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Existence of Positive Solutions to a Second -Order Multi-Point Boundary Value Problem with Delay

J. Chen, J. Tariboon and S. Koonprasert

In this paper, by using the Krasnosel'skii fixed-point theorem, we study the existence of positive solutions to the second-order delay differential equation,

$$\begin{aligned} &u''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in J = [0,1],\\ &u(t) = \beta u(\eta), \qquad -\tau \leqslant t \leqslant 0,\\ &u(1) = \alpha u(\eta), \end{aligned}$$

where α, β, η are constants with $\eta \in (0,1)$. λ is a positive real parameter.

Keywords : Positive solution; Fixed point; Multi-point boundary value problem; Delay.
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1 Introduction

Multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of the three-point boundary value problem for nonlinear ordinary differential equations was initiated by Gupta [1-2]. Since then, nonlinear multi-point boundary value problems have been studied by several authors. For details, see, for example, [3-7] and reference therein.

In this paper, we consider the existence of positive solutions for the following multi-point boundary value problem of the second order delay differential equation:

$$
\begin{aligned}
&u''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in J = [0,1], \\
&u(t) = \beta u(\eta), \qquad -\tau \leqslant t \leqslant 0, \\
&u(1) = \alpha u(\eta),
\end{aligned} \tag{1.1}
$$

where α, β, η are constants with $\eta \in (0,1)$. λ is a positive real parameter.

For the case $0 < \tau < \frac{1}{2}$, $\beta = \alpha = 0$, Bai and Xu [6] studied the existence of multiple positive solutions to BVP (1.1) with $a(t)f(t,u) = g(t,u)$ by using Krasnoselskii fixed-point theorem.

Recently, for the case $0 < \tau < 1$, $\beta = 0$, Wang and Shen [7] given some sufficient conditions with λ belonging to an open interval of eigenvalues to ensure the existence of positive solutions to BVP (1.1).

2 Preliminaries

In this section we give the following definition of positive solution of (1.1).

Definition 2.1. *$u(t)$ is called a positive solution of* (1.1) *if $u \in C[-\tau, 1] \cap C^2(0,1)$, $u(t) > 0$ for $t \in (0,1)$ and satisfies* (1.1).

Lemma 2.1. *Assume $\beta \neq \frac{1-\alpha\eta}{1-\eta}$. Then for $y \in C([0,1], R)$, the problem*

$$u''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$u(0) = \beta u(\eta), \qquad u(1) = \alpha u(\eta), \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds + \frac{\beta + (\alpha - \beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta, s) y(s) ds, \tag{2.3}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{2.4}$$

Proof. It is well known that the Green's function is $G(t,s)$ as in (2.4) for the second-order two point linear boundary value problem

$$\begin{cases} w'' + y(t) = 0, & 0 < t < 1 \\ w(0) = 0, \quad w(1) = 0, \end{cases} \tag{2.5}$$

and the solution of (2.5) is given by

$$w(t) = \int_0^1 G(t,s)y(s)ds,$$

and

$$w(0) = 0, \quad w(1) = 0, \quad w(\eta) = \int_0^1 G(\eta,s)y(s)ds. \tag{2.6}$$

We suppose that the solution of the three-point boundary value problem (2.1), (2.2) can be expressed by

$$u(t) = w(t) + A + Bt, \tag{2.7}$$

where A and B are constants that will be determined.

From (2.6), (2.7) we know that

$$\begin{aligned} u(0) &= A, \\ u(1) &= A + B, \\ u(\eta) &= w(\eta) + A + B\eta. \end{aligned}$$

Putting these into (2.2) yields

$$\begin{aligned} (1-\beta)A - \beta\eta B &= \beta w(\eta), \\ (1-\alpha)A + (1-\alpha\eta)B &= \alpha w(\eta). \end{aligned}$$

Since $\beta \neq \frac{1-\alpha\eta}{1-\eta}$, solving the system of linear equations on the unknowns A, B, we obtain

$$\begin{aligned} A &= \frac{\beta w(\eta)}{(1-\alpha\eta) - \beta(1-\eta)}, \\ B &= \frac{(\alpha-\beta)w(\eta)}{(1-\alpha\eta) - \beta(1-\eta)}. \end{aligned}$$

Hence

$$u(t) = w(t) + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} w(\eta).$$

This implies that

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta,s)y(s)ds.$$

Next, we will show that the solution $u(t)$ is unique. Assume that $v(t)$ is another solution of the three-point boundary value problem (2.1), (2.2).

Let $z(t) = v(t) - u(t)$, $t \in [0,1]$. Then, we get $z''(t) = v''(t) - u''(t) = 0$, $t \in [0,1]$ therefore

$$z(t) = C_1 t + C_2, \tag{2.8}$$

where C_1, C_2 are undetermined constants. From (2.2), we have

$$z(0) = \beta z(\eta), \qquad z(1) = \alpha z(\eta). \tag{2.9}$$

Using (2.8), we obtain

$$z(0) = C_2, \qquad z(1) = C_1 + C_2, \qquad z(\eta) = C_1\eta + C_2. \tag{2.10}$$

From (2.9), (2.10) we know that

$$\begin{aligned} -\beta\eta C_1 + (1-\beta)C_2 = 0,\\ (1-\alpha\eta)C_1 + (1-\alpha)C_2 = 0. \end{aligned}$$

Since $\beta \neq \frac{1-\alpha\eta}{1-\eta}$, then the system of linear equations on the unknown numbers C_1, C_2, has exactly one solution, therefore $z(t) \equiv 0$, $t \in [0,1]$, so $v(t) = u(t)$, $t \in [0,1]$, that is uniqueness of the solution. □

Lemma 2.2. [5] *Let* $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$. *If* $f \in C([0,1],[0,\infty))$, *then the unique solution to problem* (2.1), (2.2) *satisfies*

$$u(t) \geqslant 0, \qquad t \in [0,1].$$

Lemma 2.3. [5] *Let* $\alpha\eta \neq 1$, $\beta > \max\{\frac{1-\alpha\eta}{1-\eta}, 0\}$ *and* $f \in C([0,1],[0,\infty))$, *then problem* (2.1), (2.2) *has no nonnegative solutions.*

Hence, in this paper, we always assume the following condition is satisfied
(H): $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$, $0 < \tau < 1$, $a : (0,1) \to [0,\infty)$ is continuous and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $0 < \int_0^1 s(1-s)a(s)ds < \infty$. There exist constants $0 \leqslant b < c \leqslant 1-\tau$ such that $\int_{b+\tau}^{c+\tau} a(s)ds > 0$.

By Lemma 2.1-2.3, it is easy to see that the BVP (1.1) has a solution $u = u(t)$ if and only if u is a solution of the operator equation $u = Tu$, where

$$Tu(t) = \begin{cases} \beta u(\eta), & -\tau \leqslant t \leqslant 0, \\ \lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds & \\ +\lambda \frac{\beta+(\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(\eta,s)a(s)f(s,u(s-\tau))ds, & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{2.11}$$

Let

$$\begin{aligned}P =&\{u \in C[-\tau, 1] : u(t) \geqslant 0 \text{ for } t \in (-\tau, 1], u(t) = \beta u(\eta), -\tau \leqslant t \leqslant 0,\\ &u(1) = \alpha u(\eta)\}.\end{aligned}$$

It is clear that $C[-\tau, 1]$ with norm $\|u\| = \sup\{|u(t)| : -\tau \leqslant t \leqslant 1\}$ is a Banach space.
Put

$$p(t) = \min\left\{\frac{\beta\eta + (1-\beta)t}{\eta}, \frac{1-\alpha\eta + (\alpha-1)t}{1-\eta}\right\},$$
$$\theta = \min\left\{\frac{\beta\eta + (1-\beta)b}{\eta}, \frac{1-\alpha\eta + (\alpha-1)c}{1-\eta}\right\}\min\{\eta, 1-\eta\},$$

and a cone K in $C[-\tau, 1]$ is defined by

$$K = \left\{u \in C[-\tau, 1] : u(t) \geqslant 0 \text{ for } t \in J, \min_{b \leqslant t \leqslant c} u(t) \geqslant \theta\|u\|\right\}$$

We now state and prove the following lemmas before stating our main results.

Lemma 2.4. *Assume that* $u \in P \cap C^2(0, 1)$ *and* $u''(t) \leqslant 0$ *for* $t \in J$*, then* $u(t) \geqslant p(t)\min\{\eta, 1-\eta\}\|u\|$ *for* $t \in J$*.*

Proof. Firstly, we show that $u(\eta) \geqslant \min\{\eta, 1-\eta\}\|u\|$. By the properties of Green's function (2.4), we can find that

$$\min\{\eta, 1-\eta\}s(1-s) \leqslant G(\eta, s) \leqslant G(s, s) = s(1-s), \quad (\eta, s) \in [0, 1]\times[0, 1]. \tag{2.12}$$

By using (2.11) and (2.12), we know that for every solution $u(t)$ of BVP (1.1), one has

$$\begin{aligned}\|u\| \leqslant&\lambda\int_0^1 s(1-s)a(s)f(s, u(s-\tau))ds\\ &+ \frac{\lambda\alpha}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(\eta, s)a(s)f(s, u(s-\tau))ds.\end{aligned} \tag{2.13}$$

Multiplying both side of inequality (2.13) by $\min\{\eta, 1-\eta\}$, we get

$$\begin{aligned}\min\{\eta,1-\eta\}\|u\| \leqslant& \lambda\int_0^1 \min\{\eta,1-\eta\}s(1-s)a(s)f(s,u(s-\tau))ds\\ &+\frac{\lambda\alpha\min\{\eta,1-\eta\}}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(\eta,s)a(s)f(s,u(s-\tau))ds\\ \leqslant&\lambda\int_0^1 G(\eta,s)a(s)f(s,u(s-\tau))ds\\ &+\frac{\lambda(\beta+(\alpha-\beta)\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(\eta,s)a(s)f(s,u(s-\tau))ds=u(\eta).\end{aligned}$$

Hence, the inequality $u(\eta)\geqslant \min\{\eta,1-\eta\}\|u\|$ is true.

Next, we will prove that $u(t)\geqslant p(t)\min\{\eta,1-\eta\}\|u\|$ in two cases.
Case(i) if $0\leqslant t\leqslant \eta$, then

$$\frac{u(t)-u(0)}{t-0}\geq\frac{u(\eta)-u(0)}{\eta-0}.$$

Using $u(0)=\beta u(\eta)$, we have

$$u(t)\geqslant\frac{\beta\eta+(1-\beta)t}{\eta}u(\eta)\geqslant\frac{\beta\eta+(1-\beta)t}{\eta}\min\{\eta,1-\eta\}\|u\|.$$

Case(ii) if $\eta<t\leqslant 1$, then

$$\frac{u(t)-u(\eta)}{t-\eta}\geqslant\frac{u(1)-u(\eta)}{1-\eta},$$

since $u(1)=\alpha u(\eta)$, we get

$$u(t)\geqslant\frac{1-\alpha\eta+(\alpha-1)t}{1-\eta}u(\eta)\geqslant\frac{1-\alpha\eta+(\alpha-1)t}{1-\eta}\min\{\eta,1-\eta\}\|u\|.$$

Combining above two cases, we have that $u(t)\geq p(t)\min\{\eta,1-\eta\}\|u\|$ for $t\in J$, and the proof is complete. □

Lemma 2.5. *The fixed point of* T *is a solution of* (1.1) *and* $T:K\to K$ *is completely continuous.*

Proof. From (2.11), we have

$$\begin{aligned}&(Tu)''(t)+\lambda a(t)f(t,u(t-\tau))=0 \qquad t\in J=[0,1],\\ &(Tu)(t)=\beta(Tu)(\eta), \qquad -\tau\leqslant t\leqslant 0,\\ &(Tu)(1)=\alpha(Tu)(\eta).\end{aligned}$$

Therefore, the fixed point of T is a solution of (1.1).

Next, we will prove that $T : K \to K$. For any $u \in K$, it is easy to see that $Tu \in C[-\tau, 1]$ and $Tu \geqslant 0$ for $t \in J$. Since $(Tu)''(t) \leqslant 0$, by Lemma 2.4, we have

$$(Tu)(t) \geqslant p(t)\min\{\eta, 1-\eta\}\|Tu\|, \qquad \text{for } t \in J.$$

Hence

$$\min\{(Tu)(t) : b \leqslant t \leqslant c\} \geqslant \min\{p(t) : b \leqslant t \leqslant c\}\min\{\eta, 1-\eta\}\|Tu\| \geqslant \theta\|Tu\|.$$

So $T : K \to K$. By using Arzela-Ascoli theorem, it is easy to prove that T is completely continuous. The proof is complete. □

Lemma 2.6. *Let X be a Banach space, and let $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open subset of E with $0 \in \Omega_1, \overline{\Omega}_2 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K,$$

be a completely continuous operator such that either
(i) $\|Au\| \leqslant \|u\|, u \in K \cap \partial\Omega_1$ and $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, or
(ii)$\|Au\| \geqslant \|u\|, u \in K \cap \partial\Omega_1$ and $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main Results

Let

$$f^0 = \limsup_{u\to 0^+}\max_{t\in J}\frac{f(t,u)}{u}, \qquad f_0 = \liminf_{u\to 0^+}\min_{t\in J}\frac{f(t,u)}{u},$$
$$f^\infty = \limsup_{u\to \infty}\max_{t\in J}\frac{f(t,u)}{u}, \qquad f_\infty = \liminf_{u\to \infty}\min_{t\in J}\frac{f(t,u)}{u}.$$

Theorem 3.1. *Let (H) hold and $f_\infty > 0$, $f^0 < \infty$, then there exists at least one positive solution to* (1.1) *for*

$$\lambda \in \left(\frac{1}{f_\infty \sup_{t\in J}\left(\beta\min\{\eta,1-\eta\}\int_0^\tau G(t,s)a(s)ds + \theta\int_{b+\tau}^{c+\tau}G(t,s)a(s)ds\right)},\right.$$
$$\left.\frac{1-\alpha\eta-\beta(1-\eta)}{(1+\alpha(1-\eta)+\beta\eta)f^0\left(\beta\int_0^\tau G(s,s)a(s)ds + \int_\tau^1 G(s,s)a(s)ds\right)}\right) \quad (3.1)$$

Proof. By (3.1), there exists an $\varepsilon > 0$ such that

$$\frac{1}{(f_\infty - \varepsilon)\sup_{t\in J}\left(\beta \min\{\eta, 1-\eta\}\int_0^\tau G(t,s)a(s)ds + \theta\int_{b+\tau}^{c+\tau} G(t,s)a(s)ds\right)} \leqslant \lambda \leqslant \frac{1-\alpha\eta-\beta(1-\eta)}{(1+\alpha(1-\eta)+\beta\eta)(f^0+\varepsilon)\left(\beta\int_0^\tau G(s,s)a(s)ds + \int_\tau^1 G(s,s)a(s)ds\right)}. \tag{3.2}$$

Let ε be fixed. By $f^0 < \infty$, there exists a $r > 0$ such that for $u : 0 < u \leqslant r$,

$$f(s,u) \leqslant (f^0+\varepsilon)u. \tag{3.3}$$

Let $\Omega_1 = \{u \in C[-\tau, 1] : \|u\| < r\}$, then for $u \in K \cap \partial\Omega_1$, we have by (3.2) and (3.3) that

$$\begin{aligned}
\|Tu\| &\leqslant \lambda \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\
&\quad + \frac{\lambda(\beta+\alpha)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\
&\leqslant \lambda\frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\
&\leqslant \lambda(f^0+\varepsilon)\frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(s,s)a(s)u(s-\tau)ds \\
&= \lambda(f^0+\varepsilon)\frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\Bigg(\int_0^\tau G(s,s)a(s)\beta u(\eta)ds \\
&\quad + \int_\tau^1 G(s,s)a(s)u(s-\tau)ds\Bigg) \\
&\leqslant \lambda(f^0+\varepsilon)\frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\Bigg(\beta\int_0^\tau G(s,s)a(s)ds \\
&\quad + \int_\tau^1 G(s,s)a(s)ds\Bigg)\|u\| \leqslant \|u\|.
\end{aligned}$$

Next, by $f_\infty > 0$, there exists a $R > r$ such that $f(s,u) \geqslant (f_\infty - \varepsilon)u$ for

$u \geqslant R$. Set $\Omega_2 = \{u \in C[-\tau, 1] : \|u\| < R\}$, then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|Tu\| &\geqslant \lambda \sup_{t\in J} \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds \\
&\geqslant \lambda \sup_{t\in J} \int_0^1 G(t,s)a(s)(f_\infty - \varepsilon)u(s-\tau))ds \\
&= \lambda(f_\infty - \varepsilon) \sup_{t\in J} \left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds + \int_\tau^1 G(t,s)a(s)u(s-\tau)ds \right) \\
&= \lambda(f_\infty - \varepsilon) \sup_{t\in J} \left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds + \int_0^{1-\tau} G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_\infty - \varepsilon) \sup_{t\in J} \left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds + \int_b^c G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_\infty - \varepsilon) \sup_{t\in J} \left(\int_0^\tau G(t,s)a(s)\beta \min\{\eta, 1-\eta\}\|u\|ds \right. \\
&\quad \left. + \int_b^c G(t,s+\tau)a(s+\tau)\theta\|u\|ds \right) \\
&\geqslant \lambda(f_\infty - \varepsilon) \sup_{t\in J} \left(\beta \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)a(s)ds \right. \\
&\quad \left. + \theta \int_{b+\tau}^{c+\tau} G(t,s)a(s)ds \right) \|u\| \geqslant \|u\|.
\end{aligned}$$

Therefore, by the first part of Lemma 2.6, T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $\|u\| \geqslant r$. From Lemma 2.5, $u(t)$ is a positive solution of BVP (1.1). The proof is complete. □

Theorem 3.2. *Let* (H) *hold and* $f_0 > 0$, $f^\infty < \infty$, *then there exists at least one positive solution to* (1.1) *for*

$$\lambda \in \left(\frac{1}{f_0 \sup_{t\in J} \left(\beta \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)a(s)ds + \theta \int_{b+\tau}^{c+\tau} G(t,s)a(s)ds \right)}, \right.$$
$$\left. \frac{1 - \alpha\eta - \beta(1-\eta)}{(1+\alpha(1-\eta)+\beta\eta) f^\infty \left(\beta \int_0^\tau G(s,s)a(s)ds + \int_\tau^1 G(s,s)a(s)ds \right)} \right). \tag{3.4}$$

Proof. Suppose that λ satisfies (3.4). There exists an $\varepsilon > 0$ such that

$$\frac{1}{(f_0-\varepsilon)\sup_{t\in J}\left(\beta\min\{\eta,1-\eta\}\int_0^\tau G(t,s)a(s)ds+\theta\int_{b+\tau}^{c+\tau}G(t,s)a(s)ds\right)}$$
$$\leqslant \lambda \leqslant$$
$$\frac{1-\alpha\eta-\beta(1-\eta)}{(1+\alpha(1-\eta)+\beta\eta)(f^\infty+\varepsilon)\left(\beta\int_0^\tau G(s,s)a(s)ds+\int_\tau^1 G(s,s)a(s)ds\right)}. \tag{3.5}$$

By $f_0 > 0$, there exists a $r^* > 0$ such that for $u : 0 < u \leqslant r^*$,

$$f(s,u) \geqslant (f_0-\varepsilon)u. \tag{3.6}$$

Let $\Omega_1 = \{x \in C[-\tau,1] : \|u\| < r^*\}$, then for $u \in K \cap \partial\Omega_1$, we have by (3.5) and (3.6) that

$$\begin{aligned}
\|Tu\| &\geqslant \lambda \sup_{t\in J}\int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds \\
&\geqslant \lambda \sup_{t\in J}\int_0^1 G(t,s)a(s)(f_0-\varepsilon)u(s-\tau))ds \\
&= \lambda(f_0-\varepsilon)\sup_{t\in J}\left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds+\int_\tau^1 G(t,s)a(s)u(s-\tau)ds\right) \\
&= \lambda(f_0-\varepsilon)\sup_{t\in J}\left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds+\int_0^{1-\tau} G(t,s+\tau)a(s+\tau)u(s)ds\right) \\
&\geqslant \lambda(f_0-\varepsilon)\sup_{t\in J}\left(\int_0^\tau G(t,s)a(s)\beta u(\eta)ds+\int_b^c G(t,s+\tau)a(s+\tau)u(s)ds\right) \\
&\geqslant \lambda(f_0-\varepsilon)\sup_{t\in J}\left(\int_0^\tau G(t,s)a(s)\beta\min\{\eta,1-\eta\}\|u\|ds\right. \\
&\qquad \left.+\int_b^c G(t,s+\tau)a(s+\tau)\theta\|u\|ds\right) \\
&\geqslant \lambda(f_0-\varepsilon)\sup_{t\in J}\left(\beta\min\{\eta,1-\eta\}\int_0^\tau G(t,s)a(s)ds\right. \\
&\qquad \left.+\theta\int_{b+\tau}^{c+\tau} G(t,s)a(s)ds\right)\|u\| \geqslant \|u\|.
\end{aligned}$$

By $f^{\infty} < \infty$, we choose that $R_* > r^*$ such that for $u \geqslant R_*$,

$$f(s,u) \leqslant (f^{\infty} + \varepsilon)u.$$

There are two cases of interest: Case (i) f is bounded, and Case (ii) f is unbounded.

Case(i) Suppose that f is bounded. We can choose $N > r^*$ such that $f(s,u) \leqslant N$ for $s \in J$ and $u \in [0,\infty)$. Let

$$R^* = \max\left\{N, \lambda N \frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)ds\right\}$$

and $\Omega_2 = \{x \in C[-\tau,1] : \|u\| < R^*\}$. Then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}\|Tu\| &\leqslant \lambda \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\ &\quad + \frac{\lambda(\beta+\alpha)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\ &\leqslant \lambda N \frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)ds \leqslant R^* = \|u\|.\end{aligned}$$

Case(ii) Suppose that f is unbounded. There exists $R^{**} > R_*$ such that $f(s,u) \leqslant f(s,R^{**})$ for $s \in J$ and $0 < x \leqslant R^{**}$. Then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}\|Tu\| &\leqslant \lambda \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\ &\quad + \frac{\lambda(\beta+\alpha)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \\ &\leqslant \lambda \frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)f(s,R^{**})ds \\ &\leqslant \lambda(f^{\infty}+\varepsilon)R^{**} \frac{1+\alpha(1-\eta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)a(s)ds \leqslant R^{**} = \|u\|.\end{aligned}$$

Therefore, by the second part of Lemma 2.6, T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $\|u\| \geqslant r^*$. From Lemma 2.5, $u(t)$ is a positive solution of BVP (1.1). The proof is complete. □

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References

[1] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem under for a second order ordinary differential equation, J. Math. Anal. Appl. (168), 540-551, 1992.

[2] C.P. Gupta, A sharper condition for solvability of a three-point nonlinear boundary value problem, J. Math. Anal. Appl. (205), 586-597, 1997.

[3] J.R.L. Webb, Positive solutions of some three-point boundary value problems via fixed point index theory, Nonlinear Anal. (47), 4319-4332, 2001.

[4] R. Ma, Positive solutions of a nonlinear three-point boundary-value problem, Electron. J. Differ. Eq. (34), 1-8, 1998.

[5] H. Luo, Q. Ma, Positive solutions to a generalized second-order three-point boundary-value problem on time scales, Electron J. Diff. Eq. (17), 1-14, 2005.

[6] D. Bai, Y. Xu, Existence of positive solutions for boundary-value problems of second-order delay differential equations, (18), 621-630, 2005.

[7] W. Wang, J. Shen, Positive solutions to a multi-point boundary value problem with delay, Appl. Math. Comput., (188), 96-102, 2007.

[a]Jongyi Chen, [b]Jessada Tariboon, [c]Sanoe Koonprasert
Department of Mathematics,
Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800 , THAILAND.
Centre of Excellence in Mathematics, PERDO, Commission on Higher Education Thailand.
e-mail : [a]payi1123@hotmail.com, [b]jessadat@kmutnb.ac.th , [c]skp@kmutnb.ac.th