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# An Effective Method for Calculating the Sum of an Infinite Series of Legendre Polynomials

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**Abstract :** To evaluate series of Legendre polynomials of the form $\sum_{n=0}^{\infty} a_n P_n(z)$, we find closed forms for Legendre polynomials that are good approximations for large $n$ and are suitable for calculations on a personal computer of medium capacity. The convergence of an infinite series of Legendre polynomials is discussed.

**Keywords :** Legendre polynomial; Closed form; Infinite series; Approximation.
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## 1 Introduction

The analytical solutions of many problems in physics are frequently given in the form of infinite series. In particular, the use of series of Legendre polynomials $\sum_{n=0}^{\infty} a_n P_n(z)$ are often used in electricity problems. Since the numerical calculation of infinite series is tedious and inconvenient, especially for slowly convergent series, a high-performance computer may be needed for this purpose. Alternatively, an efficient method may reduce the computer task. In this paper, we will develop a technique of calculating the sum of an infinite series of Legendre polynomials by looking for specific closed forms that are good approximations for high-order Legendre polynomials.

## 2 Solution of Legendre Equation

Consider the Legendre equation

$$(1-z^2)y'' - 2zy' + n(n+1)y = 0 \tag{2.1}$$

where $n$ is nonnegative integer and $z$ is complex number. Solving (2.1) by employing Frobenius's method, we find that the solution is in the form of Legendre

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polynomials $P_n(z)$ and Legendre functions of the second kind $Q_n(z)$, namely

$$y = A_n P_n(z) + B_n Q_n(z). \tag{2.2}$$

Since the main purpose of this paper is to study Legendre polynomials, we will omit $Q_n(z)$ and restrict our attention to $P_n(z)$ which can be written in the form

$$P_n(z) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)! z^{n-2r}}{2^n (n-r)! r! (n-2r)!} \tag{2.3}$$

where $p = n/2$ when $n$ is an even number and $p = (n-1)/2$ when $n$ is an odd number and the sum exists for all $z$ . In addition, we can obtain another form of $P_n(z)$ from the generating function given by

$$(1 - 2hz + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(z) \tag{2.4}$$

where $|2hz - h^2| < 1$. For $|z| < 1$, this condition is satisfied for $|h| < 1$. When $|z| > 1$ we let $z = \cosh t$ and then the condition is satisfied if $|h| < e^{-t} < 1$ for $t > 0$. Consequently,

$$P_n(1) = 1,\ P_n(-1) = (-1)^n,\ P_{2n+1}(0) = 0,\ P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}. \tag{2.5}$$

Finally, we consider Laplace's integral for Legendre polynomial, $P_n(z)$, which is in the form

$$P_n(z) = \frac{1}{\pi} \int_0^{\pi} \{z + (z^2-1)^{1/2} \cos\phi\}^n d\phi. \tag{2.6}$$

# 3 Closed Form of Legendre Polynomial for Large $n$

In this section, we first develop the formulae of $P_n(z)$ obtained from the generating function and the Laplace's integral for Legendre polynomials for $|z| < 1$. Finally, we will discuss the process to obtain closed forms of $P_n(z)$ for both $|z| < 1$ and $|z| > 1$ that are useful for calculating $P_n(z)$ when $n$ is large.

$P_n(z)$ **derived from Generating Function**

We can find a form of $P_n(\cos\theta)$ by substituting $z = \cos\theta$ in (2.4) and obtain

$$\begin{aligned} P_n(\cos\theta) \quad = \quad & \frac{(2n)!}{2^{2n+1}(n!)^2}\{2\cos n\theta + \frac{1(2n)}{2(2n-1)} 2\cos(n-2)\theta \\ & + \frac{1\cdot 3(2n)(2n-2)}{2\cdot 4(2n-1)(2n-3)} 2\cos(n-4)\theta + \ldots\}. \end{aligned} \tag{3.1}$$

$P_n(z)$ **derived from Laplace's integral**
On substituting $z = \cos\theta$ in equation (2.6) we obtain

$$P_n(\cos\theta) = \cos^n\theta\{1 - \frac{n(n-1)}{2^2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{2^2\cdot 4^2}\tan^4\theta + \ldots\}. \tag{3.2}$$

## 3.1 Closed Form for $|z| < 1$

If we calculate a numerical solution of $\sum_{n=0}^{\infty} a_n P_n(z)$ which is slowly convergent by using a medium capacity computer, we find that the computer may or may not give us a value of $P_n(z)$ when $n$ is large (see Table 1). Therefore, we find a closed form for large $n$ that can be used in summing this kind of slowly convergent series.

Consider the generating function (2.4) with $z = \cos\theta$ where $0 < \theta < \pi$. We find that

$$\begin{aligned}\sum_{n=0}^{\infty} h^n P_n(\cos\theta) &= (1 - 2h\cos\theta + h^2)^{-1/2} \\ &= \sum_{r=0}^{\infty}(-\frac{1}{2}, r)\frac{(e^{-i\theta} - h)^{r-1/2}}{(2\sin\theta)^{r+1/2}} \text{ where } |e^{-i\theta} - h| < 2\sin\theta\end{aligned} \tag{3.3}$$

and where $(\alpha, r) = \dfrac{\Gamma(\alpha+1)}{r!\Gamma(\alpha+1-r)}$. In addition, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} h^n P_n(\cos\theta) &= (1 - 2h\cos\theta + h^2)^{-1/2} \\ &= \sum_{r=0}^{\infty}(-\frac{1}{2}, r)\frac{(e^{i\theta} - h)^{r-\frac{1}{2}}}{(-2i\sin\theta)^{r+1/2}} \text{ where } |e^{i\theta} - h| < 2\sin\theta.\end{aligned} \tag{3.4}$$

Consequently, we note from equations (3.3)-(3.4) that the conditions for convergence are satisfied for $|h| < 1$.

After adding equation (3.3) to (3.4), we obtain

$$2P_n(\cos\theta) = \frac{2(-1)^n}{(2\sin\theta)^{1/2}}\sum_{r=0}^{\infty}(-\frac{1}{2}, r)(r - \frac{1}{2}, n)\frac{\cos(n + 1/2 - r)\theta - \pi/4 - r\pi/2}{(2\sin\theta)^r} \tag{3.5}$$

where

$$(-\frac{1}{2}, r)(r-\frac{1}{2}, n) = \frac{\Gamma(\frac{-1}{2}+1)\Gamma(r - \frac{1}{2}+1)}{r!\Gamma(\frac{-1}{2}+1-r)n!\Gamma(r-\frac{1}{2}+1-n)} = \frac{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{r!\Gamma(\frac{1}{2}-r)n!\Gamma(r+\frac{1}{2}-n)}$$

or

$$P_n(\cos\theta) = \frac{(-1)^n\sqrt{\pi}\cos((n+\frac{1}{2})\theta-\frac{\pi}{4})}{n!(2\sin\theta)^{\frac{1}{2}}\Gamma(\frac{1}{2}-n)} + \frac{(-1)^n}{(2\sin\theta)^{\frac{1}{2}}}\sum_{r=1}^{\infty}\frac{(-1)^r\{\Gamma(r+\frac{1}{2})\}^2\cos((n+\frac{1}{2}-r)\theta-\frac{\pi}{4}-r\frac{\pi}{2})}{\sqrt{\pi}r!n!\Gamma(\frac{1}{2}+r-n)(2\sin\theta)^r}. \tag{3.6}$$

The formula on the right is difficult and time-consuming to calculate numerically. However, we have found that a good approximation can be obtained as follows. We use Stirling's formula to approximate the Gamma functions in the first term of equation (3.6). We then write

$$P_n(\cos\theta) = v(n,\theta)(1+K(n,\theta)), \tag{3.7}$$

where

$$v(n,\theta) = \frac{1}{\sqrt{2n\pi\sin\theta}}\cos((n+\tfrac{1}{2})\theta-\tfrac{\pi}{4}) \tag{3.8}$$

is obtained from the Stirling's formula approximation. We then compared the values of $P_n(\cos\theta)$ with the values of $v(n,\theta)$ and found that the values of $1+K(n,\theta)$ were approximately equal to 2 with an error of approximately 1% or less.

Therefore, a closed form approximation of a Legendre Polynomial can be expressed by

$$P_n(\cos\theta) \approx 2v(n,\theta) = (\frac{2}{n\pi\sin\theta})^{1/2}\cos((n+1/2)\theta-\pi/4) \text{ where } 0<\theta<\pi. \tag{3.9}$$

We observe from Tables 1 and 2 that the formulae for Legendre polynomials in equations (2.3), (3.1) developed from the generating function, and in (3.2) developed from the Laplace's integral may give undetermined values for some large values of $n$. We note here that *NaN* from MATLAB represents the results of mathematically undefined operations like 0/0 and *inf*-*inf* where *inf* is positive infinity that are produced by operations such as dividing by zero or from overflow. The formula from (3.9) gives good approximate values of Legendre Polynomials when compared with other formulae.

## 3.2 Closed Form for $|z| > 1$

We start by finding a closed form for $|z| > 1$ from the generating function for the Legendre Polynomials by substituting $z = \cosh t$ where $t > 0$ into (2.4). We find that

$$\sum_{n=0}^{\infty} h^n P_n(\cosh t) = \sum_{n=0}^{\infty}\{h^n\sum_{r=0}^{\infty}(-\frac{1}{2},r)(r-\frac{1}{2},n)(-1)^n\frac{e^{(n-r+\frac{1}{2})t}}{(2\sinh t)^{r+\frac{1}{2}}}\} \quad \text{where } |h| < e^{-t} < 1 \tag{3.10}$$

Table 1: Values of $P_n(z)$ when $z = \cos 3\pi/8$ calculated from MATLAB, equations (2.3), (3.1), (3.2), and (3.9), respectively

| $n$ | MATLAB | eq.(2.3) | eq.(3.1) | eq.(3.2) | eq.(3.9) |
|---|---|---|---|---|---|
| 5 | 0.2918 | 0.2918 | 0.2918 | 0.2918 | 0.3087* |
| 10 | 0.1412 | 0.1412 | 0.1412 | 0.1412 | 0.1458* |
| 15 | 0.0404 | 0.0404 | 0.0404 | 0.0404 | 0.0418* |
| 20 | -0.0362 | -0.0362 | -0.0362 | -0.0362 | -0.0362 |
| .. | ...... | ...... | ...... | ...... | ...... |
| 85 | 0.0746 | 0.0749 | 0.0746 | 0.0746 | 0.0749 |
| 90 | 0.0484 | NaN | NaN | 0.0484 | 0.0486 |
| .. | ...... | ...... | ...... | ...... | ...... |
| 400 | 0.0407 | NaN | NaN | 1.2705e+028 | 0.0407 |
| 405 | 0.0343 | NaN | NaN | NaN | 0.0343 |

* The closed form formulae are not accurate for small values of $n$ as Stirling's formula is not a good approximation to the Gamma function for small $n$.

or

$$P_n(\cosh t) = \frac{(-1)^n \pi e^{(n+\frac{1}{2})t}}{n!\sqrt{2\pi \sinh t}\,\Gamma(\frac{1}{2}-n)} + \frac{(-1)^n e^{(n+\frac{1}{2})t}}{n!\sqrt{2\pi \sinh t}} \sum_{r=1}^{\infty} \frac{\Gamma(r+\frac{1}{2})\Gamma(r+\frac{1}{2})}{r!\Gamma(r-n+\frac{1}{2})} \left(\frac{-e^{-t}}{2\sinh t}\right)^r. \tag{3.11}$$

We can obtain an approximate closed form for $P_n(\cosh t)$ by a similar method to the method we used to obtain an approximate closed for $P_n(\cos\theta)$ in equation (3.6). We use Stirling's formula to approximate the Gamma functions in the first term of equation (3.11). We then write

$$P_n(\cosh t) = v(n,t)(1 + K(n,t)), \tag{3.12}$$

where

$$v(n,t) = \frac{e^{(n+\frac{1}{2})t}}{\sqrt{2n\pi \sinh t}} \tag{3.13}$$

is obtained from the Stirling's formula approximation. We then compared the values of $P_n(\cosh t)$ with the values of $v(n,t)$ and found that the values of $1 + K(n,t)$ were approximately equal to 1 with an error of approximately 1% or less.

Therefore, a closed form approximation of a Legendre Polynomial can be expressed by

$$P_n(\cosh t) \approx v(n,t) = \frac{e^{(n+\frac{1}{2})t}}{\sqrt{2n\pi \sinh t}} \text{ where } t > 0. \tag{3.14}$$

Table 2: Values of $P_n(z)$ when $z = \cosh 3\pi/8$ calculated from equations (2.3)and (3.14), respectively

| $n$ | eq.(2.3) | eq.(3.14) |
|---|---|---|
| 5 | 9.4090e+001 | 9.5889e+001 |
| 10 | 2.4280e+004 | 2.4517e+004 |
| 15 | 7.1910e+006 | 7.2380e+006 |
| 20 | 2.2554e+009 | 2.2665e+009 |
| .. | ...... | ...... |
| 85 | 1.9830e+042 | 1.9853e+042 |
| 90 | $NaN$ | 6.9764e+044 |
| .. | ...... | ...... |
| 400 | $NaN$ | 1.3442e+203 |
| 405 | $NaN$ | 4.8304e+205 |

Since MATLAB only computes values for $P_n(z)$ for $|z| < 1$ the formulae we compare for calculating $P_n(z)$ for $|z| > 1$ are from equation (2.3) and the closed form of (3.14). The results are shown in Table 2.

To calculate $\sum_{n=0}^{\infty} a_n P_n(z)$, it is necessary for us to examine convergence of the series as follows.

## 3.3 Convergence of $\sum_{n=0}^{\infty} a_n P_n(z)$

Since $\displaystyle\sum_{n=0}^{\infty} a_n P_n(z) \le \sum_{n=0}^{\infty} |a_n|$ when $|z| < 1$ or $z = \cos\theta$ where $0 < \theta < \pi$, we can conclude that the series converges when $\sum_{n=0}^{\infty} |a_n|$ converges. For $|z| > 1$ or $z = \cosh t$ where $t > 0$, we find the condition of convergence of the series by considering

$$\sum_{n=0}^{\infty} a_n P_n(\cosh t) = \sum_{n=0}^{N-1} a_n P_n(\cosh t) + \sum_{N}^{\infty} a_n \frac{e^{(n+1/2)t}}{\sqrt{2n\pi \sinh t}}. \tag{3.15}$$

Note that the series of the left hand side converges when $\displaystyle\sum_{N}^{\infty} a_n \frac{e^{(n+1/2)t}}{\sqrt{2n\pi \sinh t}}$ converges. Hence, the condition of convergence is $1 < z < \frac{1}{2}(L + \frac{1}{L})$ where $L = \displaystyle\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

To find $\displaystyle\sum_{n=0}^{\infty} a_n P_n(z)$ when the series is slowly convergent, we need to calculate many terms of $a_n P_n(z)$. As shown in the previous section, the formulae obtained from (2.3), (3.1) and (3.2) may not give efficient results while MATLAB is limited by the condition $|z| < 1$. In this section we shall employ the closed forms of Legendre polynomial in (3.9) for $|z| < 1$ and (3.14) for $|z| > 1$ when $n$ is large.

We next designate $\sum_{n=0}^{\infty} a_n P_n(z)$ as

$$\sum_{n=0}^{\infty} a_n P_n(z) = \sum_{n=0}^{N} a_n P_n(z) + \sum_{N+1}^{\infty} a_n P_n(z). \tag{3.16}$$

In equation (3.16), for $|z| < 1$ we may use formulae of $P_n(z)$ from MATLAB, or from equations (2.3), (3.1), (3.2) to calculate the first part of the right hand side in which $n$ is small. For the second part, where $n$ is large, we use the closed form in equation ( 3.9) to compute $P_n(z)$.

Similarly, for $|z| > 1$ we will calculate $P_n(z)$ in the first part, where $n$ is small, by using (2.3) and in the second part, where $n$ is large, by using the closed form of (3.14).

In Tables 3 and 4, we show examples of calculations of the $\sum_{n=0}^{\infty} a_n P_n(z)$ by using $N = 19$ and (2.3) for the first part and (3.9) for $|z| < 1$ and (3.14)for $|z| > 1$ for the second part.

Table 3: Values of $\sum_{n=0}^{\infty} a_n P_n(z)$ when $a_n = \frac{25n}{n^2+5n+1}$ and $z = 0.1$

| $n$ | $a_n P_n(z)$ | $\sum_{n=0}^{\infty} a_n P_n(z)$ | $\|a_n P_n(z) - a_{n-1} P_{n-1}(z)\|$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0.3571 | 0.3571 | 0.3571 |
| 2 | -1.6167 | -1.2595 | 1.9738 |
| .. | ...... | ...... | ...... |
| 19 | -0.1746 | -0.9498 | 0.2309 |
| 20 | -0.0828 | -1.0326 | 0.0918 |
| .. | ...... | ...... | ...... |
| 918 | 0.0004 | -0.9072 | 0.0010 |
| 919 | 0.0006 | -0.9066 | 0.0002 |

# 4 Advantages of Using Closed Forms of Legendre Polynomial

As mentioned earlier, MATLAB only provides a command to find $P_n(z)$ when $|z| < 1$. Therefore, in the case of $|z| > 1$ it is necessary for us to use the closed form of $P_n(z)$ given in (3.14). In addition, we find that a drawback incurred from using MATLAB is that MATLAB requires considerable amount of memory and time to

Table 4: Values of $\sum_{n=0}^{\infty} a_n P_n(z)$ when $a_n = n^2/2^n$ and $z = 1.235$

| $n$ | $a_n P_n(z)$ | $\sum_{n=0}^{\infty} a_n P_n(z)$ | $\|a_n P_n(z) - a_{n-1} P_{n-1}(z)\|$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0.6175 | 0.6175 | 0.6175 |
| 2 | 1.7878 | 2.4053 | 1.1703 |
| .. | ...... | ...... | ...... |
| 19 | 3.6850 | 3.3226e+002 | 2.1751 |
| 20 | 3.9063 | 3.7132e+002 | 2.2129 |
| .. | ...... | ...... | ...... |
| 591 | 5.6595e-002 | 1.4770e+004 | 1.0166e-003 |
| 592 | 5.5596e-002 | 1.4770e+004 | 9.9890e-004 |

finish evaluating the data. For the computations in Table 1, for instance, MATLAB takes about 0.4056 seconds to calculate $P_n(z)$ while $P_n(z)$ can be calculated from our closed form of (3.9) in 0.0312 seconds. In particular, MATLAB does not seem to be appropriate for finding the sum of a slowly converging series $\sum_{n=0}^{\infty} a_n P_n(z)$.

# 5 Conclusion

In this paper, we have derived approximate closed forms of Legendre polynomials that can be easily calculated in comparison with the original forms. It is expected to be an advantage in several algorithm designs to be able to efficiently sum infinite series of this polynomial. In particular, numerical calculation of the sums of slowly converging series can be easily performed on a medium performance computer.

Moreover, the closed form methods used in this paper are very effective for comparing the rates of convergence of an infinite series of Legendre polynomials $\sum_{n=0}^{\infty} a_n P_n(z)$ and of Chebyshev polynomials $\sum_{n=0}^{\infty} a_n^{'} T_n(z)$ for the same function of $z$ when $z$ lies in the the radius of convergence for both series. This comparison is still in the research process.

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# References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York,

1972.

[2] G. E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, 2000.

[3] R. E. Attar, Legendre Polynomials and Functions, Createspace, 2009.

[4] S. S. Bayin, Mathematical Methods in Science and Engineering, Wiley, 2006.

[5] J. S. R. Chisholm and R. M. Morris, Mathematical Methods in Physics, North-Holland Publishing Company, Amsterdam, 1966.

[6] E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, 1989.

[7] C. Dunkl, M. Ismail and R. Wong, Proceeding of the International Workshop Special Functions, World Scientific, 1999.

[8] V. C. A. Ferraro, Electromagnetic Theory, Athlone Press, London, 1962.

[9] J. Irving and N. Mullineux, Mathematics in Physics and Engineering, Academic Press, New York, 1959.

[10] E. Kreyszig, Advanced Engineering Mathematics, Wiley, 2005.

[11] Z. Shanjie and J. Jianming, Computation of Special Functions, Wiley-Interscience, New York, 1996.

[12] L. C. Shen and J. A. Kong, Applied Electromagnetism, PWS Publishing Company, Boston, 1995.

[13] W. R. Wade, An Introduction to Analysis, Prentice Hall, 2009.

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