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On Generalized Stability of an n-Dimensional Quadratic Functional Equation

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In this paper, we establish the general solution and investigate the generalized stability of an n-dimensional quadratic functional equation

$$\sum_{1\le i<j\le n}(f(x_i+x_j)+f(x_i-x_j)) = 2(n-1)\sum_{i=1}^{n} f(x_i)$$

where $n>1$ is an integer.

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1 Introduction

A number of functional equation problems were concerned about the stability of homomorphism asked by S. M. Ulam [4]: *Given a group G_1, a metric group G_2 with metric $d(\cdot,\cdot)$, and a positive real number ε, does there exist a positive real number δ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \le \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $d(f(x), h(x)) \le \varepsilon$ for all $x \in G_1$?*

D. H. Hyers [3] solved the problem in the case of approximately additive mappings under the assumption that G_1 and G_2 are Banach spaces. Later, T. Aoki [1] and Th. M. Rassias [5] generalized the result of Hyers in the following theorem.

Theorem 1.1. *Let G_1 and G_2 be Banach spaces, let $\theta \in [0,\infty)$, and let $p \in [0,1)$. If a function $f : G_1 \to G_2$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in G_1$*, then there is a unique additive mapping* $A : G_1 \to G_2$ *such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in G_1$*. If, in addition,* $f(tx)$ *is continuous in* t *for each fixed* $x \in G_1$*, then* A *is linear.*

Due to the above theorem, the functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability property on (G_1, G_2). Later, many Rassias-type theorems concerning about the stability of various functional equations have been studied.

In this paper, an n-dimensional quadratic functional equation

$$\sum_{1 \leq i < j \leq n} (f(x_i + x_j) + f(x_i - x_j)) = 2(n-1) \sum_{i=1}^{n} f(x_i) \tag{1.1}$$

where $n > 1$ is an integer, as well as its stability will be studied.

2 Main Results

In this section, we will investigate the general solution and generalized stability of the functional equation (1.1).

2.1 General Solution

It is interesting to note that the functional equation (1.1) is equivalent to the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.1}$$

Thus, the solution of (1.1) can be stated as follows:

Theorem 2.1. *Let* X *and* Y *be vector spaces. A mapping* $f : X \to Y$ *satisfies the functional equation (1.1) where* $n > 1$*, for all* $x_1, \ldots, x_n \in X$*, if and only if it satisfies the quadratic functional equation (2.1) for all* $x, y \in X$*.*

Proof. (Necessity) Putting $x_1 = \ldots = x_n = 0$ in (1.1) yields

$$(n-1)\, nf(0) = 2(n-1)\, nf(0).$$

Since $n > 1$, we have

$$f(0) = 0. \tag{2.2}$$

Then, setting $x_1 = x,\ x_2 = y$, and $x_3 = \ldots = x_n = 0$ in (1.1), we have

$$f(x+y) + f(x-y) + 2(n-2)f(x) + 2(n-2)f(y) + (n-2)(n-3)f(0)$$
$$= 2(n-1)\big(f(x) + f(y) + (n-2)f(0)\big).$$

Using the equation (2.2) ensures the validity of (2.1).

(Sufficiency) Assume (2.1) holds. Then,

$$\begin{aligned}\sum_{1\le i<j\le n} \left(f(x_i+x_j) + f(x_i-x_j)\right) &= \sum_{1\le i<j\le n} \left(2f(x_i) + 2f(x_j)\right)\\ &= 2(n-1)\sum_{i=1}^{n} f(x_i).\end{aligned}$$

This completes the proof. □

2.2 Generalized Stability of the Equation

Throughout this section X and Y will be a real normed vector space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x_1, x_2, \ldots, x_n) := \sum_{1\le i<j\le n} \left(f(x_i+x_j) + f(x_i-x_j)\right) - 2(n-1)\sum_{i=1}^{n} f(x_i)$$

for all $x_1, \ldots, x_n \in X$.

Theorem 2.2. *Let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\begin{cases} \sum_{i=0}^{\infty} 4^{-i}\phi\left(2^i x, 2^i x, 0, \ldots, 0\right) & converges\ for\ all\ x \in X,\ and \\ \lim_{m\to\infty} 4^{-m}\phi\left(2^m x_1, 2^m x_2, \ldots, 2^m x_n\right) = 0 & for\ all\ x_1, \ldots, x_n \in X, \end{cases} \tag{2.3}$$

or

$$\begin{cases} \sum_{i=0}^{\infty} 4^{i}\phi\left(2^{-i} x, 2^{-i} x, 0, \ldots, 0\right) & converges\ for\ all\ x \in X,\ and \\ \lim_{m\to\infty} 4^{m}\phi\left(2^{-m} x_1, 2^{-m} x_2, \ldots, 2^{-m} x_n\right) = 0 & for\ all\ x_1, \ldots, x_n \in X. \end{cases} \tag{2.4}$$

If a function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \phi(x_1, x_2, \ldots, x_n) \tag{2.5}$$

for all $x_1, \ldots, x_n \in X$, and, in addition, $f(0) = 0$ if (2.4) holds, then there is a unique function $Q : X \to Y$ such that Q satisfies (1.1) and, for all $x \in X$,

$$\left\| f(x) + \frac{n^2-n-3}{3} f(0) - Q(x) \right\| \le \begin{cases} \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \phi\left(2^i x, 2^i x, 0, \ldots, 0\right) & if\ (2.3)\ holds \\ \frac{1}{4} \sum_{i=1}^{\infty} 4^{i} \phi\left(2^{-i} x, 2^{-i} x, 0, \ldots, 0\right) & if\ (2.4)\ holds \end{cases} \tag{2.6}$$

The function Q is given by

$$Q(x) = \begin{cases} \lim_{m\to\infty} 4^{-m} f(2^m x) & if\ (2.3)\ holds \\ \lim_{m\to\infty} 4^{m} f(2^{-m} x) & if\ (2.4)\ holds \end{cases} \tag{2.7}$$

for all $x \in X$.

Proof. We will first prove the case when condition (2.3) holds. Let $g : X \to Y$ be the function defined by $g(x) := f(x) + \frac{n^2-n-3}{3} f(0)$ for all $x \in X$. Putting $x_1 = x_2 = x$ and $x_3 = \ldots = x_n = 0$ in (2.5) yields

$$\begin{aligned} \|g(2x) - 4g(x)\| &= \left\| f(2x) - 4f(x) - \left(n^2 - n - 3\right) f(0) \right\| \\ &= \|Df(x, x, 0, \ldots, 0)\| \\ &\le \phi(x, x, 0, \ldots, 0) \end{aligned}$$

for all $x \in X$. Dividing the above relation by 4 yields

$$\left\| \frac{g(2x)}{4} - g(x) \right\| \le \frac{1}{4} \phi(x, x, 0, \ldots, 0). \tag{2.8}$$

Therefore,

$$\begin{aligned} \left\| \frac{g(2^m x)}{4^m} - g(x) \right\| &= \left\| \sum_{i=0}^{m-1} \left(\frac{g\left(2^{i+1} x\right)}{4^{i+1}} - \frac{g\left(2^i x\right)}{4^i} \right) \right\| \\ &\le \sum_{i=0}^{m-1} \frac{1}{4^i} \left\| \frac{g\left(2 \cdot 2^i x\right)}{4} - g\left(2^i x\right) \right\| \\ &\le \frac{1}{4} \sum_{i=0}^{m-1} 4^{-i} \phi\left(2^i x, 2^i x, 0, \ldots, 0\right) \end{aligned} \tag{2.9}$$

for any positive integer m and for all $x \in X$.

We will show that the sequence $\left\{\frac{g(2^m x)}{4^m}\right\}$ converges for all $x \in X$. For every positive integer l and m, we consider

$$\begin{aligned}\left\|\frac{g\left(2^{l+m}x\right)}{4^{l+m}} - \frac{g(2^m x)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{g\left(2^l \cdot 2^m x\right)}{4^l} - g\left(2^m x\right)\right\| \\ &\le \frac{1}{4^m}\sum_{i=0}^{l-1} 4^{-i}\frac{1}{4}\phi\left(2^i \cdot 2^m x, 2^i \cdot 2^m x, 0, \ldots, 0\right) \\ &= \frac{1}{4}\sum_{i=0}^{l-1} 4^{-(i+m)}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right).\end{aligned}$$

By condition (2.3), the right-hand side approaches 0 when m tends to infinity. Thus, the sequence $\left\{\frac{g(2^m x)}{4^m}\right\}$ is a Cauchy sequence. Since a Banach space is complete, we can define

$$Q(x) \quad := \quad \lim_{m\to\infty}\frac{g\left(2^m x\right)}{4^m}$$

for all $x \in X$. Consequently, by passing to the limit in (2.9) when m goes to infinity, it follows that

$$\|Q(x) - g(x)\| \le \frac{1}{4}\sum_{i=o}^{\infty} 4^{-i}\phi\left(2^i x, 2^i x, 0, \ldots, 0\right)$$

for all $x \in X$. This inequality implies the validity of (2.6). Moreover, let $x_1, \ldots, x_n$ be any points in X. We have

$$\begin{aligned}\|DQ(x_1, \ldots, x_n)\| &= \lim_{m\to\infty} 4^{-m}\left\|Dg\left(2^m x_1, \ldots, 2^m x_n\right)\right\| \\ &\le \lim_{m\to\infty} 4^{-m}\left(\left\|Df\left(2^m x_1, \ldots, 2^m x_n\right)\right\| + \left|\frac{n(n-1)\left(n^2-n-3\right)}{3}\right|\|f(0)\|\right) \\ &\le \lim_{m\to\infty} 4^{-m}\phi\left(2^m x_1, \ldots, 2^m x_n\right).\end{aligned}$$

Using condition (2.3) the right-hand side tends to 0. Hence, Q satisfies (1.1) for all $x_1, \ldots, x_n \in X$ which implies that Q is a quadratic function. It should be noted that $Q(ax) = a^2 Q(x)$ for every positive integer a and for all $x \in X$. [2]

Now, we prove the uniqueness of Q. Let $Q' : X \to Y$ be another function satisfying (1.1) and (2.3). Therefore,

$$\begin{aligned}\|Q'(x) - Q(x)\| &= \frac{1}{4^m}\left\|Q'\left(2^m x\right) - Q\left(2^m x\right)\right\| \\ &\le \frac{1}{4^m}\left\|Q'\left(2^m x\right) - g\left(2^m x\right)\right\| + \frac{1}{4^m}\left\|g\left(2^m x\right) - Q\left(2^m x\right)\right\| \\ &\le 2 \cdot \frac{1}{4^m}\frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right) \\ &\le \frac{1}{2}\sum_{i=0}^{\infty} 4^{-(i+m)}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right)\end{aligned}$$

for all $x \in X$. By condition (2.3), the right-hand side goes to 0 as m tends to infinity, and it follows that $Q'(x) = Q(x)$ for all $x \in X$. Hence, Q is unique.

For the case that condition (2.4) holds, we can state the proof in a similar manner as in the case which the condition (2.3) holds with the additional condition, $f(0) = 0$. Starting by replacing x with $\frac{x}{2}$ in (2.8) and multiplying by 4, we get

$$\left\| g(x) - 4g\left(\frac{x}{2}\right) \right\| \le \phi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

for all $x \in X$. This inequality can be extended by mathematical induction to

$$\begin{aligned} \left\| g(x) - 4^m g\left(\tfrac{x}{2^m}\right) \right\| &\le \sum_{i=0}^{m-1} 4^i \phi\left(\tfrac{x}{2^{i+1}}, \tfrac{x}{2^{i+1}}, 0, \ldots, 0\right) \\ &\le \tfrac{1}{4} \sum_{i=1}^{m} 4^i \phi\left(2^{-i}x, 2^{-i}x, 0, \ldots, 0\right) \end{aligned}$$

for any positive integer m and for all $x \in X$.

We can show that a sequence $\left\{4^m g\left(\frac{x}{2^m}\right)\right\}$ converges for all $x \in X$ and let

$$Q(x) := \lim_{m \to \infty} 4^m f\left(2^{-m}x\right)$$

for all $x \in X$. The rest of the proof is similar to the corresponding part of the proof of the previous case. Thus, it will be omitted. □

Corollary 2.3. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \ldots, x_n)\| \le \varepsilon$$

for all $x_1, \ldots, x_n \in X$ and for some real number $\varepsilon > 0$, then there exists a unique function $Q : X \to Y$ such that Q satisfies (1.1) and

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \le \frac{\varepsilon}{3}$$

for all $x \in X$.

Proof. We choose $\phi(x_1, \ldots, x_n) = \varepsilon$ for all $x_1, \ldots, x_n \in X$. Being in condition (2.3) in Theorem 2.2, it follows that

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \le \frac{1}{4} \sum_{i=0}^{\infty} \frac{\varepsilon}{4^i} = \frac{\varepsilon}{3}$$

for all $x \in X$ as desired. □

Corollary 2.4. *Given positive real number* ε *and* p *with* $p \neq 2$. *If a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, \ldots, x_n)\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$, *then there exists a unique function* $Q : X \to Y$ *such that* Q *satisfies (1.1) and*

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{\varepsilon}{|2 - 2^{p-1}|} \|x\|^p$$

for all $x \in X$.

Proof. We choose $\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. If $0 < p < 2$, then condition (2.3) in Theorem 2.2 is fulfilled, and consequently

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \varepsilon \cdot 2 \left\| 2^i x \right\|^p = \frac{\varepsilon}{2 - 2^{p-1}} \|x\|^p$$

for all $x \in X$. If $p > 2$, the condition (2.4) in Theorem 2.2 is fulfilled, and consequently

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{1}{4} \sum_{i=1}^{\infty} 4^{i} \varepsilon \cdot 2 \left\| 2^{-i} x \right\|^p = \frac{\varepsilon}{2^{p-1} - 2} \|x\|^p$$

for all $x \in X$. □

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