Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2010) : 51–59
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Planar m-Bubbles with m-1 Equal Highest Pressures

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Abstract : The planar soap bubble problem asks for the least-perimeter way to enclose and separate open regions $R_1, R_2, \ldots, R_m$ of m given areas on the plane. In this work, we study properties for minimizing bubbles in case that the pressure of R_m is lower than the equal pressures of $R_1, R_2, \ldots$ and R_{m-1}. For $m = 4$, we show that a minimizing bubble with nonnegative pressures and without empty chambers has at most one internal component of the region R_4.

Keywords : Soap bubble; Minimizing enclosure; Least perimeter.
2000 Mathematics Subject Classification : 53A10; 51M25.

1 Introduction

It is convincing that the soap bubble is the best way, using the least surface area, to enclose and separate the given volumes of air. The ancient Greeks believed that the circle can enclose and separate a single given area on the plane using the least perimeter but the rigorous proof appeared in the late nineteenth century. The planar soap bubble problem arks for the least-perimeter way to enclose and separate open regions $R_1, R_2, \ldots, R_m$ of given areas $A_1, A_2, \ldots, A_m$ on the plane. Intuitively, we believe that the problem have natural solutions which keep each region in a single connected component. For $m = 2$, the problem was solved, in 1993, by Foisy, Alfaro, Brock, Hodges and Zimba [4]. In 1998, Vaughn [11] solved the problem for three areas in case equal pressures and no empty chambers. The problem for three areas was solved completely by Wichilamala [12, 13] in 2002. For $m = 4, 5$ and 6, the problems in case equal pressures and no empty chambers was solved by Sroysang and Wichiramala [9, 10]. The problem for four areas in other cases was considered firstly by Keawkhao and Wichiramala [5, 6]. In addition, for the case of three equal highest pressures and no empty chambers,

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if the lower-pressure region R_4 is connected, then it is external. In [8], we ignore above assumption and show that a minimizing bubble B must have at least one external component of R_4. Now, we will show a new result that B has at most one internal component of R_4.

2 Preliminaries

Definition 2.1. *A set E on $\mathbb{R}^2$ is called an **enclosure** of areas $A_1, A_2, \ldots, A_m$ if E is closed and bounded with finite one-dimensional Hausdorff measure and $\mathbb{R}^2 \backslash E$ contains open regions $R_1, R_2, \ldots, R_m$ of areas $A_1, A_2, \ldots, A_m$, respectively. The set $\mathbb{R}^2 \backslash \overline{R_1 \cup \ldots \cup R_m}$ is called the **exterior region**, denoted by R_0. Each connected component of a region is called a **component**. A component is **external** if it meets R_0 and is **internal** if not. A bounded component of R_0 is called an **empty chamber**.*

Definition 2.2. *An enclosure E is **minimizing** if E has least Haussdorf measure among enclosures of given areas, and E is **standard** if every region is connected and every two regions may meet at most once along a single edge.*

Theorem 2.3. [1, 3, 7] *For $A_1, A_2, \ldots, A_m > 0$, there is a minimizing enclosure of areas $A_1, A_2, \ldots, A_m$. Let E be a minimizing enclosure. Then*

(1) *E is composed of finitely many circular/straight edges separating different regions and meeting only in threes at $120°$ angles,*

(2) *all edges in E form a connected graph, and*

(3) *there are $p_1, p_2, \ldots, p_m \in \mathbb{R}$, which will be called the **pressures** of the region R_i, such that every edge between R_i and R_j has curvature $|p_i - p_j|$ (bulges into the lower pressure region) where the pressure p_0 of the the region R_0 is set to be zero.*

An enclosure of m regions with properties (1), (2) and (3) is called an ***m*-bubble**.

Proposition 2.4. [2] *For a minimizing bubble, any two components may meet at most once, along a single edge.*

Corollary 2.5. [4] *For $m \geq 3$, a minimizing m-bubble has no 2-sided component.*

Definition 2.6. *The **sign** of curvature of a directed edge is considered positive[negative] if the edge is turning left[right]. The **turning angle** of a directed edge of a component is the product of its signed curvature and its length.*

Lemma 2.7. [12, 13] *The sum of turning angles of all edges in an n-sided component of a bubble is $\frac{6-n}{3}\pi$ if the component is bounded, and $\frac{-6-n}{3}\pi$ if the component is unbounded.*

Definition 2.8. *A component is **convex** if all edges have nonnegative curvatures.*

Theorem 2.9. [12, 13] *A minimizing m-bubble has at most $m-1$ disjoint non-hexagonal convex components and a convex component away from them.*

Theorem 2.10. [5, 6] *In a minimizing 4-bubble with pressures $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers, if R_4 is connected then it is external.*

Theorem 2.11. [5, 6] *In a minimizing 4-bubble with pressures $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers, every component of R_4 has at most nine sides.*

Theorem 2.12. [8] *A minimizing 4-bubble with pressures $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers must have at least one external component of the region R_4.*

In [5, 6], the components (a), (b) and (c) in Figure 1 are called according to their absolute turning angles of the circular edges, a ***π-cell***, a ***$\frac{2\pi}{3}$-cell*** and a ***$\frac{\pi}{3}$-cell***, respectively. Moreover, the the 4-sided components (d) and (e) in Figure 1 are called a ***parallel component*** and a ***nonparallel component***, respectively. Note that the circular edges of a nonparallel component is cocircular if their curvatures are the same. In a 4-bubble with pressures $p_1 = p_2 = p_3 > p_4$, an internal nonhexagonal component of each highest pressure region is of a type in Figure 1.

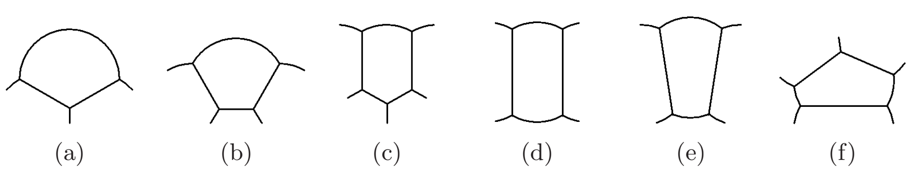

Figure 1: A π-cell, a $\frac{2\pi}{3}$-cell, a $\frac{\pi}{3}$-cell, a parallel component, a nonparallel component and a 5-sided component.

3 Main Results

In this section, we present some properties of m-bubbles with pressures $p_1 = p_2 = \ldots = p_{m-1} > p_m$ and then show that a minimizing 4-bubble with pressures $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers has at most one internal component of the region R_4.

Theorem 3.1. *Assume that an m-bubble with pressures $p_1 = p_2 = \ldots = p_{m-1} > p_m$ has an internal component C of the region R_m where $m \geq 4$. Let n be the number of sides of C and let $t_1, t_2, \ldots, t_n$ be the turning angles of all edges of C. Then $n > 6\left(\frac{|t_i|}{\pi} + 1\right)$ for all i.*

Proof. Since $p_1 = p_2 = \ldots = p_{m-1} > p_m$, it follows that $t_i < 0$ for all i. By Lemma 2.7, $\dfrac{6-n}{3}\pi = \sum_{i=1}^{n} t_i < 0$. We scale the bubble so that all edges of C have curvature one. For each i, the edge of C with turning angle t_i has length $|t_i|$. Note that the length of an edge of C is less than the sum of the length of other edges of C. For each i, we obtain that $|t_i| < \sum_{\substack{j=1 \\ j\neq i}}^{n} |t_j| = -|t_i| + \sum_{j=1}^{n} |t_j| = -|t_i| + \left|\sum_{j=1}^{n} t_j\right| = -|t_i| + \left|\dfrac{6-n}{3}\pi\right| = -|t_i| + \dfrac{n-6}{3}\pi$, and then $2|t_i| < \dfrac{n-6}{3}\pi$. Thus $6\left(\dfrac{|t_i|}{\pi} + 1\right) < n$ for all i. □

Corollary 3.2. *Assume that an 4-bubble with pressures $p_1 = p_2 = p_3 > p_4$ has an internal component C of the region R_4. Let n be the number of sides of C. Then*

(1) *if $n = 7$, then all edges of C have absolute turning angle less than $\frac{\pi}{6}$,*
(2) *if $n = 8$, then all edges of C have absolute turning angle less than $\frac{\pi}{3}$, and*
(3) *if $n = 9$, then all edges of C have absolute turning angle less than $\frac{\pi}{2}$.*

Theorem 3.3. *Assume that a minimizing m-bubble with pressures $p_1 = p_2 = \ldots = p_{m-1} > p_m$ has an internal 5-sided component D adjacent to two components of the region R_m where $m \geq 4$. For each $i \in \{1, 2, ..., m-1\}$, if D is adjacent to a nonparallel component of the region R_i, then D is not adjacent to another component of R_i.*

Proof. Let $i \in \{1, 2, ..., m-1\}$. Assume that D is adjacent to a nonparallel component E of the region R_i. Since $p_1 = p_2 = \ldots = p_{m-1} > p_m$, it follows that the circular edges of E have the same positive curvature. Thus the circular edges of E are cocircular and then the circular edges of D are also cocircular as in Figure 2. In fact, both circles have the same radius.

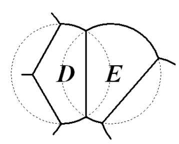

Figure 2: D is adjacent to a nonparallel component E of the region R_i.

Suppose that D is adjacent to other component of R_i. There are two possibilities shown in Figure 3. Note that e and e' are of the same length.

For each possibility, we may move the edge e as shown in Figure 4 and then create an enclosure preserving both the length and the areas.

By Theorem 2.3, the new enclosure is not minimizing and hence the original bubble is not minimizing, a contradiction. □

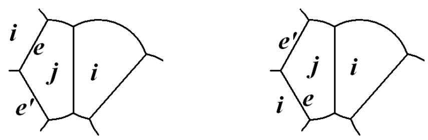

Figure 3: Two possibilities of components D and E.

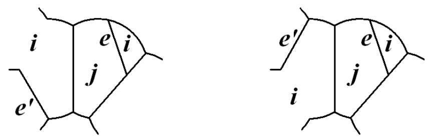

Figure 4: New enclosures preserving both the length and the areas.

Theorem 3.4. *Assume that the region R_4 of a minimizing 4-bubble with pressures $p_1 = p_2 = p_3 > p_4 \geq 0$ has at least two internal components C and C'. Each internal component adjacent to both C and C' must be 5-sided.*

Proof. Suppose the contrary that there is a 4-sided internal component adjacent to both C and C'. By Theorem 2.3, there are a 5-sided internal component D and a 4-sided internal component E such that they are adjacent to both C and C'. By Theorem 3.3, E must be a parallel component as in Figure 5.

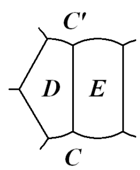

Figure 5: A parallel component E adjacent to D.

Note that the edge between C and E has turning angle $\frac{\pi}{3}$. By Theorem 2.10 and Corollary 3.2, we obtain that C has nine sides. Since $p_1 = p_2 = p_3 > p_4 \geq 0$, it follows that all components around C or around C' are nonhexagonal and convex. Hence we have five nonhexagonal convex components G, G_2, G_3, G_4 and E as in Figure 6.

By Lemma 2.7 and Corollary 3.2, F must have five sides. Then G_1 and G_2 are disjoint. Since all the nine components around C are convex, the convex components G_2, G_3, G_4 and E are disjoint. Similarly, G_1 and E are disjoint. Hence we have five disjoint nonhexagonal convex components as in Figure 6, contradicting Theorem 2.9. □

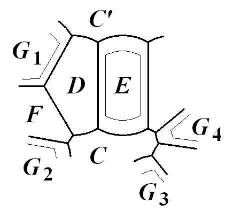

Figure 6: Five disjoint nonhexagonal convex components.

Theorem 3.5. *A minimizing* 4*-bubble with pressures* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers has at most one internal component of the region* R_4*.*

Proof. Suppose that a minimizing bubble has at least two internal components of R_4. By Lemma 2.7 and Theorem 2.11, each internal component of R_4 has seven, eight or nine sides. Let C_1 and C_2 be internal components of R_4.

Case 1. There is a component adjacent to both C_1 and C_2.

Note that each component adjacent to both C_1 and C_2 is internal. By Theorem 3.4, each internal component adjacent to both C_1 and C_2 has five sides. Thus there is only one possibility shown in Figure 7.

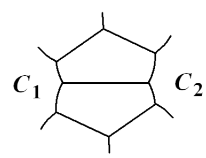

Figure 7: The possibility of a component between two internal components of R_4.

Hence we have five nonhexagonal convex components as in Figure 8. By Corollary 3.2, E_1 and E_2 must be 5-sided. Then D_1 and D_2 are disjoint. Similarly, F_1 and F_2 are disjoint. Therefore we have five disjoint nonhexagonal convex components, contradicting Theorem 2.9.

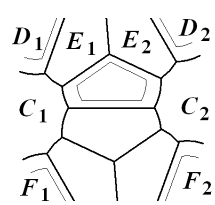

Figure 8: The convex components around C_1 and C_2.

Case 2. There is no component adjacent to both C_1 and C_2.

If each component adjacent to C_1 does not meet a components adjacent to C_2, then we have at least six disjoint nonhexagonal convex components, contradicting Theorem 2.9. Thus there are a component D_1 adjacent to C_1 and a component D_2 adjacent to C_2 such that D_1 is adjacent to D_2 as in Figure 9.

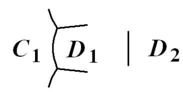

Figure 9: D_1 is adjacent to D_2.

Since D_2 is not adjacent to C_1, it follows that D_1 has at least four sides. Similarly, D_2 has at least four sides. Now, we consider all possibilities for D_1 and D_2 as in Figure 10.

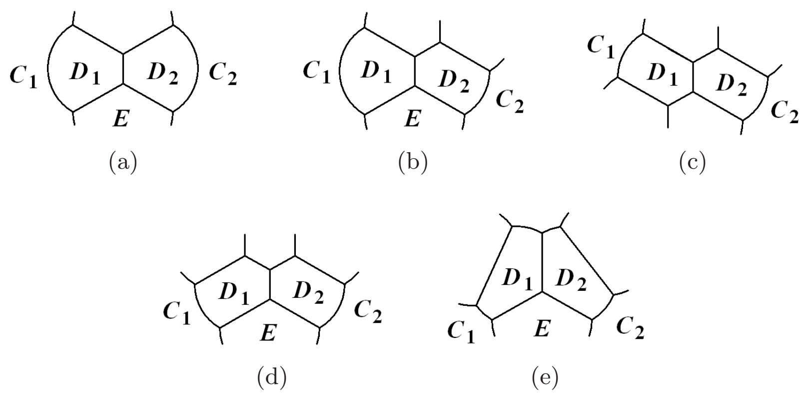

Figure 10: The possibilities for D_1 and D_2.

Configurations (a), (b), (d) and (e) in Figure 10 are impossible as each of them has a component E between C_1 and C_2. Thus D_1 and D_2 are $\frac{\pi}{3}$-cells as configuration (c) in Figure 10. By Theorem 2.10 and Corollary 3.2, we obtain that C_1 and C_2 has nine sides. Hence C_2 is surrounded by nine nonhexagonal convex components. Therefore there exist four of the nine components that are disjoint and do not meet D_1. In total, we have five disjoint nonhexagonal convex components, contradicting Theorem 2.9. □

Corollary 3.6. *A minimizing* 4*-bubble with pressures* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers must have at least one external component of the region* R_4 *and at most one internal component of* R_4*.*

Acknowledgement(s) : This research is (partially) supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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