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A Common Fixed Point Theorem for a Pair of Nonself Multi-valued Mappings

M. Imdad and Ladlay Khan

Abstract : A common fixed point theorem for a pair of nonself multi-valued mappings in complete metrically convex metric spaces is proved which generalizes some earlier known results due to Khan et al. [9], Bianchini [2], Chatterjea [3], Khan et al. [10] and others. An illustrative example is also discussed.

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1 Introduction

The study of fixed point theorems for nonself multi-valued contractions on metrically convex metric spaces was initiated by Assad and Kirk [1]. In recent years, several fixed point theorems for such maps were proved which include relevant results due to Rhoades [12, 13], Hadžič and Gajic [4], Iséki [5], Itoh [6], Khan [8] and others.

The purpose of this paper is to extend a fixed point theorem due to Khan et al. [9] proved for nonself single valued mappings to a pair of multi-valued nonself mappings. For the sake of completeness, we state Theorem 1 due to Khan et al. [9].

Theorem 1.1 *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $T : K \to X$ be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq a\ max\{d(x, Tx), d(y, Ty)\} + b\ \{d(x, Ty) + d(y, Tx)\} \qquad (1)$$

for every $x, y \in K$, where a and b are non-negative reals such that

$$\max\left\{\frac{a+b}{1-b}, \frac{b}{1-a-b}\right\} = h > 0, \max\left\{\frac{1+a+b}{1-b}h, \frac{1+b}{1-a-b}h\right\} = h',$$

and

$$\max\{h, h'\} = h'' < 1.$$

Further, if for every $x \in \delta K$, $Tx \in K$, then T has a unique fixed point in K.

2 Preliminaries

Let (X, d) be a metric space. Then following Nadler[11], we recall

(i) $CB(X) = \Big\{A : A$ is nonempty closed and bounded subset of $X\Big\}$,

(ii) $C(X) = \Big\{A : A$ is nonempty compact subset of $X\Big\}$.

(iii) For nonempty subsets A, B of X,

$$H(A,B) = \max\{(\sup\ d(a,B) : a \in A), (\sup\ d(A,b) : b \in B)\}.$$

It is well known (cf. Kuratowski [7]) that $CB(X)$ is a metric space with the distance H which is known as Hausdorff-Pompeiu metric on X.

Before proving our main result, we collect the relevant definitions and lemmas for our subsequent discussion.

Definition 2.1 Let (X, d) be a metric space and K a nonempty subset of X. Let $F, T : K \to CB(X)$ satisfy the condition

$$H(Fx,Ty) \leq a\ \max\left\{\frac{1}{2}d(x,y), d(x,Fx), d(y,Ty)\right\} + b\ \Big\{d(x,Ty) + d(y,Fx)\Big\} \quad (2)$$

for all $x, y \in K$ with $x \neq y,\ a, b \geq 0$ such that $2a + 3b < 1$. Then F is called *generalized T-contraction mapping* on K.

Definition 2.2 A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

Lemma 2.3 ([1]) *Let K be a nonempty closed subset of a metrically convex metric space (X, d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that $d(x,z) + d(z,y) = d(x,y)$.*

Lemma 2.4 ([11]) *Let $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \epsilon$. If $A, B \in C(X)$, then one can choose $b \in B$ such that $d(a,b) \leq H(A,B)$.*

3 Main result

In an attempt to extend Theorem 1.1 for a pair of multi-valued nonself mappings, we prove the following.

Theorem 3.1 *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. If F is generalized T-contraction mapping of K into X satisfying* *(iv)* $x \in \delta K \Rightarrow Fx \subseteq K, Tx \subseteq K.$

Then there exists $z \in K$ *such that* $z \in Fz$ *and* $z \in Tz$.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Assume $\alpha = h(1+h)$, let $x_\circ \in \delta K$ and $x_1 = y_1 \in F(x_0)$. Using Lemma 2.4, one can choose $y_2 \in T(x_1)$ such that

$$d(y_1, y_2) \leq H(F(x_0), T(x_1)) + \alpha.$$

Suppose $y_2 \in K$, then set $y_2 = x_2$. In case $y_2 \notin K$ then (due to Lemma 2.1) there exists a point $x_2 \in \delta K$ such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_n \in F(x_{n-1}),$ if n is odd and

(vi) $y_n \in T(x_{n-1}),$ if n is even

(vii) $y_n \in K \Rightarrow y_n = x_n$ or $y_n \notin K \Rightarrow x_n \in \delta K$ and
$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n),$

(viii) $d(y_n, y_{n+1}) \leq H(F(x_{n-1}), T(x_n)) + \alpha^n$ if n is odd

(ix) $d(y_n, y_{n+1}) \leq H(T(x_{n-1}), F(x_n)) + \alpha^n$ if n is even.

We denote

$$P = \Big\{x_i \in \{x_n\} : x_i = y_i\Big\},\ Q = \Big\{x_i \in \{x_n\} : x_i \neq y_i\Big\}.$$

One can note that two consecutive terms cannot lie in Q.

Now, we distinguish the following three cases.

Case 1. If $x_n, x_{n+1} \in P$, then

$$\begin{aligned} d(x_n, x_{n+1}) = d(y_n, y_{n+1}) &\leq H(Fx_{n-1}, Tx_n) + \alpha^n \\ &\leq a\ \max\Big\{\frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, Fx_{n-1}), d(x_n, Tx_n)\Big\} \\ &\quad + b\ \Big\{d(x_{n-1}, Tx_n) + d(x_n, Fx_{n-1})\Big\} + \alpha^n \\ &\leq a\ \max\Big\{\frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} \\ &\quad + b\ d(x_{n-1}, x_{n+1}) + \alpha^n, \end{aligned}$$

which in turn yields

$$d(x_n, x_{n+1}) \leq \begin{cases} \left(\frac{a+b}{1-b}\right) d(x_{n-1}, x_n) + \frac{\alpha^n}{1-b}, & \text{if } d(x_{n-1}, x_n) \geq d(x_{n+1}, x_n), \\ \\ \left(\frac{b}{1-b-a}\right) d(x_{n-1}, x_n) + \frac{\alpha^n}{1-b-a}, & \text{if } d(x_{n-1}, x_n) \leq d(x_{n+1}, x_n), \end{cases}$$

or

$$d(x_n, x_{n+1}) \leq h\ d(x_{n-1}, x_n) + \max\left\{\frac{1}{1-b}, \frac{1}{1-b-a}\right\}\alpha^n,$$

or

$$d(x_n, x_{n+1}) \leq h\ d(x_{n-1}, x_n) + \frac{\alpha^n}{1-b-a},$$

where $h = \max\left\{(\frac{a+b}{1-b}), (\frac{b}{1-b-a})\right\} < 1$, since $2a + 3b < 1$.

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}),$$

or

$$d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}) = d(y_n, y_{n+1}),$$

and hence

$$d(x_n, x_{n+1}) \leq d(y_n, y_{n+1}) \leq H(Fx_{n-1}, Tx_n) + \alpha^n.$$

Now, proceeding as Case 1, one can have

$$d(x_n, x_{n+1}) \leq h\ d(x_{n-1}, x_n) + \frac{\alpha^n}{1-b-a}.$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$ then $x_{n-1} \in P$. Proceeding as in Case 1, one gets

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, y_{n+1}) &\leq d(x_n, y_n) + d(y_n, y_{n+1}) \\ &\leq d(x_n, y_n) + a\ \max\left\{\frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, Fx_{n-1}), d(x_n, Tx_n)\right\} \\ &\quad + b\ \left\{d(x_{n-1}, Tx_n) + d(x_n, Fx_{n-1})\right\} + \alpha^n \\ &\leq d(x_n, y_n) + a\ \max\left\{\frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, y_n), d(x_n, x_{n+1})\right\} \\ &\quad + b\ \{d(x_{n-1}, x_{n+1}) + d(x_n, y_n)\} + \alpha^n, \end{aligned}$$

which in turn yields

$$d(x_n, x_{n+1}) \leq \begin{cases} \left(\frac{1+a+b}{1-b}\right) d(x_{n-1}, y_n) + \frac{\alpha^n}{1-b}, & \text{if } d(x_{n-1}, y_n) \geq d(x_{n+1}, x_n), \\ \left(\frac{1+b}{1-b-a}\right) d(x_{n-1}, y_n) + \frac{\alpha^n}{1-b-a}, & \text{if } d(x_{n-1}, y_n) \leq d(x_{n+1}, x_n). \end{cases}$$

Now, proceeding as earlier, one also obtains

$$d(x_{n-1}, y_n) \leq \begin{cases} \left(\frac{a+b}{1-b}\right) d(x_{n-1}, x_{n-2}) + \frac{\alpha^{n-1}}{1-b}, & \text{if } d(x_{n-1}, x_{n-2}) \geq d(x_{n-1}, y_n), \\ \left(\frac{b}{1-b-a}\right) d(x_{n-1}, x_{n-2}) + \frac{\alpha^{n-1}}{1-b-a}, & \text{if } d(x_{n-1}, x_{n-2}) \leq d(x_{n-1}, y_n). \end{cases}$$

Therefore combining above inequalities, we have

$$d(x_n, x_{n+1}) \leq k\ d(x_{n-1}, x_{n-2}) + \frac{\alpha^{n-1}}{1-b-a} + \frac{\alpha^n}{1-b-a}, \text{ as } k \leq 2a + 3b < 1.$$

Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq \begin{cases} h\ d(x_n, x_{n-1}) + \frac{\alpha^n}{1-b-a} \text{ or} \\ \\ k\ d(x_{n-2}, x_{n-1}) + \frac{\alpha^{n-1}}{1-b-a} + \frac{\alpha^n}{1-b-a}. \end{cases}$$

Now, on the lines of Itoh[6], it can be shown that $\{x_n\}$ is Cauchy and hence converges to a point $z \in K$. Then as noted in [4], there exists at least one subsequence $\{x_{n_k}\}$ which is contained in P and converges to some $z \in K$. Now, using (2.1.1), one can write

$$\begin{aligned} d(x_{n_k}, Fz) &\leq H(Tx_{n_k-1}, Fz) \\ &\leq a\ \max\left\{\frac{1}{2}d(x_{n_k-1}, z), d(x_{n_k-1}, Tx_{n_k-1}), d(z, Fz)\right\} \\ &+ b\left\{d(x_{n_k-1}, Fz) + d(z, Tx_{n_k-1})\right\}, \end{aligned}$$

which on letting $k \to \infty$ reduces to

$$d(z, Fz) \leq a\ max\{0, 0, d(z, Fz)\} + b\ d(z, Fz),$$

yielding thereby $z \in Fz$ which shows that z is a fixed point of F. Similarly, one can show that $z \in Tz$. This completes the proof. □

Remark 3.2 By choosing $F = T$ in the Theorem 3.1, one deduces a multi-valued analogue of Theorem 1 due to Khan et al. [9] and Theorem 1 due to Khan et al. [10].

Remark 3.3 By setting $F = T$ and $b = 0$ in Theorem 3.1, one obtains a result which can be realized as a multi-valued analogue of a result due to Bianchini [2] to nonself multi-valued mappings in metrically convex spaces.

Remark 3.4 Similarly, by restricting $F = T$ and $a = 0$ in Theorem 3.1, one deduces a result which can be realized as a multi-valued analogue of a result due to Chatterjea [3] to nonself multi-valued mappings in metrically convex spaces.

The following theorem is naturally predictable.

Theorem 3.5 *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $F, T : K \to C(X)$ be a pair of maps which satisfy (2) and (iv). Then there exists $z \in K$ such that $z \in Fz \cap Tz$.*

4 An Illustrative Example

Since every single valued mapping can always be realized as a multi-valued mapping, therefore we adapt the following example to demonstrate Theorem 3.1.

Example 4.1 Consider $X = \mathbb{R}$ equipped with the natural distance and $K = [0, 3]$. Define $F, T : K \to CB(X)$ by

$$Fx = \begin{cases} \{\frac{-x}{8}\}, & \text{if } 0 < x \leq 2, \\ \{0\}, & \text{if } x \in (2, 3] \cup \{0\}, \end{cases}$$

and

$$Tx = \begin{cases} \{\frac{-x}{12}\}, & \text{if } 0 < x \leq 2, \\ \{0\}, & \text{if } x \in (2, 3] \cup \{0\}. \end{cases}$$

Note that for boundary points '0' and '3' satisfy the required condition (iv) of Theorem 3.1 because,

$$0 \in \delta K \Rightarrow F0 = \{0\} \subseteq K, T0 = \{0\} \subseteq K,$$

$$3 \in \delta K \Rightarrow F3 = \{0\} \subseteq K, T3 = \{0\} \subseteq K.$$

Moreover, for the verification of contraction condition (2.1.1), the following cases arise :
Case 1. If $x, y \in (0, 2]$, then

$$\begin{aligned} H(Fx, Ty) = d(Fx, Ty) &= \left| \frac{-x}{8} + \frac{y}{12} \right| = \frac{1}{24}|3x - 2y| = \frac{1}{24}|2x + x - 2y| \\ &= \frac{1}{24}|2x - 2y + x| = \frac{1}{24}\Big[2\max\Big\{|2x - 2y|, |x|\Big\}\Big] = \frac{1}{12}\max\Big\{|2x - 2y|, |x|\Big\} \\ &= \max\left\{\frac{1}{6}|x - y|, \frac{1}{12}|x|\right\} \leq \max\left[\frac{1}{3}\left\{\frac{1}{2}|x - y|\right\}, \frac{1}{3}\left(\frac{9}{8}|x|\right)\right] \\ &\leq \frac{1}{3}\max\left\{\frac{1}{2}d(x, y), d(x, Fx), d(y, Ty)\right\} + b\Big\{d(x, Ty) + d(y, Fx)\Big\}. \end{aligned}$$

Case 2. If $0 < x \leq 2$ and $y \in (2, 3] \cup \{0\}$, then

$$\begin{aligned} H(Fx, Ty) = d(Fx, Ty) &= \left| \frac{-x}{8} - 0 \right| = \frac{1}{8}|x| = \frac{1}{9}\left(\frac{9}{8}|x|\right) < \frac{1}{3}\left(\frac{9}{8}|x|\right) \\ &< \frac{1}{3}\max\left\{\frac{1}{2}d(x, y), d(x, Fx), d(y, Ty)\right\} + b\Big\{d(x, Ty) + d(y, Fx)\Big\}. \end{aligned}$$

Thus the contraction condition (2.1.1) is satisfied for $a = \frac{1}{3}$ and $0 < b < \frac{1}{9}$ which completes the verification of all the conditions of the Theorem 3.1. Note that '0' is the common fixed point of (F, T).

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M. Imdad and Ladlay Khan
Department of Mathematics
Aligarh Muslim University
Aligarh 202 002, India.
e-mail : mhimdad@yahoo.co.in, k_ladlay@yahoo.com