T_i : i \in J\}$ of $N$ self-mappings of $C$ with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy

(1) condition $(B)$ on $C$ [5] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that $\max\limits_{1\leqslant l\leqslant N} \{\|x - T_l x\|\} \geqslant f(d(x, F))$;

(2) condition $(\overline{C})$ on $C$ [4] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that $\{\|x - T_l x\|\} \geqslant f(d(x, F))$ for at least one $T_l$, $l = 1, 2, \ldots, N$.

Note that condition $(B)$ and condition $(\overline{C})$ are equivalent (see [4]).

In the sequel, the following lemmas are needed to prove our main results.

**Lemma 1.6** ([9]). *Let $\{a_n\}$ and $\{u_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + u_n)a_n, \ \ \forall n = 1, 2, \ldots .$$

*If $\sum_{n=1}^{\infty} u_n < \infty$, then*
*(1) $\lim_{n\to\infty} a_n$ exists .*
*(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.*

**Lemma 1.7** ([6]). *Let $X$ be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

*for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.*

**Lemma 1.8** ([6]). *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space$X$. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed at zero.*

**Lemma 1.9** ([15]). *Let $X$ be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in $X$ . Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to $u$ and $v$, respectively, then $u = v$.*

# 2 Main Results

In this section, we prove weak and strong convergence of the implicit iteration process (1.4) to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. First of all, we give necessary and sufficient conditions for convergence in a Banach space.

**Theorem 2.1.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $\{T_i : i \in J\}$ be $N$ asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0, \infty)$ (i.e., $\|T_i^n x - q_i\| \leq (1 + u_{in})\|x - q_i\|$ for all $x \in C$, $q_i \in F(T_i)$, $i \in J$) such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (here $F(T_i)$ denotes the set of fixed points of $T_i$). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0, 1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1 - s)$ for some $s \in (0, 1)$ and for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F)$ denotes the distance of $x$ to set $F$, i.e., $d(x, F) = \inf_{y \in F} d(x, y)$.*

**Proof.** The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. Let $p \in F$. Then from (1.4) we obtain that

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1 - \alpha_n - \beta_n) T_i^k x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|T_i^k x_{n-1} - p\| + (1 - \alpha_n - \beta_n)\|T_i^k x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n (1 + u_{ik})\|x_{n-1} - p\| \\
&\quad + (1 - \alpha_n - \beta_n)(1 + u_{ik})\|x_n - p\| \\
&\leq (\alpha_n + \beta_n + \beta_n u_{ik})\|x_{n-1} - p\| \\
&\quad + (1 - \alpha_n - \beta_n - \beta_n u_{ik} + u_{ik})\|x_n - p\|.
\end{aligned}$$

Transposing and simplifying above inequality, and noticing that $0 < s < \alpha_n + \beta_n < 1 - s < 1$, we have

$$\begin{aligned}
(\alpha_n + \beta_n + \beta_n u_{ik})\|x_n - p\| &\leq (\alpha_n + \beta_n + \beta_n u_{ik})\|x_{n-1} - p\| + u_{ik}\|x_n - p\| \\
&\leq (\alpha_n + \beta_n + \beta_n u_{ik})\|x_{n-1} - p\| \\
&\quad + u_{ik}\left(\frac{\alpha_n + \beta_n + \beta_n u_{ik}}{s}\right)\|x_n - p\|.
\end{aligned}$$

This implies that

$$\frac{s - u_{ik}}{s}\|x_n - p\| \leq \|x_{n-1} - p\|. \tag{2.1}$$

Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in J$, we have that $\lim_{k\to\infty} u_{ik} = 0$ and hence there exists a natural number $n_0$, as $k > n_0/N + 1$, i.e., $n > n_0$ such that $s - u_{ik} > 0$ and $u_{ik} < \frac{s}{2}$. Thus, from (2.1), for $n > n_0$, we obtain that

$$\|x_n - p\| \leq \frac{s}{s - u_{ik}}\|x_{n-1} - p\|. \tag{2.2}$$

Let $1+v_{ik}=\frac{s}{s-u_{ik}}=1+\frac{u_{ik}}{s-u_{ik}}$. Then we see that $v_{ik}=(\frac{1}{s-u_{ik}})u_{ik}<\frac{2}{s}u_{ik}$ and that $\sum_{k=1}^{\infty}v_{ik}<\frac{2}{s}\sum_{k=1}^{\infty}u_{ik}<\infty$ for all $i\in J$. By (2.2), for $n>n_0$, we get that

$$\|x_n-p\| \leq (1+v_{ik})\|x_{n-1}-p\|. \tag{2.3}$$

Thus, by Lemma 1.6 we have that $\lim_{n\to\infty}\|x_n-p\|$ exists. Since, for $n>n_0$, $d(x_n,F)\leq(1+v_{ik})d(x_{n-1},F)$ and by assumption $\liminf_{n\to\infty}d(x_n,F)=0$ we conclude that $\lim_{n\to\infty}d(x_n,F)=0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. Notice that when $x>0$, $1+x\leq e^x$, from (2.3) for any $p\in F$ we have

$$\begin{aligned}\|x_{n+m}-p\| &\leq \exp\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}v_{ik}\}\|x_n-p\|\\ &< M\|x_n-p\| \end{aligned} \tag{2.4}$$

for all natural number $m,n$, where $M=\exp\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}v_{ik}\}+1<\infty$. Since $\lim_{n\to\infty}d(x_n,F)=0$, given any $\epsilon>0$, there exists a natural number $n_1$ such that $d(x_n,F)<\frac{\epsilon}{2M}$ for all $n\geq n_1$. So we can find $p_1\in F$ such that $\|x_{n_1}-p_1\|\leq d(x_{n_1},p_1)\leq\frac{\epsilon}{2M}$ by the definition of $d(x_n,F)$. By (2.4), for all $n\geq n_1$ and $m\geq 1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-p_1\|+\|x_n-p_1\|\\ &< M\|x_{n_1}-p_1\|+M\|x_{n_1}-p_1\|\\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since $X$ is complete. Let $\lim_{n\to\infty}x_n=p$. Then since $C$ is closed, $p\in C$. It remains to show that $p\in F$. Notice that

$$|d(p,F)-d(x_n,F)|\leq\|p-x_n\|$$

for all $n$. Thus, since $\lim_{n\to\infty}x_n=p$ and $\lim_{n\to\infty}d(x_n,F)=0$, we obtain that $p\in F$. This completes the proof. $\square$

The following Corollary follows from Theorem 2.1.

**Corollary 2.2.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $\{T_i : i\in J\}$ be $N$ asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in}\in[0,\infty)$ such that $\sum_{n=1}^{\infty}u_{in}<\infty$ for all $i\in J$ and $F=\bigcap_{i=1}^{N}F(T_i)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that $\alpha_n,\beta_n,\alpha_n+\beta_n$ are in $(s,1-s)$ for some $s\in(0,1)$ and for all $n\geq 1$. From an arbitrary $x_0\in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to a common fixed point $p\in F$ if and only if there exists some infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p$.*

The main purpose of this paper is to prove strong and weak convergent results for the process (1.4). In order to prove our main results, the following lemma is needed.

**Lemma 2.3.** *Let $X$ be a uniformly convex Banach space and $C$ a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0, 1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1 - s)$ for some $s \in (0, 1)$ and for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in J$.*

**Proof.** We have $\{x_n\}$ is bounded (see proof Theorem 2.1). So, there exists $r > 0$ such that $\{x_n\} \subset B(0, r)$ for all $n \geq 1$, where $B(0, r)$ is the closed ball of $X$ with center zero and radius $r$. By Lemma 1.7, we get for any $q \in F$

$$\begin{aligned}
\|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + (1 - \alpha_n - \beta_n) T_i^k x_n - q\|^2 \\
&= \|\alpha_n (x_{n-1} - q) + \beta_n (T_i^k x_{n-1} - q) + (1 - \alpha_n - \beta_n)(T_i^k x_n - q)\|^2 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \|T_i^k x_{n-1} - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)\|T_i^k x_n - q\|^2 - \alpha_n \beta_n g(\|T_i^k x_{n-1} - x_{n-1}\|) \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
\|x_n - q\|^2 &\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \|T_i^k x_{n-1} - q\|^2 + (1 - \alpha_n - \beta_n)\|T_i^k x_n - q\|^2 \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n) g(\|T_i^k x_n - x_{n-1}\|), \qquad (2.6)
\end{aligned}$$

where $\|T_i^k x_{n-1} - x_{n-1}\| = \|T_n^k x_{n-1} - x_{n-1}\|$ and $\|T_i^k x_n - x_{n-1}\| = \|T_n^k x_n - x_{n-1}\|$, $n = (k-1)N + i$, $i \in J$. Since $T_n$ is asymptotically quasi-nonexpansive, it follows from (2.5) and (2.6) that

$$\begin{aligned}
\|x_n - q\|^2 &\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n (1 + u_{ik})^2 \|x_{n-1} - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)(1 + u_{ik})^2 \|x_n - q\|^2 - \alpha_n \beta_n g(\|T_i^k x_{n-1} - x_{n-1}\|) \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \|x_{n-1} - q\|^2 + \beta_n v_{ik} \|x_{n-1} - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \beta_n v_{ik} + v_{ik})\|x_n - q\|^2 - \alpha_n \beta_n g(\|T_i^k x_{n-1} - x_{n-1}\|) \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
\|x_n - q\|^2 &\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \|x_{n-1} - q\|^2 + \beta_n v_{ik} \|x_{n-1} - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \beta_n v_{ik} + v_{ik})\|x_n - q\|^2 \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n) g(\|T_i^k x_n - x_{n-1}\|), \qquad (2.8)
\end{aligned}$$

respectively, where $v_{ik} = 2u_{ik} + u_{ik}^2$. Hence $\sum_{k=1}^{\infty} v_{ik} < \infty$ for all $i \in J$. Using (2.7), (2.8) and $s < \alpha_n + \beta_n \leq \alpha_n + \beta_n + \beta_n v_{ik}$, we have

$$\begin{aligned}
(\alpha_n + \beta_n + \beta_n v_{ik})\|x_n - q\|^2 &\leq (\alpha_n + \beta_n + \beta_n v_{ik})\|x_{n-1} - q\|^2 \\
&\quad + v_{ik} \frac{(\alpha_n + \beta_n + \beta_n v_{ik})}{s} \|x_n - q\|^2 \\
&\quad - \alpha_n \beta_n g(\|T_i^k x_{n-1} - x_{n-1}\|) \qquad (2.9)
\end{aligned}$$

and

$$
\begin{aligned}
(\alpha_n+\beta_n+\beta_n v_{ik})\|x_n-q\|^2 \leq (\alpha_n+\beta_n+\beta_n v_{ik})\|x_{n-1}-q\|^2 \\
+ v_{ik}\frac{(\alpha_n+\beta_n+\beta_n v_{ik})}{s}\|x_n-q\|^2 \\
- \alpha_n(1-\alpha_n-\beta_n)g(\|T_i^k x_n - x_{n-1}\|). \qquad (2.10)
\end{aligned}
$$

Hence $\|x_n-q\|^2 \leq \|x_{n-1}-q\|^2 + \frac{v_{ik}}{s}\|x_n-q\|^2$. Therefore, as in Theorem 2.1, we can show that $\lim_{n\to\infty}\|x_n-q\|^2$ exists and let $\lim_{n\to\infty}\|x_n-q\|^2 = d$. From (2.9), (2.10), $s < \alpha_n \leq \alpha_n+\beta_n < 1-s$ and $s < \beta_n$ for all $n \in N$, we obtain two important inequalities:

$$
\begin{aligned}
\frac{s^2}{1-s+\beta_n v_{ik}} g(\|T_i^k x_{n-1} - x_{n-1}\|) < \frac{\alpha_n\beta_n}{\alpha_n+\beta_n+\beta_n v_{ik}} g(\|T_i^k x_{n-1} - x_{n-1}\|) \\
\leq \|x_{n-1}-q\|^2 - \|x_n-q\|^2 + \frac{v_{ik}}{s}\|x_n-q\|^2 \qquad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{s^2}{1-s+\beta_n v_{ik}} g(\|T_i^k x_n - x_{n-1}\|) < \frac{\alpha_n(1-\alpha_n-\beta_n)}{\alpha_n+\beta_n+\beta_n v_{ik}} g(\|T_i^k x_n - x_{n-1}\|) \\
\leq \|x_{n-1}-q\|^2 - \|x_n-q\|^2 + \frac{v_{ik}}{s}\|x_n-q\|^2. \qquad (2.12)
\end{aligned}
$$

Hence, by (2.11) and (2.12), we have

$$
g(\|T_i^k x_{n-1} - x_{n-1}\|) \leq \frac{1-s+\beta_n v_{ik}}{s^2}(\|x_{n-1}-q\|^2 - \|x_n-q\| + \frac{v_{ik}}{s}\|x_n-q\|^2) \qquad (2.13)
$$

and

$$
g(\|T_i^k x_n - x_{n-1}\|) \leq \frac{1-s+\beta_n v_{ik}}{s^2}(\|x_{n-1}-q\|^2 - \|x_n-q\| + \frac{v_{ik}}{s}\|x_n-q\|^2). \qquad (2.14)
$$

Since $\sum_{k=1}^{\infty} v_{ik} < \infty$, inequalities (2.13) and (2.14) become

$$
g(\|T_i^k x_{n-1} - x_{n-1}\|) \leq \frac{1-s+K}{s^2}(\|x_{n-1}-q\|^2 - \|x_n-q\|^2 + \frac{v_{ik}}{s}\|x_n-q\|^2) \qquad (2.15)
$$

and

$$
g(\|T_i^k x_n - x_{n-1}\|) \leq \frac{1-s+K}{s^2}(\|x_{n-1}-q\|^2 - \|x_n-q\|^2 + \frac{v_{ik}}{s}\|x_n-q\|^2) \qquad (2.16)
$$

for some constant $K > 0$. Hence, by (2.15) and (2.16), we have

$$\sum_{n=1}^{m} g(\|T_i^k x_{n-1} - x_{n-1}\|) \leq \frac{1-s+K}{s^2} \sum_{n=1}^{m} (\|x_{n-1} - q\|^2 - \|x_n - q\|^2 + v_{ik} M) \tag{2.17}$$

and

$$\sum_{n=1}^{m} g(\|T_i^k x_n - x_{n-1}\|) \leq \frac{1-s+K}{s^2} \sum_{n=1}^{m} (\|x_{n-1} - q\|^2 - \|x_n - q\|^2 + v_{ik} M), \tag{2.18}$$

where $M = \frac{2r}{s} < \infty$, $r$ is the ball radius. Since $\sum_{k=1}^{\infty} v_{ik} < \infty$, by letting $m \to \infty$ in (2.17) and (2.18) we get $\sum_{n=1}^{\infty} g(\|T_i^k x_{n-1} - x_{n-1}\|) < \infty$ and $\sum_{n=1}^{\infty} g(\|T_i^k x_n - x_{n-1}\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|T_i^k x_{n-1} - x_{n-1}\|) = 0 = \lim_{n\to\infty} g(\|T_i^k x_n - x_{n-1}\|)$. Since $g$ is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \|T_n^k x_{n-1} - x_{n-1}\| = 0 = \lim_{n\to\infty} \|T_n^k x_n - x_{n-1}\|. \tag{2.19}$$

Using (2.19), we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|\beta_n (T_i^k x_{n-1} - x_{n-1}) + (1 - \alpha_n - \beta_n)(T_i^k x_n - x_{n-1})\| \\
&\leq \beta_n \|T_i^k x_{n-1} - x_{n-1}\| + (1 - \alpha_n - \beta_n)\|T_i^k x_n - x_{n-1}\| \\
&= \beta_n \|T_n^k x_{n-1} - x_{n-1}\| + (1 - \alpha_n - \beta_n)\|T_n^k x_n - x_{n-1}\| \\
&\quad \to 0 \ \text{(as } n \to \infty),
\end{aligned}$$

as well as $\lim_{n\to\infty} \|x_n - x_{n+l}\| = 0$ for all $l < 2N$. Hence for $n > N$,

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\
&\leq \sigma_n + L(\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| + \\
&\quad \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|).
\end{aligned}$$

Clearly, $n \equiv (n - N)(mod\, N)$. Thus $T_n = T_{n-N}$ and above inequality becomes

$$\|x_{n-1} - T_n x_n\| \leq \sigma_n + L^2 \|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_n - x_{(n-N)-1}\|,$$

which yields that $\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0$. From

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|,$$

it follows that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Hence for all $l \in J$

$$\begin{aligned}
\|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\
&\leq (1 + L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\|,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0$ for all $l \in J$. Since for each $l \in J$, $\{\|x_n - T_{n+l}\|\}$ is a subset of $\cup_{l=1}^{N}\{\|x_n - T_{n+l}\|\}$, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$$

for all $l \in J$. This completes the proof. □

**Theorem 2.4.** *Let $X$ be a uniformly convex Banach space and $C$ a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member $T$ in $\{T_i : i \in J\}$ which is either semi-compact or satisfies condition $(\overline{C})$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$ and for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

**Proof.** We may assume that $T_1$ is either semi-compact or satisfies condition $(\overline{C})$ without loss of generality. By Lemma 2.3, we have $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$. If $T_1$ is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$ as $j \to \infty$. Now Lemma 2.3 guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l \in J$ and so $\|x^* - T_l x^*\| = 0$ for all $l \in J$. This implies that $x^* \in F$. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, Theorem 2.1 guarantees that $\{x_n\}$ converges strongly to some common fixed point in $F$. If $T_1$ satisfies condition $(\overline{C})$, then we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Now apply Theorem 2.1. This completes the proof. □

The following result is directly obtained by Theorem 2.4.

**Theorem 2.5.** *Let $X$ be a uniformly convex Banach space and $C$ a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ asymptotically nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member $T$ in $\{T_i : i \in J\}$ which is either semi-compact or satisfies condition $(\overline{C})$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$ and for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

In the next result, we prove weak convergence of the sequence $\{x_n\}$ defined by (1.4) in a uniformly convex Banach space satisfying *Opial's condition*.

**Theorem 2.6.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition and $C$ a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$ and for all $n \geq 1$. Suppose that $I - T_i$ for all $i \in J$ are demi-closed at $0$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

**Proof.** By Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in J$. Since $X$ is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to x^*$ weakly as $n \to \infty$, with out loss of generality. Since $I - T_i$ for all $i \in J$ are demi-closed at 0, we have $x^* \in F(T_i)$ for all $i \in J$. Hence $x^* \in F$. Suppose that there exist subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to $y^*$ and $z^*$, respectively. Again, as above, we can prove that $y^*, z^* \in F$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ (see proof Theorem 2.1), we have $\lim_{n\to\infty}\|x_n - y^*\|$ and $\lim_{n\to\infty}\|x_n - z^*\|$ exists. It follows from Lemma 1.9 that $y^* = z^*$. Therefore $\{x_n\}$ converges weakly to a common fixed point $x^*$ in $F$. □

Since every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian and asymptotically quasi-nonexpansive, so with the help of Lemma 1.8, the following result is directly obtained by Theorem 2.6.

**Theorem 2.7.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition and $C$ a nonempty closed convex subset of $X$. Let $\{T_i : i \in J\}$ be $N$ asymptotically nonexpansive self-mappings of $C$ with $u_{in} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in J$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that $\alpha_n, \beta_n, \alpha_n + \beta_n$ are in $(s, 1-s)$ for some $s \in (0,1)$ and for all $n \geq 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

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