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Endo-Regularity of Cycle Book Graphs

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Abstract : A graph G is endo-regular (endo-completely-regular, endo-orthodox) if the monoid of all endomorphisms on G is regular (completely regular, orthodox). In this paper, we characterized endo-regular (endo-completely-regular, endo-orthodox) cycle book graphs.

Keywords : Cycle book graph, Endomorphism, Regular, Complectly regular, Orthodox.
2000 Mathematics Subject Classification : 05C25; 05C38

1 Introduction and Preliminaries

In [2], W. Li characterized regular endomorphisms on arbitrary graphs. The characterizations of endo-regular and endo-orthodox connected bipartite graphs were explicitly found in [6] and [1], respectively. A characterization of endo-completely regular of even cycles in found in [4]. And in [5], J. Thamkeaw and Sr. Arworn showed that every odd cycle book graphs are endo-regular.

As usual we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph G, respectively, where $V(G) \neq \emptyset$ and $E(G) \subseteq \{\{u, v\} | \ u \neq v \ in \ V(G)\}$. The *distance* between u and v, $d(u, v)$ is the smallest length of $u - v$ path in G. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by $diam(G)$. The graph with vertex set $\{0, 1, \ldots, n\}$ and edge set $\{\{0, 1\}, \{1, 2\}, \ldots, \{n-1, n\}\}$ is called a path of length n, denoted by P_n. Therefore, the path P_n has $n+1$ vertices and n edges. The graph with vertex set $\{0, 1, ..., n-1\}$, such that $n \geq 3$, and edge set $\{\{i, i+1\} | i = 0, 1, ..., n-1\}$ (with addition modulo n) is called a cycle of length n, denoted by C_n. Therefore, the cycle C_n has n vertices and n edges.

A (*graph*) *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e. $\forall u, v \in V(G)$, $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is also a homomorphism. A homomorphism (resp. isomorphism) f from G to itself is called an *endomorphism* (resp. *automorphism*) of G. The sets of all endomorphisms and automorphisms of G are denoted by $End(G)$ and $Aut(G)$, respectively.

An element a of a semigroup S (or a monoid S) is called an *idempotent* of S if $a^2 = a$. A *regular element* of S is an element $a \in S$ such that $a = aa'a$ for some $a' \in S$, such a' is called a *pseudo inverse* to a. A semigroup S is called *regular* if every element of S is regular. An element $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$ is called an *inverse* to a. A *regular element* a of S is called *completely regular* if there exists a pseudo inverse a' to a such that $aa' = a'a$. In this case we call a' a *commuting pseudo inverse* to a. A semigroup S is called *completely regular* if every element of S is completely regular. A regular semigroup S is called *orthodox* if the set of all idempotent elements of S ($Idpt(S)$) forms a semigroup under the operation of S.

We called a graph G *endo-regular* (*endo-completely-regular*, *endo-orthodox*), if the monoid $End(G)$ is regular (completely regular, orthodox).

The following lemmas are useful for this paper.

Lemma 1.1. [6] *Let G be a connected bipartite graph. Then G is endo-regular if and only if G is one of following graphs:*

1. *completely bipartite graph $K_{m,n}$, (including K_1, K_2, cycle C_4 and tree T with $diam(T) = 2$).*
2. *tree T with $diam(T) = 3$,*
3. *cycle C_6 and C_8,*
4. *path with 5 vertices, i.e. P_4.*

Lemma 1.2. [5] *Every odd cycle book graph is endo-regular.*

Lemma 1.3. [4] *Every even cycle is not endo-completely-regular.*

Lemma 1.4. [3] *A semigroup S is completely regular if and only if S is a union of (disjoint) groups.*

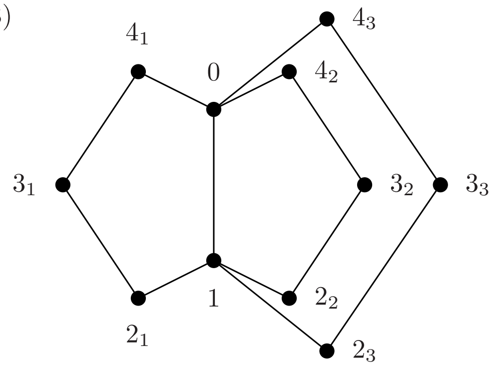
Lemma 1.5. [1] *Let G be a bipartite graph. Then G is endo-orthodox if and only if G is one of the following graphs*: $K_1, K_2, P_2, P_3, C_4, 2K_1$ *and* $K_1 \cup K_2$.

2 Endo-Regularity of Cycle Book Graphs

For each $i = 1, 2, \ldots, m$, let G_i be a graph which isomorphic to cycle C_n with the following vertex set $V(G_i) = \{0_i, 1_i, 2_i, \ldots, (n-1)_i\}$, and edge set $E(G_i) = \{\{x_i, (x+1)_i\} | x = 0, 1, 2, \ldots, n-1\}$ where $0_i = 0, 1_i = 1$ for all $i = 1, \ldots, m$ and $+$ is the addition modulo n. (Note that for all $i \neq j, V(G_i) \cap V(G_j) = \{0, 1\}$).

Let $B_n(m)$ be a C_n book graph of m page with the vertex set $V(B_n(m)) = \bigcup_{i=1}^{m} V(G_i)$ and the set $E(B_n(m)) = \bigcup_{i=1}^{m} E(G_i)$.

Example $B_5(3)$



This section is the main results, the characterization of endo-regular, endo-completely-regular and endo-orthodox of cycle book graphs.

Theorem 2.1. *A cycle book graph is endo-regular if and only if it is an odd cycle book graph or one page cycle C_4, C_6, or C_8 book graph.*

Proof. For any cycle book graph $B_n(m)$:

1. If n is odd, then by Lemma 1.2, $B_n(m)$ is a endo-regular.
2. If n is even, then $B_n(m)$ is bipartite graph. From Lemma 1.1, $B_n(m)$ is an endo-regular if and only if $B_n(m)$ is C_4, C_6, or C_8.

□

From $K_3 \cong B_3(1)$ and $End(K_3)$ is a group, then $B_3(1)$ is endo-completely-regular and endo-orthodox. Next, we will characterized the endo-completely-regular and endo-orthodox cycle book graphs.

Lemma 2.2. *For even cycle book graphs,*

1. *$B_{2n}(m)$ is not endo-completely-regular for all positive integers m, n, $n \geq 2$.*
2. *$B_{2n}(m)$ is endo-orthodox if and only if $n = 2$ and $m = 1$.*

Proof. (1) From Theorem 2.1, $B_{2n}(m)$ is endo-regular if and only if $B_{2n}(m)$ is a cycle C_{2n} where $n \in \{2, 3, 4\}$. But from Lemma 1.3, those C_{2n} is not endo-completely-regular.

(2) Since $B_{2n}(m)$ are bipartite graphs and from Lemma 1.5, $B_{2n}(m)$ is endo-orthodox if and only if $B_{2n}(m)$ is a cycle C_4, i.e, $n = 2$ and $m = 1$. □

For every one page odd cycle book graph $B_{2n+1}(1)$, they are isomorphic to C_{2n+1}. Then $End(B_{2n+1}(1))$ forms a group for all positive intergers n. Therefore, all $B_{2n+1}(1)$ are endo-completely-regular and endo-orthodox.

For $B_{2n+1}(2)$:

Lemma 2.3. *The endomorphism monoid of any odd cycle book graph $B_{2n+1}(2)$ is a union of (disjoint) groups, which is isomorphic to $(S_2 \times S_2) \cup G_1 \cup G_2$ when $G_1 \cong G_2 \cong Aut(C_{2n+1})$ and S_2 is the permutation group of order 2.*

Proof. Consider the following subsets of $End(B_3(2))$: Let $S = \left\{ \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 1 & 2_1 & 2_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 1 & 2_2 & 2_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 0 & 2_1 & 2_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 0 & 2_2 & 2_1 \end{pmatrix} \right\}$.

Let $G_1 = \left\{ \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 1 & 2_1 & 2_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 2_1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 2_1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 2_1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 2_1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 0 & 2_1 & 2_1 \end{pmatrix} \right\}$.

And $G_2 = \left\{ \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 1 & 2_2 & 2_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 2_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 2_2 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 0 & 2_2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 2_2 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2_1 & 2_2 \\ 1 & 0 & 2_2 & 2_2 \end{pmatrix} \right\}$.

We found that those subset S, G_1, G_2 form groups with the composition, and $S \cong S_2 \times S_2$, and $G_1 \cong G_2 \cong Aut(C_3)$. Moreover, $End(B_3(2))$ is isomorphic to the monoid of the union of those groups, $S \cup G_1 \cup G_2$. Therefore, $End(B_3(2)) \cong S_2 \times S_2 \cup G_1 \cup G_2$.
Then by Lemma 1.4, $B_3(2)$ is endo-completely-regular.

For the other cycle book graphs $B_{2n+1}(2)$ where $n > 1$, we can define the subsets S, G_1, G_2 of the monoid $End(B_{2n+1}(2)$ by the same way as we did for the monoid of the book graph $B_3(2)$. Then $End(B_{2n+1}(2)) \cong S_2 \times S_2 \cup G_1 \cup G_2$.

□

Corollary 2.4. *Every odd cycle book graph of two pages $B_{2n+1}(2)$ is endo-completely-regular.*

Corollary 2.5. *Every odd cycle book graph of two pages $B_{2n+1}(2)$ is endo-orthodox.*

Proof. Form Lemma 2.3, $End(B_{2n+1}(2)) \cong S \cup G_1 \cup G_2$, where $G_1 \cong G_2 \cong Aut(C_{2n+1})$ and $S \cong S_2 \times S_2$. Then there are only three idempotents in $End(B_{2n+1}(2)$ which are the identities elements of the group S, G_1, G_2, say i, i_1, i_2, respectively, i.e.

$$Idpt(End(B_{2n+1}(2))) = \Big\{ i = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 \end{pmatrix},$$
$$i_1 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_1 & 3_1 & \dots & 2n_1 \end{pmatrix},$$
$$i_2 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 \\ 0 & 1 & 2_2 & 3_2 & \dots & 2n_2 & 2_2 & 3_2 & \dots & 2n_2 \end{pmatrix} \Big\}.$$

Therefore $Idpt(B_{2n+1}(2) = \{i, i_1, i_2\}$ and the table of composite:

$\circ$	i	i_1	i_2
i	i	i_1	i_2
i_1	i_1	i_1	i_1
i_2	i_2	i_2	i_2

forms a monoid. □

For $B_{2n+1}(m)$, $m \geq 3$,

Lemma 2.6. *For any positive integer $m, n, m \geq 3$,*

1. *there exists $f \in End(B_{2n+1}(m))$ such that f is not completely regular, and*
2. *there exist idempotents $g, h \in End(B_{2n+1}(m))$ such that gh is not idempotent.*

Proof. Consider $End(B_3(3))$,

(1) Let $f \in End(B_{2n+1}(3))$ be such that
$f = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 \end{pmatrix}.$
We will show that f is not completely regular. Since $B_{2n+1}(3)$ is endo-regular, there exists pseudo-inverse $g \in End(B_{2n+1}(3))$ of f.
Then g is one of the following forms
$g_1 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_3 & 3_3 & \dots & 2n_3 & 2_i & 3_i & \dots & 2n_i \end{pmatrix},$
or
$g_2 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 & 2_i & 3_i & \dots & 2n_i \end{pmatrix},$
where $i \in \{1, 2, 3\}$.
Consider
$fg_1 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & f(2_i) & f(3_i) & \dots & f(2n_i) \end{pmatrix},$
$g_1 f = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_1 & 3_1 & \dots & 2n_1 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix},$ and
$fg_2 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & f(2_i) & f(3_i) & \dots & f(2n_i) \end{pmatrix},$
$g_2 f = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_2 & 3_2 & \dots & 2n_2 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix}.$
Thus, for any cases $fg \neq gf$, i.e. f is not completely regular element.

(2) Let $g, h \in End(B_{2n+1}(3))$ such that
$g = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_1 & 3_1 & \dots & 2n_1 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix}$ and
$h = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_3 & 3_3 & \dots & 2n_3 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix}.$
Thus $g^2 = g$ and $h^2 = h$.
But $gh = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_3 & 3_3 & \dots & 2n_3 & 2_1 & 3_1 & \dots & 2n_1 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix}.$
$(gh)^2 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & \dots & 2n_1 & 2_2 & 3_2 & \dots & 2n_2 & 2_3 & 3_3 & \dots & 2n_3 \\ 0 & 1 & 2_3 & 3_3 & \dots & 2n_3 & 2_3 & 3_3 & \dots & 2n_3 & 2_3 & 3_3 & \dots & 2n_3 \end{pmatrix}.$
Thus $(gh)^2 \neq gh$. □

Corollary 2.7. *For any positive integers $m, n, m \geq 3$, $B_{2n+1}(3)$ is not neither endo-completely-regular nor endo-orthodox.*

Theorem 2.8. *The following statements are true:*

1. *A cycle book graph is endo-completely-regular if and only if it is an odd cycle book graph $B_{2k+1}(m)$ where $m \leq 2$.*
2. *A cycle book graph is endo-orthodox if and only if it is $B_4(1)$, or odd cycle book $B_{2k+1}(m)$ where $m \leq 2$.*

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