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A Comparison of Scale Parameter Estimators in the 2-Parameter Exponential Distribution Based on Multiple Criteria Decision Making

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Abstract : In this paper we study the problem of estimation of the scale parameter (θ) of the 2-parameter exponential distribution with prior information (θ_0). The estimators of θ are maximum likelihood estimator ($\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)})$) and shrinkage estimator ($\hat{\theta}_{(p)} = \theta_0 + \alpha(p)(\hat{\theta} - \theta_0)$, where $p = \pm 1,\ \pm 2$). The comparison is based on the Multiple Criteria Decision Making (MCDM) procedure to obtain the best estimator. The results reveal that the best estimators of θ is $\hat{\theta}_{(1)}$.

Keywords : Scale parameter estimator; 2-parameter exponential distribution; Multiple Criteria Decision Making.
2000 Mathematics Subject Classification : 62F10; 60E05; 62C25.

1 Introduction

The 2-parameter exponential distribution has been used frequently in lifetime testing and reliability theory. It is formulated as :

$$f(x;\theta,\gamma) = \frac{1}{\theta}e^{-\frac{x-\gamma}{\theta}};\ \text{for } x \geqslant \gamma, \theta > 0,$$

where θ is the scale apartmeter and γ is the location parameter. The location parameter is interpreted as the minimum (or guaranteed) time before which no failure occurs, the scale parameter is the mean life, measured from the location parameter. The parameter estimation is an interesting problem in the statistical inference. Kourouklis [3] proposed a class of shrinkage estimators $\hat{\theta}_{(p)}$ for the scale parameter and the population mean of the 2-parameter exponential distribution, given a prior estimate of the scale parameter (θ_0). These estimators had been motivated by the work of Jani [2].

In the 2-parameter exponential distribution,the maximum likelihood estimator (MLE) of θ is formulated as :

$$\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)}), \text{where } x_{(1)} \text{ is minimum order statistic,}$$

and the class of shrinkage estimators for θ is $\hat{\theta}_{(p)} = \theta_0 + \alpha(p)(\hat{\theta} - \theta_0)$, where $\alpha(p) = \frac{\Gamma(n-1-p)}{\Gamma(n-1-2p)(n-1)^p}$; for $p \in (-\infty, \frac{1}{2}(n-1))$

In this article these estimators are compared wherein p = -2, -1, 1 and 2 as described above on the basis of the mean square errors ($MSEs$) using the Multiple Criteria Decision Making (MCDM) method for ranking those estimators from the best to the worst. This method is briefly described in section 2. Section 3 describes the main results of this paper.

2 A brief description of MCDM procedure

Multiple Criteria Decision Making (MCDM) is a technique that can be used for assessments and decision making where the multiple criteria are presented. A typical MCDM problem involves a number of alternatives to be selected and a number of criterions or indicators for assessing these alternatives. Each alternative has a value for each indicator and can be selected based on its values. Lertprapai et al. [4] presented a comprehensive review on the MCDM procedure as follows:

For a 'discrete' data matrix $X = (x_{ij}) : K \times N$ where $x'_{ij}s$ represents the risk of ith source for jth category, it is necessary to compare the K rows simultaneously with respect to all the N columns, MCDM is a novel statistical procedure to integrate the multiple risks $(x_{i1}, x_{i2}, ..., x_{iN})$, $i = 1, 2, ..., K$ for the ith alternative into a single meaningful and overall risk factor [1] and [5]. The K estimators are then compared on the basis of these integrated risk factors. If M is the number of positive meaning criterions.The risk integration is done by defining an ideal row (IDR) with the best observed value for each column as:

$$IDR = (\min_i x_{i1,...,} \min_i x_{iM}, \min_i x_{iM+1}..., \min_i x_{iN}) = (u_1, ..., u_N),$$

and a negative-ideal row (NIDR) with the worst observed value for each column as:

$$NIDR = (\max_i x_{i1}, ..., \max_i x_{iM}, \max_i x_{iM+1}, ..., \max_i x_{iN}) = (v_1, ..., v_N).$$

For any given row i, now the distance of each row is computed from the ideal row and from the negative ideal row based on a suitably chosen norm. It is computed under L_1-norm [7] as:

$$L_1(i, IDR) = \sum_{j=1}^{N} \frac{|u_j - x_{ij}|w_j}{\sum_{i=1}^{K} |x_{ij}|}$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} \frac{|v_j - x_{ij}|w_j}{\sum_{i=1}^{K} |x_{ij}|}$$

where w_js is appropriate weight. The various rows are now compared based on the overall index which is computed as:

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, i = 1, ..., K \tag{2.1}$$

Similarly, under Ł$_2$ -norm ,

$$L_2(i, IDR) = \left[\sum_{j=1}^{N} \frac{[x_{ij} - u_j]^2 w_j^2}{\sum_{i=1}^{K} x_{ij}^2} \right]^{1/2}$$

$$L_2(i, NIDR) = \left[\sum_{j=1}^{N} \frac{[v_j - x_{ij}]^2 w_j^2}{\sum_{i=1}^{K} x_{ij}^2} \right]^{1/2}$$

and the rows are compared based on :

$$L_2(Index_i) = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, i = 1, ..., K. \tag{2.2}$$

A 'continuous' version of this setup would involve $x'_{ij}s$ where the index j would vary continuously. In the context of this problem five estimators of the scale parameter (θ) in the 2-parameter exponential distribution are compared (see section 3). In this cases, obviously $K = 5$, so $x'_{ij}s$ is chosen to represent the mean square of errors of the five estimators for various values of $r = \frac{(\theta_0 - \theta)}{\theta}$, -1 $< r <$ 1. In this case, L_1 and L_2-norm would be redefined as:

$$L_1(i, IDR) = \int_{-1}^{1} |x_i(r) - u(r)|w(r)dr \tag{2.3}$$

$$L_1(i, NIDR) = \int_{-1}^{1} |v(r) - x_i(r)|w(r)dr \tag{2.4}$$

$$L_2(i, IDR) = \sqrt{\int_{-1}^{1} [x_i(r) - u(r)]^2 [w(r)]^2 dr} \tag{2.5}$$

$$L_2(i, NIDR) = \sqrt{\int_{-1}^{1} [v(r) - x_i(r)]^2 [w(r)]^2 dr} \tag{2.6}$$

where $u(r) = \min_i \{x_i(r)\}$ and $v(r) = \max_i \{x_i(r)\}$.

3 Main Results

3.1 Mean Square Errors ($MSEs$)

With reference to Kourouklis [3], the MSEs of $\hat{\theta}_1$ and $\hat{\theta}_{(p)}$ are presented in details as follows:

$MSE(\hat{\theta}_1; \theta) = E(\hat{\theta}_1 - \theta)^2 = \frac{\theta^2}{n}$,

$MSE(\hat{\theta}_{(p)}; \theta) = E(\hat{\theta}_{(p)} - \theta)^2 = \left[(1 - \alpha(p))^2 \cdot r^2 + \frac{\alpha^2(p)}{n-1}\right] \cdot \theta^2;\ r \equiv \frac{\theta_0}{\theta} - 1.$

In term of $MSE(\hat{\theta})$, a common term (θ^2) is ignored, so the results are in the form : $MSE(\hat{\theta}_1) = \frac{1}{n}$, $MSE(\hat{\theta}_{(p)}; r) = (1 - \alpha(p))^2 \cdot r^2 + \frac{\alpha^2(p)}{n-1}$.

Let $p = \pm 2,\ \pm 1$, the value of α is formulated as $\alpha(p) = \frac{\Gamma(n-1-p)}{\Gamma(n-1-2p)(n-1)^p}$. Therefore we have

$\alpha(-2) = \frac{(n-1)^2}{(n+1)(n+2)}$, $\alpha(-1) = \frac{n-1}{n}$, $\alpha(+1) = \frac{n-3}{n-1}$, $\alpha(+2) = \frac{(n-4)(n+5)}{(n-1)^2}$.

We now let $\hat{\theta}_1$, $\hat{\theta}_{(-2)}$, $\hat{\theta}_{(-1)}$,$\hat{\theta}_{(1)}$ and $\hat{\theta}_{(2)}$ as T_1, T_2, T_3, T_4 and T_5 respectively. In this paper, MSEs of each estimator are computed and compared based on the MCDM method. The range $-1 < r < 1$ when $n = 10$ and 15 are considered.

Case I : $n = 10$

$MSE(T_1) = \frac{1}{10}$, $MSE(T_2) = \frac{289}{1936} r^2 + \frac{81}{1936}$, $MSE(T_3) = \frac{1}{100} r^2 + \frac{9}{100}$,
$MSE(T_4) = \frac{4}{81} r^2 + \frac{49}{729}$, $MSE(T_5) = \frac{289}{729} r^2 + \frac{100}{6561}$.

Their graphical patterns for $n = 10$ are presented in Figure 1.

Case II : $n = 15$

$MSE(T_1) = \frac{1}{15}$, $MSE(T_2) = \frac{361}{4624} r^2 + \frac{343}{9248}$, $MSE(T_3) = \frac{1}{225} r^2 + \frac{14}{225}$,
$MSE(T_4) = \frac{1}{49} r^2 + \frac{18}{343}$, $MSE(T_5) = \frac{1849}{9604} r^2 + \frac{3025}{134456}$.

Their graphical patterns for n = 15 are presented in Figure 2.

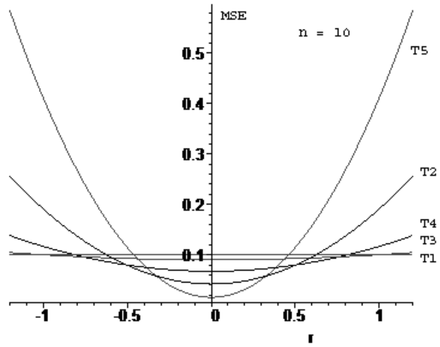

Figure 1 : Graphical illustration of MSE of $\hat{\theta}$ for $n = 10$.

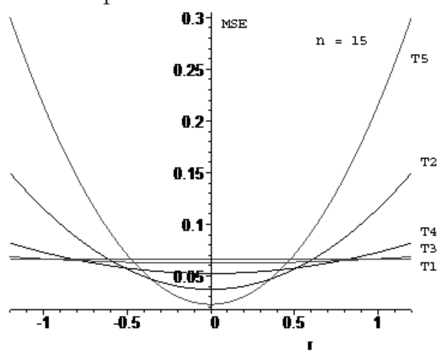

Figure 2 : Graphical illustration of MSE of $\hat{\theta}$ for n = 15.

3.2 Analysis IDR and $NIDR$

For $-1 < r < 1$, the intersection of the five graphs can separate the interval of r into 20 intervals as :

$-1 < c_9' < c_8' < c_7' < c_6' < c_5' < c_4' < c_3' < c_2' < c_1' < 0 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8 < c_9 < 1$.

Since these graphs are symmetry at $r = 0$, so the ideal row $(u_j(r))$ and the negative-ideal row $(v_j(r))$ are demonstrated only the positive intervals $(0 < r < 1)$ are shown in Table 1.

Table 1 : The IDR and NIDR for each interval for $n = 10$ and 15.

n	$0 < r < c_1$		$c_1 < r < c_2$		$c_2 < r < c_3$		$c_3 < r < c_4$		$c_4 < r < c_5$	
	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5
10	T_5	T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2	T_5
15	T_5	T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2	T_5

Table 1 (continued) : The IDR and NIDR for each interval for $n = 10$ and 15.

n	$c_5 < r < c_6$		$c_6 < r < c_7$		$c_7 < r < c_8$		$c_8 < r < c_9$		$c_9 < r < 1$	
	u_6	v_6	u_7	v_7	u_8	v_8	u_9	v_9	u_{10}	v_{10}
10	T_4	T_5	T_4	T_5	T_4	T_5	T_3	T_5	T_3	T_5
15	T_4	T_5	T_4	T_5	T_4	T_5	T_3	T_5	T_3	T_5

3.2.1 Analysis based on the L_1-norm

For i = 1, 2, 3, 4 and 5, applying equations (2.3) and (2.4), we get

$$\begin{aligned} L_1(i, IDR) =& 2\{\int_0^{c_1}[x_i(r) - u_1(r)]w(r)dr + \int_{c_1}^{c_2}[x_i(r) - u_2(r)]w(r)dr + \int_{c_2}^{c_3}[x_i(r) - u_3(r)]w(r)dr \\ &+ \int_{c_3}^{c_4}[x_i(r) - u_4(r)]w(r)dr + \int_{c_4}^{c_5}[x_i(r) - u_5(r)]w(r)dr + \int_{c_5}^{c_6}[x_i(r) - u_6(r)]w(r)dr \\ &+ \int_{c_6}^{c_7}[x_i(r) - u_7(r)]w(r)dr + \int_{c_7}^{c_8}[x_i(r) - u_8(r)]w(r)dr + \int_{c_8}^{c_9}[x_i(r) - u_9(r)]w(r)dr \\ &+ \int_{c_9}^{1}[x_i(r) - u_{10}(r)]w(r)dr\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} L_1(i, NIDR) =& 2\{\int_0^{c_1}[v_1(r) - x_i(r)]w(r)dr + \int_{c_1}^{c_2}[v_2(r) - x_i(r)]w(r)dr + \int_{c_2}^{c_3}[v_3(r) - x_i(r)]w(r)dr \\ &+ \int_{c_3}^{c_4}[v_4(r) - x_i(r)]w(r)dr + \int_{c_4}^{c_5}[v_5(r) - u_i(r)]w(r)dr + \int_{c_5}^{c_6}[v_6(r) - x_i(r)]w(r)dr \\ &+ \int_{c_6}^{c_7}[v_7(r) - x_i(r)]w(r)dr + \int_{c_7}^{c_8}[v_8(r) - x_i(r)]w(r)dr + \int_{c_8}^{c_9}[v_9(r) - x_i(r)]w(r)dr \\ &+ \int_{c_9}^{1}[v_{10}(r) - x_i(r)]w(r)dr\}. \end{aligned}$$

The overall index then can be computed from equation (2.1).

3.2.2 Analysis based on the L_2-norm

For i = 1, 2, 3, 4 and 5, applying equations 2.5 and 2.6, we get

$$L_2(i, IDR) = \sqrt{\begin{array}{l} 2\{\int_0^{c_1}[x_i(r) - u_1(r)]^2[w(r)]^2dr + \int_{c_1}^{c_2}[x_i(r) - u_2(r)]^2[w(r)]^2dr \\ + \int_{c_2}^{c_3}[x_i(r) - u_3(r)]^2[w(r)]^2dr + \int_{c_3}^{c_4}[x_i(r) - u_4(r)]^2[w(r)]^2dr \\ + \int_{c_4}^{c_5}[x_i(r) - u_5(r)]^2[w(r)]^2dr + \int_{c_5}^{c_6}[x_i(r) - u_6(r)]^2[w(r)]^2dr \\ + \int_{c_6}^{c_7}[x_i(r) - u_7(r)]^2[w(r)]^2dr + \int_{c_7}^{c_8}[x_i(r) - u_8(r)]^2w(r)dr \\ + \int_{c_8}^{c_9}[x_i(r) - u_9(r)]^2[w(r)]^2dr + \int_{c_9}^{1}[x_i(r) - u_{10}(r)]^2[w(r)]^2dr\} \end{array}}$$

and

$$L_2(i,NIDR) = \sqrt{\begin{array}{l} 2\{\int_0^{c_1}[v_1(r)-x_i(r)]^2[w(r)]^2dr+\int_{c_1}^{c_2}[v_2(r)-x_i(r)]^2[w(r)]^2)dr \\ +\int_{c_2}^{c_3}[v_3(r)-x_i(r)]^2[w(r)]^2dr+\int_{c_3}^{c_4}[v_4(r)-x_i(r)]^2[w(r)]^2dr \\ +\int_{c_4}^{c_5}[v_5(r)-u_i(r)]^2[w(r)]^2dr+\int_{c_5}^{c_6}[v_6(r)-x_i(r)]^2[w(r)]^2dr \\ +\int_{c_6}^{c_7}[v_7(r)-x_i(r)]^2[w(r)]^2drr+\int_{c_7}^{c_8}[v_8(r)-x_i(r)]^2[w(r)]^2r \\ +\int_{c_8}^{c_9}[v_9(r)-x_i(r)]^2[w(r)]^2dr+\int_{c_9}^{1}[v_{10}(r)-x_i(r)]^2[w(r)]^2dr\} \end{array}}.$$

Under L_2-norm, the overall index can also be computed from equation (2.2.)

3.3 Choice of Weight Function

There are three choices of weight function. The first weight function is defined by $w_1(r) = 1$. Refer to Filar et al [1], the second one denoted by $w_2(r)$, is based on the notion of entropy among $x_1(r), x_2(r), x_3(r), x_4(r)$ and $x_5(r)$ for various values of r, and the third one, denoted by $w_3(r)$, is based on the coefficient of variation of $x_1(r), x_2(r), x_3(r), x_4(r)$ and $x_5(r)$ for various values of r. It turns out that $w_2(r) = \frac{1-\phi(r)}{\int_{-1}^{1}(1-\phi(r))dr}$, where $\phi(r) = -\sum_{i=1}^{5}\left[x_i(r)/\sum_{i=1}^{5}x_i(r)\cdot\ln\left(x_i(r)/\sum_{i=1}^{5}x_i(r)\right)\right]/[\ln 5]$

and

$w_3 = sd/\bar{x}$, where $sd = \sqrt{\left[\sum_{i=1}^{5}x_i^2(r) - 5\left(\sum_{i=1}^{5}x_i/5\right)^2\right]/4}$ and $\bar{x} = \sum_{i=1}^{5}x_i/5.$

3.4 Comparison of the estimators

The ranks of the five estimators of θ based on L_1 and L_2-norm using the weight function $w_1(r)$, $w_2(r)$, and $w_3(r)$, for $n = 10$, 15 are shown in Table 2.

Table 2 : The ranking of estimators of θ using weights $w_1(r), w_2(r)$ and $w_3(r)$*

n	T	L_1-norm			L_2-norm		
		$w_1(r)$	$w_2(r)$	$w_3(r)$	$w_1(r)$	$w_2(r)$	$w_3(r)$
10	T_1	4	3	3	4	3	3
	T_2	2	4	4	3	4	4
	T_3	3	2	2	2	2	2
	T_4	1	1	1	1	1	1
	T_5	5	5	5	5	5	5
15	T_1	4	3	3	3	3	3
	T_2	2	4	4	4	4	4
	T_3	3	2	2	2	2	2
	T_4	1	1	1	1	1	1
	T_5	5	5	5	5	5	5

* 1 = best, 5 = worst

In Table 2 shows that most of all, T_4 is the best in any weight function while T_3, T_1, T_2 and T_5 are lower in rank respectively.

The average of L_1 and L_2-norm with of three weight functions and the ranks of the five estimators are shown in Table 3.

Table 3 : Conclusion of the ranking of estimators of *theta* for $n = 10$ and 15^*.

n	T	$L_1(index)$			$L_2(index)$			Average	Rank
		$w_1(r)$	$w_2(r)$	$w_3(r)$	$w_1(r)$	$w_2(r)$	$w_3(r)$		
10	T_1	0.3070	0.2320	0.2405	0.2672	0.2152	0.2128	0.2458	3
	T_2	0.2278	0.2621	0.2498	0.258	0.2767	0.2706	0.2576	4
	T_3	0.2441	0.1926	0.1951	0.2337	0.1923	0.1886	0.2077	2
	T_4	0.1531	0.1499	0.1416	0.1671	0.1480	0.1421	0.1503	1
	T_5	0.7537	0.7857	0.7901	0.7453	0.7864	0.7904	0.7753	5
15	T_1	0.3301	0.1993	0.2398	0.2886	0.2339	0.2079	0.2499	3
	T_2	0.2604	0.2969	0.2850	0.2922	0.3120	0.3065	0.2922	4
	T_3	0.2720	0.1689	0.2007	0.2587	0.1580	0.1877	0.2077	2
	T_4	0.1851	0.1406	0.1541	0.2006	0.1286	0.1502	0.1599	1
	T_5	0.7230	0.8153	0.7855	0.7215	0.8269	0.7946	0.7778	5

* 1 = best, 5 = worst

In Table 3, we found that T_4 is the best while T_3, T_1, T_2 and T_5 are lower in rank respectively.

4 Conclusion

MCDM method is used for comparing the estimators of scale parameter in 2 - parameter exponential distribution with prior information. Based on L_1 and L_2 - norm, we conclude that the shrinkage estimator where $p = 1$ is quite preferable while the worst one is the shrinkage estimator where $p = 2$ under the three weights $w_1(r)$, $w_2(r)$, and $w_3(r)$.

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