



Finding the Number of Cycle Egamorphisms

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A mapping f from graph G to graph H is called an *egamorphism* (or *weak homomorphism*, *contraction*) from G to H , if f preserves or contracts the edges. This paper is to find the number of egamorphism from C_m to C_n , where $m, n \in \mathbb{Z}^+$ and $m, n > 2$.

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1 Introduction and Preliminaries

The motivation of this paper is come from the determination of number of cycle homomorphisms [8]. Otherwise the number of path homomorphisms [3] and the number of endomorphisms [2]. In [7] using the congruence classes to fine the number of path and cycle endomorphisms.

As usual we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph G , respectively, where $V(G) \neq \emptyset$ and $E(G) \subseteq \{\{u, v\} \mid u \neq v \text{ in } V(G)\}$. The graph with vertex set $\{0, 1, \dots, n\}$ and edge set $\{\{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}\}$ is called a path of length n , denoted by P_n . Therefore, the path P_n has $n + 1$ vertices and n edges. The graph with vertex set $\{0, 1, \dots, n-1\}$, such that $n \geq 3$, and edge set $\{\{i, i+1\} \mid i = 0, 1, \dots, n-1\}$ (with addition modulo n) is called a cycle of length n , denoted by C_n . Therefore, the cycle C_n has n vertices and n edges.

A *homomorphism* of a graph G to a graph H is a mapping $f : V(G) \rightarrow V(H)$ which preserves edges, i.e. $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. A homomorphism from G to itself is called an *endomorphism* of G . By $Hom(G, H)$ and $End(G)$ we denote the set of all homomorphisms

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from G to H and endomorphism of G , respectively. For each bijective $f \in \text{End}(G)$, if $f^{-1} \in \text{End}(G)$, f said to be *automorphism*, denoted by $\text{Aut}(G)$.

From [6], if G and H are two graphs, a map $f : V(G) \rightarrow V(H)$ is an *egamorphism* if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$ (f is also *weak homomorphism* in [5] and *contraction* in [9]). By $\text{Ega}(G, H)$ and $\text{Ega}(G)$ we denote the set of all egamorphisms from G to H and egamorphisms from G to itself, respectively.

We now define notation about the set which will be used in later. For any given $m, n \in \mathbb{Z}^+ \cup \{0\}$ and $r, i, j, k \in C_n$. Let $\text{Ega}^i(P_m, C_n)$ be the set of all egamorphisms and maps f from P_m to C_n such that $f(0) = i$. Let $\text{Ega}_j^i(P_m, C_n)$ be the set of all egamorphisms and maps f from P_m to C_n such that $f(0) = i$ and $f(m) = j$. So the set of all egamorphisms from P_m to C_n can be written as $\text{Ega}(P_m, C_n)$. Similarly, if $\text{Ega}^i(C_m, C_n)$ is the set of all egamorphisms and maps f from C_m to C_n such that $f(0) = i$, and $\text{Ega}_j^i(C_m, C_n)$ is the set of all egamorphisms and maps f from C_m to C_n such that $f(0) = i$ and $f(m-1) = j$. And easy to proof that,

Proposition 1.1. *For any given path P_m and cycle C_n . Then, the following properties are also held.*

1. If $j \neq k$, then $\text{Ega}_k^i(P_m, C_n) \cap \text{Ega}_j^i(P_m, C_n) = \emptyset$ and $\text{Ega}_k^i(C_m, C_n) \cap \text{Ega}_j^i(C_m, C_n) = \emptyset$
2. if $i \neq r$, then $\text{Ega}^i(P_m, C_n) \cap \text{Ega}^r(P_m, C_n) = \emptyset$ and $\text{Ega}^i(C_m, C_n) \cap \text{Ega}^r(C_m, C_n) = \emptyset$,
3. $|\text{Ega}^i(P_m, C_n)| = \sum_{j=0}^{n-1} |\text{Ega}_j^i(P_m, C_n)|$,
4. $|\text{Ega}^i(C_m, C_n)| = \sum_{j=0}^{n-1} |\text{Ega}_j^i(C_m, C_n)|$,
5. $|\text{Ega}(P_m, C_n)| = \sum_{i=0}^{n-1} |\text{Ega}^i(P_m, C_n)|$,
6. $|\text{Ega}(C_m, C_n)| = \sum_{i=0}^{n-1} |\text{Ega}^i(C_m, C_n)|$.

From [1], trinomial coefficients are used as major tools for our results, so its outline are introduced in brief here. For any given $m \in \mathbb{Z}^+ \cup \{0\}$, $k \in \mathbb{Z}$,

defined

$$\binom{m}{k}_2 = \begin{cases} \sum_{j=0}^m (-1)^j \binom{m}{j} \binom{2m-2j}{m-k-j}, & -m \leq k \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

From the definition, it can be written in factorial forms

$$\binom{m}{k}_2 = \binom{m-1}{k-1}_2 + \binom{m-1}{k}_2 + \binom{m-1}{k+1}_2$$

we now properly suggested next lemma for utilizing to prove our results.

Lemma 1.2. *Let $m, q \in \mathbb{Z}^+ \cup \{0\}$, for any given positive integer n such that $m = nq$. Then*

$$\binom{m}{j-i+n(q+1)}_2 = \binom{m}{j-i-n(q+1)}_2 = 0,$$

for all $0 \leq i, j \leq n-1$.

2 The Number of Egamorphisms from Paths to Cycles

First, we will show how to find the number of egamorphisms from paths to cycles. They are the important tools to determine the number of cycle egamorphisms. It's easy to get that,

Lemma 2.1. *Let G, H be graphs, $u \in V(G)$ and $v_1, v_2 \in V(H)$. If there exists $\alpha \in \text{Aut}(H)$ such that $\alpha(v_1) = v_2$, then $|\text{Ega}_{u \rightarrow v_1}(G, H)| = |\text{Ega}_{u \rightarrow v_2}(G, H)|$.*

Lemma 2.2. *Let G be a graph and $a, b \in V(G)$. If there exists $\varphi \in \text{Aut}(G)$ such that $\varphi(a) = b$, then $|\text{Ega}^a(P_m, G)| = |\text{Ega}^b(P_m, G)|$ and $|\text{Ega}_a^v(P_m, G)| = |\text{Ega}_b^v(P_m, G)|$ for all $v \in V(G)$.*

Corollary 2.3. *For any given path P_m and cycle C_n . Then $|\text{Ega}^i(P_m, C_n)| = |\text{Ega}^j(P_m, C_n)|$ and $|\text{Ega}_j^i(P_m, G)| = |\text{Ega}_k^i(P_m, G)|$ for all $v \in V(G)$.*

Theorem 2.4. *Let G be a graph and m be a positive integer, then*

$$|\text{Ega}_j^i(P_m, G)| = \sum_{\{j, j'\} \in E(G)} |\text{Ega}_{j'}^i(P_{m-1}, G)| + |\text{Ega}_j^i(P_{m-1}, G)|, \text{ where } j \neq j'.$$

Proof. Let $m, n \in \mathbb{Z}^+$, $n \geq 3$ and $i \in V(G)$.

Define $\varphi : \text{Ega}_j^i(P_m, G) \rightarrow \bigcup_{\{j, j'\} \in E(G)} \text{Ega}_{j'}^i(P_{m-1}, G) \cup \text{Ega}_j^i(P_{m-1}, G)$,

where $j \neq j'$, by $f \in \text{Ega}_j^i(P_m, G)$, then $\varphi(f) = f|_{\{0, 1, 2, \dots, m-1\}} = f'$.

Clearly, $f' \in \bigcup_{\{j, j'\} \in E(G)} \text{Ega}_{j'}^i(P_{m-1}, G) \cup \text{Ega}_j^i(P_{m-1}, G)$, where $j \neq j'$.

(i) We have to show that φ is injective.

Let $f_1, f_2 \in \text{Ega}_j^i(P_m, G)$ such that $\varphi(f_1) = \varphi(f_2)$.

Then $f_1(x) = f_1'(x) = f_2'(x) = f_2(x)$ for all $x \in P_{m-1}$.

But $f_1, f_2 \in \text{Ega}_j^i(P_m, G)$, $f_1(m) = f_2(m) = j$.

Therefore $f_1 = f_2$.

(ii) We want to show that φ surjective.

Let $g \in \bigcup_{\{j, j'\} \in E(G)} \text{Ega}_{j'}^i(P_{m-1}, G) \cup \text{Ega}_j^i(P_{m-1}, G)$, where $j \neq j'$.

If $g \in \bigcup_{\{j, j'\} \in E(G)} \text{Ega}_{j'}^i(P_{m-1}, G)$, where $j \neq j'$, $g(0) = i$ and $g(m-1) = j'$.

$\exists f \in \text{Ega}_j^i(P_m, G)$ such that $f(x) = g(x)$ for all $0 \leq x \leq m-1$ and $f(m) = j$.

If $g \in \text{Ega}_j^i(P_{m-1}, G)$, $g(0) = i$ and $g(m-1) = j$.

$\exists f \in \text{Ega}_j^i(P_m, G)$ such that $f(x) = g(x)$ for all $0 \leq x \leq m-1$ and $f(m) = j$.

By (i) and (ii) we get that $|\text{Ega}_j^i(P_m, G)| = \sum_{\{j, j'\} \in E(G)} |\text{Ega}_{j'}^i(P_{m-1}, G)| +$

$|\text{Ega}_j^i(P_{m-1}, G)|$, where $j \neq j'$. \square

By Theorem 2.4, we get one result as follow.

Corollary 2.5. *For any $m, n \in \mathbb{Z}^+$, $n \geq 3$, $i, j \in V(C_n)$, then*

$$|\text{Ega}_j^i(P_m, C_n)| = \begin{cases} \sum_{j'=0, 1, n-1} |\text{Ega}_{j'}^i(P_{m-1}, C_n)|, & j = 0; \\ \sum_{j'=j-1, j, j+1} |\text{Ega}_{j'}^i(P_{m-1}, C_n)|, & 0 < j < n-1; \\ \sum_{j'=0, n-2, n-1} |\text{Ega}_{j'}^i(P_{m-1}, C_n)|, & j = n-1. \end{cases}$$

We will suggest an important lemma for proving theorem of finding the number of egamorphisms.

Lemma 2.6. *For all $n \in \mathbb{Z}^+$, $n \geq 3$, $i, j \in V(C_n)$, $|\text{Ega}_j^i(P_0, C_n)| =$*

$$\begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

By definition of egamorphism, we then obvious the proof of this lemma. By Corollary 2.5 and Lemma 2.6. These pretty results are useful for using to calculate the number of egamorphisms. The next corollary is given the number of egamorphisms from paths to cycles.

Corollary 2.7. *For any $m \in \mathbb{Z}^+ \cup \{0\}$, $n \in \mathbb{Z}^+$ and $n \geq 3$. Then $|Ega^i(P_m, C_n)| = 3^m$, for all $i \in V(C_n)$ and $|Ega(P_m, C_n)| = n3^m$.*

Proof. Let $m \in \mathbb{Z}^+ \cup \{0\}$, $n \in \mathbb{Z}^+$, $n \geq 3$ and $i \in C_n$.

We will show that $|Ega^i(P_m, C_n)| = 3^m$.

If $m = 0$, then

$$|Ega_j^i(P_0, C_n)| = 0, \text{ if } i \neq j \text{ and } |Ega_j^i(P_0, C_n)| = 1, \text{ if } i = j.$$

$$\text{Then } |Ega^i(P_0, C_n)| = \sum_{j=0}^{n-1} |Ega_j^i(P_0, C_n)| = 1 = 3^0.$$

Therefore $|Ega^i(P_0, C_n)| = 3^0$.

Assume $|Ega^i(P_k, C_n)| = 3^k$ at $k \in \mathbb{Z}^+ \cup \{0\}$, $i \in V(C_n)$.

We will show that $|Ega^i(P_{k+1}, C_n)| = 3^{k+1}$.

Using the fact that $|Ega^i(P_{k+1}, C_n)| = \sum_{j=0}^{n-1} |Ega_j^i(P_{k+1}, C_n)|$, we have

$$\begin{aligned} |Ega^i(P_{k+1}, C_n)| &= |Ega_0^i(P_{k+1}, C_n)| + |Ega_1^i(P_{k+1}, C_n)| + \dots + |Ega_{n-1}^i(P_{k+1}, C_n)|, \\ &= 3|Ega_0^i(P_k, C_n)| + 3|Ega_1^i(P_k, C_n)| + \dots + 3|Ega_{n-1}^i(P_k, C_n)|, \\ &= 3 \sum_{j=0}^{n-1} |Ega_j^i(P_k, C_n)|, \\ &= 3|Ega^i(P_k, C_n)| \\ &= 3 \cdot 3^k \\ &= 3^{k+1}. \end{aligned}$$

Thus $|Ega^i(P_m, C_n)| = 3^m$ for all $m \in \mathbb{Z}^+ \cup \{0\}$ and $i \in V(C_n)$.

Therefore $|Ega(P_m, C_n)| = n3^m$ for all $m \in \mathbb{Z}^+ \cup \{0\}$. \square

The next is our main result which is generated by trinomial coefficients. It can be written as the following formula.

Theorem 2.8. *Let $m \in \mathbb{Z}^+ \cup \{0\}$, $n \in \mathbb{Z}^+$, $n \geq 3$ and $i, j \in V(C_n)$. Then*

$$|Ega_j^i(P_m, C_n)| = \sum_{t=0}^{q+1} \binom{m}{j-i \pm nt}_2,$$

where $m = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$.

Proof. Let $P(m) : |Ega_j^i(P_m, C_n)| = \sum_{t=0}^{q+1} \binom{m}{j-i \pm nt}_2$, where $m = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$.

If $m = 0$, then $q = 0, r = 0$

(i) If $j = i$ ($j - i = 0$), then $|Ega_j^i(P_0, C_n)| = 1 = \binom{0}{0-n}_2 + \binom{0}{0}_2 + \binom{0}{0+n}_2 = \sum_{t=0}^{q+1=1} \binom{0}{j-i\pm nt}_2$.

(ii) If $j \neq i$ ($j - i \neq 0$), then $|Ega_j^i(P_0, C_n)| = 0 = \binom{0}{j-i-n}_2 + \binom{0}{j-i}_2 + \binom{0}{j-i+n}_2 = \sum_{t=0}^{q+1=1} \binom{0}{j-i\pm nt}_2$, (because $-n < j - i < n$).

By (i) and (ii) therefore $P(0)$ is true for all $i, j \in V(C_n)$.

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+ \cup \{0\}$,

therefore $|Ega_j^i(P_k, C_n)| = \sum_{t=0}^{q+1} \binom{k}{j-i\pm nt}_2$, for all $i, j \in V(C_n)$,

where $k = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$.

We want to show that $P(k + 1)$ is true. Since $k = nq + r$, then $k + 1 = nq + r + 1$.

Case 1 : If $0 \leq r \leq n - 2$, then $k + 1 = nq + r + 1$ where $0 \leq r + 1 \leq n - 1$.

Let $j \in V(C_n)$ such that $0 < j < n - 1$. Then

$$\begin{aligned} |Ega_j^i(P_{k+1}, C_n)| &= |Ega_{j-1}^i(P_k, C_n)| + |Ega_j^i(P_k, C_n)| + |Ega_{j+1}^i(P_k, C_n)|, \\ &= \sum_{t=0}^{q+1} \binom{k}{j-i\pm nt-1}_2 + \sum_{t=0}^{q+1} \binom{k}{j-i\pm nt}_2 + \sum_{t=0}^{q+1} \binom{k}{j-i\pm nt+1}_2, \\ &= \sum_{t=0}^{q+1} \left[\binom{k}{j-i\pm nt-1}_2 + \binom{k}{j-i\pm nt}_2 + \binom{k}{j-i\pm nt+1}_2 \right], \\ &= \sum_{t=0}^{q+1} \binom{k+1}{j-i\pm nt}_2. \end{aligned}$$

By Lemma 2.2, $|Ega_j^i(P_{k+1}, C_n)| = |Ega_0^i(P_{k+1}, C_n)| = |Ega_{n-1}^i(P_{k+1}, C_n)|$.

Therefore, $|Ega_j^i(P_{k+1}, C_n)| = \sum_{t=0}^{q+1} \binom{k+1}{j-i\pm nt}_2$, for all $i, j \in V(C_n)$, in the case of $k = nq + r$, $0 \leq r \leq n - 2$.

Case 2 : If $r = n - 1$, then $k + 1 = n(q + 1)$.

Let $j \in V(C_n)$ such that $0 < j < n - 1$. Similarly to **Case 1**, we get

$$\begin{aligned} \text{that } |Ega_j^i(P_{k+1}, C_n)| &= \sum_{t=0}^{q+1} \binom{k+1}{j-i\pm nt}_2. \text{ From Lemma 1.2, } \sum_{t=0}^{q+1} \binom{k+1}{j-i\pm nt}_2 = \\ &= \sum_{t=0}^{q+2} \binom{k+1}{j-i\pm nt}_2. \end{aligned}$$

$$\text{Thus } |Ega_j^i(P_{k+1}, C_n)| = \sum_{t=0}^{q+2} \binom{k+1}{j-i\pm nt}_2.$$

By Lemma 2.2, $|Ega_j^i(P_{k+1}, C_n)| = |Ega_0^i(P_{k+1}, C_n)| = |Ega_{n-1}^i(P_{k+1}, C_n)|$.

Therefore, $|Ega_j^i(P_{k+1}, C_n)| = \sum_{t=0}^{q+2} \binom{k+1}{j-i\pm nt}_2$, for all $i, j \in V(C_n)$, where

$k + 1 = n(q + 1)$, for some $q \in \mathbb{Z}^+ \cup \{0\}$.

By **Case 1** and **Case 2**, we get that $P(k+1)$ is true.

Therefore, by mathematical induction, $|Ega_j^i(P_m, C_n)| = \sum_{t=0}^{q+1} \binom{m}{j-i \pm nt}_2$, for all $m \in \mathbb{Z}^+ \cup \{0\}$, $n \in \mathbb{Z}^+$, $n \geq 3$, $i, j \in V(C_n)$, where $m = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$. \square

This result used for finding the number of the value of $Ega_j^i(P_m, C_n)$ at i and j .

3 The Number of Cycle Egamorphisms

In this section, we presented the theorem which is use to find the number of cycle egamorphisms.

Lemma 3.1. *For all $m, n \in \mathbb{Z}^+$, $m, n \geq 3$ and $i, j \in V(C_n)$, if $|Ega_j^i(C_m, C_n)| \neq 0$, then $\{i, j\} \in E(C_n)$ or $i = j$.*

Lemma 3.2. *For all $m, n \in \mathbb{Z}^+$, $m, n \geq 3$ and $i, j \in V(C_n)$, if $\{i, j\} \in E(C_n)$ or $i = j$, then $|Ega_j^i(P_{m-1}, C_n)| = |Ega_j^i(C_m, C_n)|$.*

The proofs of two lemmas above are clear. In case of strictly for $m, n \geq 3$

Theorem 3.3. *Let $m, n \in \mathbb{Z}^+$, $m, n \geq 3$. If $m - 1 = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$, then*

$$|Ega_j^i(C_m, C_n)| = \sum_{t=0}^{q+1} \binom{m}{\pm nt}_2, \text{ for all } i \in V(C_n).$$

Proof. Let $m, n \in \mathbb{Z}^+$, $m, n \geq 3$ such that $m - 1 = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$. Let $i \in V(C_n)$. Thus $|Ega_j^i(C_m, C_n)| = \sum_{j=0}^n |Ega_j^i(C_m, C_n)|$.

For $0 < i < n - 1$.

By Lemma 3.1, $|Ega_j^i(C_m, C_n)| = 0$ for all $j = 0, 1, \dots, (i - 2), (i + 2), \dots, n$. Thus

$$\begin{aligned}
|Ega^i(C_m, C_n)| &= |Ega_{i-1}^i(C_m, C_n)| + |Ega_i^i(C_m, C_n)| + |Ega_{i+1}^i(C_m, C_n)|, \\
&= |Ega_{i-1}^i(P_{m-1}, C_n)| + |Ega_i^i(P_{m-1}, C_n)| + |Ega_{i+1}^i(P_{m-1}, C_n)|, \\
&= \sum_{t=0}^{q+1} \binom{m-1}{-1 \pm nt}_2 + \sum_{t=0}^{q+1} \binom{m-1}{\pm nt}_2 + \sum_{t=0}^{q+1} \binom{m-1}{1 \pm nt}_2, \\
&= \sum_{t=0}^{q+1} \left[\binom{m-1}{-1 \pm nt}_2 + \binom{m-1}{\pm nt}_2 + \binom{m-1}{1 \pm nt}_2 \right], \\
&= \sum_{t=0}^{q+1} \binom{m}{\pm nt}_2.
\end{aligned}$$

From 2.2, $|Ega^i(C_m, C_n)| = |Ega^0(C_m, C_n)| = |Ega^{n-1}(C_m, C_n)|$. Altogether $|Ega^i(C_m, C_n)| = \sum_{t=0}^{q+1} \binom{m}{\pm nt}_2$, for all $i \in V(C_n)$. \square

Proposition 3.4. *Let $m, n \in \mathbb{Z}^+$, $m, n \geq 3$. If $m - 1 = nq + r$, for some $q \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq r \leq n - 1$, then*

1. $|Ega(C_m, C_n)| = n \sum_{t=0}^{q+1} \binom{m}{\pm nt}_2$,
2. $|Ega(C_m)| = n \left[\binom{m}{0}_2 + 2 \right]$.

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