



One Dimensional Scattering Problems: A Pedagogical Presentation of the Relationship between Reflection and Transmission Amplitudes

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Abstract : In this article we provide a pedagogical introduction to scattering theory in one space dimension. This is an elegant topic that is mathematically simple and physically transparent. We shall apply the Schrödinger equation to a generic system to identify the Bogoliubov coefficients. Furthermore, we shall then derive a number of significant relationships between reflection and transmission amplitudes.

Keywords : Schrödinger equation, Scattering problems, Bogoliubov coefficients, Reflection and transmission amplitudes.

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1 Introduction

In this article we shall present a simple pedagogical introduction to quantum scattering theory in one space dimension. This is a beautiful subject that is mathematically simple and physically transparent. Moreover, it still leads to important and significant novel results [1, 2, 3, 4, 5, 6, 7].

One-dimensional scattering problems appear in a vast variety of physical contexts, textbook presentations [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], and research monographs [24, 25, 26, 27, 28, 29, 30, 31]. For instance, in acoustics one might be interested in the propagation of sound waves down a long pipe, while in electromagnetism one might be interested in the physics of wave-guides. Another important context which we want to stress in this arti-

cle is that in quantum physics the canonical examples related to one-dimensional scattering theory are barrier penetration and reflection. In contrast, in classical physics an equivalent problem is the analysis of parametric resonances [1]. When considering the basic ideas of “reflection and transmission amplitudes”, we shall introduce a useful technique to derive a connection between reflection and transmission coefficients, showing that they are related via a conceptually simple formalism. This technique has been used multiple times in several recent related articles [1, 2, 3, 4, 5, 6, 7].

In particular, at the end of this article we shall illustrate how to derive the “transfer matrix” in terms of the transmission and reflection amplitudes due to scattering by a finite-width potential well. Specifically, we are interested in the Schrödinger equation as shown below in equation (2.1) in conditions where the potential $V(x)$ is zero outside of a finite interval. Purely for mathematical convenience we are most interested in considering potentials of compact support. (Though much of what we will have to say will also apply to potentials with suitably rapid falloff properties as one moves to spatial infinity.)

2 Reflection and Transmission Amplitudes

Let us consider the one-dimensional time-independent Schrödinger equation [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x). \quad (2.1)$$

If the potential asymptotes to a constant,

$$V(x \rightarrow \pm\infty) \rightarrow V_{\pm\infty}, \quad (2.2)$$

then in each of the two asymptotic regions there are two independent solutions to the Schrödinger equation

$$\psi_{\pm}(x \rightarrow \pm\infty) \approx \frac{\exp(\pm ik_{\pm\infty}x)}{\sqrt{k_{\pm\infty}}}. \quad (2.3)$$

Here the \pm distinguishes right-moving modes e^{+ikx} from left-moving modes e^{-ikx} , while the $\pm\infty$ specifies which of the asymptotic regions we are in. Furthermore

$$k_{\pm\infty} = \sqrt{\frac{2m(E - V_{\pm\infty})}{\hbar}}. \quad (2.4)$$

To even begin to set up a scattering problem the minimum requirements are that potential asymptote to some constant, and this assumption will be made henceforth. The so-called Jost solutions (see for example [26]) are exact solutions $\mathcal{J}_{\pm}(x)$ of the Schrödinger equation that satisfy

$$\mathcal{J}_+(x \rightarrow -\infty) \rightarrow \frac{\exp(+ik_{-\infty}x)}{\sqrt{k_{-\infty}}}, \quad (2.5)$$

$$\mathcal{J}_+(x \rightarrow +\infty) \rightarrow \alpha_+ \frac{\exp(+ik_{+\infty}x)}{\sqrt{k_{+\infty}}} + \beta_+ \frac{\exp(-ik_{+\infty}x)}{\sqrt{k_{+\infty}}}, \quad (2.6)$$

and

$$\mathcal{J}_-(x \rightarrow +\infty) \rightarrow \frac{\exp(-ik_{+\infty}x)}{\sqrt{k_{+\infty}}}, \quad (2.7)$$

$$\mathcal{J}_-(x \rightarrow -\infty) \rightarrow \alpha_- \frac{\exp(-ik_{-\infty}x)}{\sqrt{k_{-\infty}}} + \beta_- \frac{\exp(+ik_{-\infty}x)}{\sqrt{k_{-\infty}}}. \quad (2.8)$$

There are unfortunately at least four distinct sets of conventions and formulations in common use, depending on whether or not one absorbs factors of $\sqrt{k_{\pm\infty}}$ into the reflection and transmission amplitudes r and t respectively, and on whether one chooses to focus on left-moving or right-moving waves as being primary. We shall discuss three of these formulations in some detail.

2.1 Formulation 1

Let us, for the current section, adopt the convention of *not* absorbing the factors of $\sqrt{k_{\pm\infty}}$ into r and t . We start by introducing a minor variant of Messiah's notation [20]

$$\mathcal{J}_+(x \rightarrow -\infty) \rightarrow t_+ \exp(+ik_{-\infty}x), \quad (2.9)$$

$$\mathcal{J}_+(x \rightarrow +\infty) \rightarrow \exp(+ik_{+\infty}x) + r_+ \exp(-ik_{+\infty}x), \quad (2.10)$$

By comparing these two different forms for the asymptotic form of the Jost function we see that in this situation the ratios of the amplitudes are given by

$$\frac{1}{\sqrt{k_{-\infty}}} : \frac{\alpha_+}{\sqrt{k_{+\infty}}} : \frac{\beta_+}{\sqrt{k_{+\infty}}} = t_+ : 1 : r_+. \quad (2.11)$$

Thus we obtain

$$r_+ = \frac{\beta_+}{\sqrt{k_{+\infty}}} \frac{\sqrt{k_{+\infty}}}{\alpha_+} = \frac{\beta_+}{\alpha_+}. \quad (2.12)$$

We also derive (in this set of conventions)

$$t_+ = \frac{1}{\sqrt{k_{-\infty}}} \frac{\sqrt{k_{+\infty}}}{\alpha_+} = \sqrt{\frac{k_{+\infty}}{k_{-\infty}}} \frac{1}{\alpha_+}. \quad (2.13)$$

Thus we have demonstrated that α_+ and β_+ , the (right-moving) Bogoliubov coefficients, are related to the (left-moving) reflection and transmission amplitudes by

$$r_+ = \frac{\beta_+}{\alpha_+}; \quad t_+ = \sqrt{\frac{k_{+\infty}}{k_{-\infty}}} \frac{1}{\alpha_+}. \quad (2.14)$$

Without further calculation we can also deduce

$$r_- = \frac{\beta_-}{\alpha_-}; \quad t_- = \sqrt{\frac{k_{+\infty}}{k_{-\infty}}} \frac{1}{\alpha_-}. \quad (2.15)$$

The explicit occurrence of $k_{+\infty}$ and $k_{-\infty}$ in these equations is an annoyance, which is why many authors adopt the alternative normalization to be discussed below [1, 2, 3, 4, 5, 6, 7].

In Bogoliubov language the present conventions correspond to an incoming flux of right-moving particles (incident from the left) being amplified to amplitude α_+ at a cost of a backflow of amplitude β_+ . In scattering language one should consider the complex conjugate \mathcal{J}_+^* — this is equivalent to an incoming flux of left-moving particles (incident from the right) of amplitude α_+^* being partially transmitted (amplitude unity) and partially scattered (amplitude β_+^*). If the potential has even parity, then the left-moving Bogoliubov coefficients are just the complex conjugates of the right-moving coefficients, however if the potential is asymmetric a more subtle analysis is called for.

The second interesting issue is that we can deal exclusively with α_+ and β_+ , dropping the suffix for brevity — if information about α_- and β_- is desired simply work with the reflected potential $V(-x)$. It should also be borne in mind that the *phases* of β and β^* are physically meaningless in that they can be arbitrarily changed simply by moving the origin of coordinates (or equivalently, physically moving the location of the potential). The phases of α and α^* on the other hand do contain real and significant physical information.

For completely arbitrary potentials, with no parity restriction (so the potential is neither even nor odd), a Wronskian analysis yields (see for example reference [20, pages 106-108], noting that an overall minus sign between Messiah and the conventions above neatly cancels):

$$k_{-\infty} [1 - |r_+|^2] = k_{+\infty} |t_+|^2; \quad (2.16)$$

$$k_{-\infty} |t_-|^2 = k_{+\infty} [1 - |r_-|^2]; \quad (2.17)$$

$$k_{-\infty} t_- = k_{+\infty} t_+; \quad (2.18)$$

$$k_{-\infty} r_+ t_+^* = -k_{+\infty} r_- t_-^*; \quad (2.19)$$

with equivalent relations for α and β . Then

$$T_+ = \frac{k_{+\infty}}{k_{-\infty}} |t_+|^2 = \frac{k_{-\infty}}{k_{+\infty}} |t_-|^2 = T_- \quad (2.20)$$

and so the barrier transmission probability is independent of direction. We also have

$$\text{phase}(t_+) = \text{phase}(t_-), \quad (2.21)$$

and

$$\text{phase}(r_+/t_+) = \pi - \text{phase}(r_-/t_-), \quad (2.22)$$

with equivalent relations for α and β .

2.2 Formulation 2

If we now adopt the (to our minds) more useful convention, by absorbing suitable factors of $k_{+\infty}$ and $k_{-\infty}$ into the definitions of r and t , then things simplify considerably. We restart the calculation by now defining a slightly different set of reflection and transmission amplitudes r and t via the equations

$$\mathcal{J}_+(x \rightarrow -\infty) \rightarrow t_+ \frac{\exp(+ik_{-\infty}x)}{\sqrt{k_{-\infty}}}, \quad (2.23)$$

$$\mathcal{J}_+(x \rightarrow +\infty) \rightarrow \frac{\exp(+ik_{+\infty}x)}{\sqrt{k_{+\infty}}} + r_+ \frac{\exp(-ik_{+\infty}x)}{\sqrt{k_{+\infty}}}, \quad (2.24)$$

By comparing these two different forms for the asymptotic form of the Jost function we see that in this situation the ratios of the amplitudes are given by the much simpler formulae

$$1 : \alpha_+ : \beta_+ = t_+ : 1 : r_+. \quad (2.25)$$

We now have

$$r_+ = \frac{\beta_+}{\alpha_+}, \quad (2.26)$$

and

$$t_+ = \frac{1}{\alpha_+}. \quad (2.27)$$

We see that by putting the factors of $\sqrt{k_{\pm\infty}}$ into the asymptotic form of the Jost functions, where they really belong, the formulae for r and t are suitably simplified.

For completely arbitrary potentials, with no parity restriction (so the potential is neither even nor odd), a modified Wronskian analysis now yields (in analogy with that reported by Messiah [20, pages 106-108]):

$$|t_+|^2 = 1 - |r_+|^2; \quad (2.28)$$

$$|t_-|^2 = 1 - |r_-|^2; \quad (2.29)$$

$$t_- = t_+; \quad (2.30)$$

$$r_+ t_+^* = -r_- t_-^*; \quad (2.31)$$

with equivalent relations for α and β . Then

$$T_+ = |t_+|^2 = |t_-|^2 = T_- \quad (2.32)$$

and so the barrier transmission probability is independent of direction. Because they are independent of any overall scaling by a real number, we also retain the previous results

$$\text{phase}(t_+) = \text{phase}(t_-), \quad (2.33)$$

and

$$\text{phase}(r_+/t_+) = \pi - \text{phase}(r_-/t_-), \quad (2.34)$$

with equivalent relations for α and β . It is this modified set of conventions, because they have much nicer normalization properties, that we shall prefer for the bulk of the paper.

2.3 Formulation 3

The Schrödinger equation also can be analyzed in terms of a different formalism based on the functions u and v , as defined by Messiah [20], and their complex conjugates u^* and v^* . Note that the *Wronskian* of any two such solutions is independent of x . In particular, it takes on the same value in the two asymptotic regions. Our approach can be seen as equating these two values; we shall now derive a relation between the coefficients r_+ , t_+ , r_- , t_- , or their complex conjugates. Six such relations can be formed with the four functions u , v , u^* and v^* . From what we have seen earlier it is clear that they are very basic relations which must be maintained whatever the form of the potential function $V(x)$. See for instance reference [20, pages 106–108]. Specifically, we derive (in Messiah-like conventions)

$$\frac{i}{2} W(u, u^*) = k_{+\infty} (1 - |r_+|^2) = k_{-\infty} |t_+|^2; \quad (2.35)$$

$$\frac{i}{2} W(v, v^*) = k_{-\infty} (1 - |r_-|^2) = k_{+\infty} |t_-|^2; \quad (2.36)$$

$$\frac{i}{2} W(u, v) = k_{+\infty} t_- = k_{-\infty} t_+; \quad (2.37)$$

$$\frac{i}{2} W(u, v^*) = -k_{+\infty} r_+ t_-^* = k_{-\infty} r_-^* t_+. \quad (2.38)$$

The equations (2.35) and (2.36) are called the *relations of conservation of flux*. They must always be true, and this should be verified in special cases. This name comes from the following statements regarding the wave function ψ of an *unbound state* in the asymptotic region. We let $A \exp(ikx) + B \exp(-ikx)$ be the expression of the wave function ψ in one of the asymptotic regions, for $-\infty$ case.

The *total flux of particles* when passing a given point is the difference between the flux $(\hbar k/m)|A|^2$ of particles traveling in the positive sense, and the flux $(\hbar k/m)|B|^2$ of particles traveling in the negative sense. This flux is equal, to within a constant, to the *Wronskian* $W(\psi, \psi^*)$ [20]:

$$\frac{\hbar k}{m} \{|A|^2 - |B|^2\} = \frac{i}{2} \frac{\hbar k}{m} W(\psi, \psi^*) \quad (2.39)$$

The equality of the *Wronskian* $W(\psi, \psi^*)$ at both ends of the interval $(-\infty, +\infty)$, implies that the number of particles entering the interaction region per unit time is equal to the number which leave it. In accordance with this interpretation, one or the other of equation (2.35) and (2.36) can be written as:

$$\text{incident flux} - \text{reflected flux} = \text{transmitted flux}. \quad (2.40)$$

Considering the same interpretation, we now can define the *transmission coefficient* (transmission probability) T as follows:

$$T = \frac{\text{transmitted flux}}{\text{incident flux}}. \quad (2.41)$$

We have in particular

$$T_+ = \frac{k_{-\infty}}{k_{+\infty}} |t_{+\infty}|^2, \quad T_- = \frac{k_{+\infty}}{k_{-\infty}} |t_{-\infty}|^2. \quad (2.42)$$

This result again shows that the absolute values of the two sides of equation (2.37) are equal, and one again obtains the equality

$$T_- = T_+. \quad (2.43)$$

Thus the transmission coefficient of a wave at a given energy is independent of the direction of travel. This is the *reciprocity property of the transmission coefficient*. It is just as hard to traverse a potential barrier in one direction as in the other.

The equality of the absolute values of the two ways of representing the Wronskian appearing in equation (2.38), coupled with the conservation relations (2.35) and (2.36), again yields the reciprocity relation (2.41), and we also obtain relations between the phases of the reflection and transmission amplitudes:

$$\begin{aligned} \text{phase}(t_+) &= \text{phase}(t_-); \\ \text{phase}\left(\frac{r_+}{t_+}\right) &= \pi - \text{phase}\left(\frac{r_-}{t_-}\right). \end{aligned}$$

The most interesting point for these relations is the fact that the phases are related to “retardation” effects in the propagation of the wave packets, with equivalent relations for α and β . As previously, we can re-scale r and t by absorbing appropriate factors of $\sqrt{k_{\pm\infty}}$, and so simplify the discussion as in the previous section. (We will not repeat the details of the analysis, as it is straightforward.)

3 Bogoliubov transformation

To see why the Bogoliubov transformation is important, and how it relates to the transmission and reflection amplitudes, let us consider the canonical commutation relation for bosonic creation and annihilation operators

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (3.1)$$

Define a new pair of operators

$$\hat{b} = u \hat{a} + v \hat{a}^\dagger; \quad (3.2)$$

$$\hat{b}^\dagger = u^* \hat{a}^\dagger + v^* \hat{a}; \quad (3.3)$$

where the equation (3.3) is the hermitian conjugate of the equation (3.2). This transformation is a canonical transformation of these operators. It is easy to find the implied constraints on the constants u and v . For instance, if the transformation remains canonical, then by expanding the commutator we see

$$[\hat{b}, \hat{b}^\dagger] = [u \hat{a} + v \hat{a}^\dagger, u^* \hat{a}^\dagger + v^* \hat{a}] = \{|u|^2 - |v|^2\} [\hat{a}, \hat{a}^\dagger]. \quad (3.4)$$

Therefore, it can be seen that

$$|u|^2 - |v|^2 = 1 \quad (3.5)$$

is the condition for which the transformation is canonical. Note that since the form of this condition is reminiscent of the hyperbolic identity

$$\cosh^2 r - \sinh^2 r = 1, \quad (3.6)$$

between cosh and sinh, the constants u and v are usually parameterized as

$$u = \exp(i\theta) \cosh r; \quad (3.7)$$

$$v = \exp(i\theta) \sinh r. \quad (3.8)$$

4 Transfer matrix representation

We can also investigate quantum mechanical tunneling by the so-called “transfer matrix method” or “transfer matrix representation”. Ultimately, of course, this is still equivalent to extracting the transmission coefficient from the solution to the one-dimensional, time-independent Schrödinger equation. As before, the transmission coefficient is the ratio of the flux of particles that penetrate a potential barrier to the flux of particles incident on the barrier. It is related to the probability that tunneling will occur [33]. We again consider a one-dimensional problem which is characterized by an incident beam of particles that is either transmitted or reflected as a result of scattering from an object. For current purposes it is easiest to work with potentials of compact support, where $V(x) = 0$ except in some finite region $[a, b]$.

As long as the potential $V(x)$ is of compact support, it splits the space in three parts ($x < a$, $x \in [a, b]$, and $x > b$). In both $(-\infty, a]$ and $[b, \infty)$ the potential energy is zero. Moreover, in each of these two regions the solution of the Schrödinger equation can be presented as a *superposition* of exponentials by

$$\psi_L(x) = A_r \exp(ikx) + A_l \exp(-ikx), \quad x < a, \quad \text{and} \quad (4.1)$$

$$\psi_R(x) = B_r \exp(ikx) + B_l \exp(-ikx), \quad x > b, \quad (4.2)$$

where $A_{l/r}$ and $B_{l/r}$ are at this stage unspecified, and $k = \sqrt{2mE}/\hbar$. But because ψ_L and ψ_R are solutions to the Schrödinger equation that can be extended to the entire real line, and because the Schrödinger equation is a second-order differential equation so that its solution space is two-dimensional, there must be some linear relation between the coefficients appearing in ψ_L and ψ_R — specifically, there must be a 2×2 matrix M such that

$$\begin{bmatrix} B_l \\ B_r \end{bmatrix} = M \begin{bmatrix} A_l \\ A_r \end{bmatrix}. \quad (4.3)$$

The 2×2 matrix M depends, in a complicated way, on the potential $V(x)$ in the region $[a, b]$. In the transfer matrix approach we shall seek to extract as

much information as possible without explicitly calculating M . To now derive amplitudes for reflection and transmission for incidence from the left, we put $A_r = 1$ (incoming particles), $A_l = r$ (reflection), $B_l = 0$ (no incoming particle from the right) and $B_r = t$ (transmission) in equations (4.1) and (4.2). Then

$$\psi_L(x) = \exp(ikx) + r_L \exp(-ikx), \quad (4.4)$$

where r_L is the left-moving reflection amplitude and on the right of the potential

$$\psi_R(x) = t_L \exp(ikx). \quad (4.5)$$

where t_L is the left-moving transmission amplitude. This tells us that

$$\begin{bmatrix} t_L \\ 0 \end{bmatrix} = M \begin{bmatrix} 1 \\ r_L \end{bmatrix}. \quad (4.6)$$

But since the Schrödinger equation (2.1) is real, the complex conjugate of any solution is also a solution. Therefore the solution which on the left has the form

$$\psi_L = \exp(-ikx) + r_L^* \exp(+ikx), \quad (4.7)$$

must on the right have the form

$$\psi_R(x) = t_L^* \exp(-ikx), \quad (4.8)$$

and so we also have

$$\begin{bmatrix} 0 \\ t_L^* \end{bmatrix} = M \begin{bmatrix} r_L^* \\ 1 \end{bmatrix}. \quad (4.9)$$

These two matrix equations now imply

$$M = \frac{1}{1 - r_L^* r_L} \begin{bmatrix} t_L & -t_L r_L^* \\ -t_L^* r_L & t_L^* \end{bmatrix}. \quad (4.10)$$

But by conservation of flux we must have

$$|t_L|^2 + |r_L|^2 = 1. \quad (4.11)$$

We just have seen an important connection between reflection and transmission amplitudes. In addition, it is also interesting to show how to derive the above equation by the following argument. From the equation (4.4), we can see that this corresponds to a flux in the positive x direction. For $x < a$ this is of magnitude

$$\begin{aligned} \mathfrak{J} &= \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right), \\ &= \frac{\hbar}{2mi} \left((\exp(-ikx) + r_L^* \exp(+ikx)) \times (ik \exp(ikx) - r_L ik \exp(-ikx)) \right. \\ &\quad \left. - (\text{complex conjugate}) \right), \\ &= \frac{\hbar}{2mi} \left(2ik - 2ik|r_L|^2 \right), \\ &= \frac{\hbar k}{m} \left(1 - |r_L|^2 \right). \end{aligned} \quad (4.12)$$

In contrast, for $x > b$ we similarly derive from equation (4.5) the fact that we can write the flux as

$$\begin{aligned}
\mathfrak{J} &= \frac{\hbar}{2mi} \left((t_L^* \exp(-ikx) \times ik(t_L \exp(ikx))) \right. \\
&\quad \left. - (t_L \exp(ikx) \times -ik(t_L^* \exp(-ikx))) \right), \\
&= \frac{\hbar}{2mi} \left(ik|t_L|^2 + ik|t_L|^2 \right), \\
&= \frac{\hbar k}{m} \left(|t_L|^2 \right). \tag{4.13}
\end{aligned}$$

The *probability current* \mathfrak{J} of the wave function $\psi(x)$ is defined as

$$\mathfrak{J} = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right), \tag{4.14}$$

in the position basis and satisfies the quantum mechanical continuity equation

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} \mathfrak{J}(x, t) = 0, \tag{4.15}$$

where $\rho(x, t)$ is probability density. Since there is no time dependence in the problem, the conservation law in equation (4.15) implies that $\mathfrak{J}(x)$ is independent of x . Hence the flux on the left must be equal to the flux on the right, that is, we expect that

$$\begin{aligned}
\frac{\hbar k}{m} \left(1 - |r_L|^2 \right) &= \frac{\hbar k}{m} \left(|t_L|^2 \right). \\
1 - |r_L|^2 &= |t_L|^2.
\end{aligned}$$

therefore

$$|t_L|^2 + |r_L|^2 = 1, \tag{4.16}$$

so

$$\frac{1}{1 - r_L^* r_L} = \frac{1}{1 - |r_L|^2} = \frac{1}{|t_L|^2}. \tag{4.17}$$

Finally we see that the transfer matrix can be explicitly represented in the form

$$M = \frac{1}{|t_L|^2} \begin{bmatrix} t_L & -t_L r_L^* \\ -t_L^* r_L & t_L^* \end{bmatrix} = \begin{bmatrix} 1/t_L^* & -r_L^*/t_L^* \\ -r_L/t_L & 1/t_L \end{bmatrix}. \tag{4.18}$$

Similarly, we now consider a wave moving in from the right

$$\exp(-ikx), \tag{4.19}$$

which then hits the potential, is partially reflected and partially transmitted. In this case, on the *right* of the potential we have

$$\psi_R(x) = \exp(-ikx) + r_R \exp(+ikx), \quad (4.20)$$

where r_R is the right-moving reflection amplitude and on the *left* of the potential

$$\psi_L(x) = t_R \exp(-ikx), \quad (4.21)$$

where t_R is the left-moving transmission amplitude. This tells us that

$$\begin{bmatrix} r_R \\ 1 \end{bmatrix} = M \begin{bmatrix} 0 \\ t_R \end{bmatrix}. \quad (4.22)$$

Again, since the Schrödinger equation is real, the complex conjugate of any solution is also a solution. Therefore a related interesting solution which on the left can be cast in the form

$$\psi_L(x) = t_R^* \exp(+ikx), \quad (4.23)$$

must on the right have the form

$$\psi_R(x) = \exp(+ikx) + r_R^* \exp(-ikx), \quad (4.24)$$

whence

$$\begin{bmatrix} 1 \\ r_R^* \end{bmatrix} = M \begin{bmatrix} t_R^* \\ 0 \end{bmatrix}. \quad (4.25)$$

But now these two matrix equations imply

$$M = \begin{bmatrix} 1/t_R^* & r_R/t_R \\ r_R^*/t_R^* & 1/t_R \end{bmatrix}. \quad (4.26)$$

Combining the information from left moving and right moving cases we have first that

$$t_L = t_R. \quad (4.27)$$

So we again derive the equality of the transmission amplitudes.

Similarly we see that

$$\frac{r_R}{t_R} = -\frac{r_L^*}{t_L^*}, \quad (4.28)$$

implying

$$r_R = -r_L^* \frac{t_L}{t_L^*}; \quad |r_R| = |r_L|. \quad (4.29)$$

Note that we *cannot* in general deduce $r_L = r_R$. Indeed, in general this is false.

So for *any* potential (regardless of whether or not it possesses parity symmetry) we have

$$T = |t_L|^2 = |t_R|^2; \quad R = |r_L|^2 = |r_R|^2, \quad (4.30)$$

implying (in the same manner as the previous argument) that the transmission and reflection coefficients are independent on whether or not the particle is incident from the left or the right — and we have very carefully *not* made any assumption here about any symmetry for the potential $V(x)$ itself. We conclude

$$M = \begin{bmatrix} 1/t^* & -r_L^*/t^* \\ -r_L/t & 1/t \end{bmatrix} = \begin{bmatrix} 1/t^* & r_R/t \\ r_R^*/t^* & 1/t \end{bmatrix}. \quad (4.31)$$

Note the key step in this general derivation: In any region where the potential is zero we simply need to solve

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = E \psi(x), \quad (4.32)$$

for which the two independent solutions are

$$\exp(\pm ikx); \quad k = \frac{\sqrt{2mE}}{\hbar}, \quad (4.33)$$

or more explicitly

$$\exp\left(\pm i \frac{\sqrt{2mE}}{\hbar} x\right). \quad (4.34)$$

To the *left* of the potential we have

$$\psi_L(x) = a \exp(ikx) + b \exp(-ikx), \quad (4.35)$$

while to the *right* of the potential we have

$$\psi_R(x) = c \exp(ikx) + d \exp(-ikx). \quad (4.36)$$

Even without knowing anything more about the potential $V(x)$, the linearity of the Schrödinger ODE guarantees that there will be some 2×2 transfer matrix M such that

$$\begin{bmatrix} c \\ d \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix}. \quad (4.37)$$

This transfer matrix relates the situation to the *left* of the potential with the wave-function to the *right* of the potential. We could now use this formalism, for instance, to think about the propagation of electrons down a wire (approximately one-dimensional) with $V(x)$ used to describe various barriers placed in the path of the electron. Similar matrices also occur in optics, where they are referred to as “Jones matrices”.

The components of the transfer matrix M will be some complicated nonlinear function of the potential $V(x)$, but by linearity of the Schrödinger ODE these matrix components must be independent of the parameters a, b, c , and d . In some particularly simple situations we may be able to calculate the matrix M explicitly, but in general it can only be approximated or bounded [1, 2, 3, 4, 5, 6, 7]. From the above discussion we now understand, from several different points of view, the basic concepts of transmission and reflection amplitudes. The probability that a given incident particle is reflected is called the “reflection coefficient”, $R = |r|^2$. While the probability that it is transmitted is called the “transmission coefficient”, $T = |t|^2$.

5 Discussion

In this article, we have presented basic aspects of scattering theory in one dimension in (we hope) a pedagogically clear manner. For a one-dimensional model, only one of the three coordinates of 3-dimensional physical space is explicitly involved. Specifically, we considered potentials of compact support, when the potential $V(x)$ is mathematically zero outside of a finite interval. We have just seen an important connection between reflection and transmission amplitudes, and how to derive this relation directly by using scattering theory.

We introduced the probability current to express the reflection and transmission coefficients. The probability current is based on the axiom that the intensity of a beam is the product of the speed of its particles and their linear number density. It is then a mathematical theorem that this probability current is conserved. We then introduced important ideas of reflection and transmission of waves, and have seen that in principle they are completely specified by the potential function $V(x)$. For instance, the linearity of the Schrödinger ODE guarantees that there will be some 2×2 transfer matrix. Moreover, this transfer matrix can be represented by investigating quantum mechanical tunneling by extracting the transmission coefficient from the solution to the one-dimensional, time-independent Schrödinger equation. This general formalism has served as a backdrop for our further investigations reported in references [1, 2, 3, 4, 5, 6, 7].

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