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Generalized Frames

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Abstract : In this paper, we use regular infinite matrices $(a_{nk})_{n,k=1}^{\infty}$ to define A-frame and prove some analogous results of Duffin and Schaeffer [5].

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1 Introduction

The concept of frames were introduced by Duffin and Schaeffer [5] in the context of non- harmonic Fourier series. A sequence $\{x_n\}$ in a Hilbert Space H is a frame if there exists numbers $C_1, C_2 > 0$, such that for all $x \in H$, we have

$$C_1 \parallel x \parallel^2 \leq \sum_n \mid \langle x, x_n \rangle \mid^2 \leq C_2 \parallel x \parallel^2 .$$

The two constants C_1 and C_2 are called frame bounds. The frame is tight if $C_1 = C_2$ and is said to be exact if it ceases to be a frame by removing any of it's elements.

Frames have many of the properties of bases, but lack very important one i.e., uniqueness. Frames need not to be linearly independent. This turns out to be useful in image and signal processing applications, since the redundancy or transmission errors. As with the Riesz bases, perturbing a frame by a small amount, also yield a frame. Chui and Shi's over sampling theorems provide methods to generate wavelet frames. The theory of frames are discussed in variety of sources, including [1, 3, 4, 6, 9]. Here in this paper we define A-frame for an infinite non-negative regular matrix and obtained analogous results of Duffin and Schaeffer.

2 Definitions and Notations

In the sequel, $\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}$ and $\mathbb{C}$ will respectively denote the integers, the strictly positive integers, the real numbers and the complex numbers and H will denote a(separable) Hilbert space with inner product $\langle ., . \rangle$ and norm $\parallel . \parallel = \langle ., . \rangle^{\frac{1}{2}}$.

A sequence $\{x_n, n \in \mathbb{Z}^+\} \subset H$ is called a *frame* if there exists two constants C_1 and C_2 such that for every $x \in H$

$$C_1 \parallel x \parallel^2 \leq \sum_{n \in \mathbb{Z}^+} \mid \langle x, x_n \rangle \mid^2 \leq C_2 \parallel x \parallel^2$$

The constants C_1 and C_2 are called *frame bounds* of the frame. The supremum of all such C_1 and infimum of all such C_2 are called *best bounds* of the frame. If only the right hand inequality is satisfied for all $x \in H$, then $\{x_n, n \in \mathbb{Z}^+\}$ is called *Bessel sequence with bounds* C_2. One of the properties that are important is the following:

$\{x_n, n \in Z^+\}$ is a Bessel sequence with bound M if and only if for every finite sequence of scalers $\{c_k\}$

$$\parallel \sum_k c_k x_k \parallel^2 \leq M \sum_k \parallel c_k \parallel^2 \tag{2.1}$$

(see [9, page 155]). As remarked by Chui and Shi's in [2], it is straight forward consequence of this statement that $\{x_n, n \in \mathbb{Z}^+\}$ is a Bessel sequence with bound M if and only if, (2.1) is satisfied for every sequence $\{c_k\} \in \ell^2$.

Two sequences $\{a_n, n \in \mathbb{Z}^+\}$ and $\{b_n, n \in \mathbb{Z}^+\}$ in H are called *bi-orthogonal* if $\langle a_m, a_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta.

Example 2.1 Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Then

(i) $\{e_1, e_1, e_2, e_2, ...\}$ is a tight exact frame with bounds $C_1 = C_2 = 2$.

(ii) $\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, ...\}$ is a complete orthogonal sequence, but not a frame.

(iii) $\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, ...\}$ is a tight inexact frame with bounds $C_1 = C_2 = 2$.

Definition 2.2 Given a sequence $\{x_n, n \in Z^+\} \subset H$. Let $S : H \to H$ be defined by

$$S(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.$$

A number of important properties of frames are mentioned below (see [1]).

Theorem 2.3 *Let* $\{x_n, n \in Z^+\} \subset H$

(a) The following are equivalent :

(i) $\{x_n, n \in Z^+\}$ *is a frame for* H *with frame bounds* C_1 *and* C_2.

(ii) $S : H \to H$ *is a topological isomorphism with norm bounds*

$$\parallel S \parallel \leq C_2$$

and

$$\parallel S \parallel \leq C_1^{-1}.$$

(b) In case of either condition in part (a), we have that

$$C_1 I \leq S \leq C_2 I, \qquad C_2^{-1} I \leq S^{-1} \leq C_1^{-1} I,$$

$\{S^{-1}x_n\}$ *is a frame for* H *with frame bounds* C_2^{-1} *and* C_1^{-1} *and for all* $x \in H$.

$$f = \sum \langle x, S^{-1}x_n \rangle x_n, \tag{2.2}$$

and

$$f = \sum \langle x, x_n \rangle S^{-1}(x_n). \tag{2.3}$$

If $\{x_n, n \in Z^+\}$ is a frame, S is called the *frame operator*, $\{S^{-1}x_n\}$ is called *dual frame* of $\{x_n\}$, (2.2) is the *frame decomposition* of x and (2.3) is the *dual frame decomposition* of x (I is the identity map, $S \leq C_2 I$ means that $\langle (C_2 I - S)x, x \rangle \geq 0$ for each x $\in H$)

A sequence is called minimal, if no element of the sequence is closure of the linear span of of other elements.

Theorem 2.4 *Let* $\{x_n, n \in \mathbb{Z}^+\} \subset H$ *be a frame for* H *with frame bounds* C_1 *and* C_2.

- *(a) For each* $C = \{c_n\} \in \ell^2$ *such that* $x = \sum c_n x_n$ *converges in* H *and* $\| x \|^2 \leq C_2 \| c \|^2_{\ell^2}$.
- *(b) Let* $\{x_n, n \in Z^+\}$ *be a frame and let* v *be any arbitrary vector, then there exist a moment sequence* $\{y_n, n \in Z^+\}$ *such that* $v = \sum_{n=1}^{\infty} x_n y_n$ *and*

$$C_2^{-1} \| v \|^2 \leq \sum_{n=1}^{\infty} | y_n |^2 \leq C_1 \| v \|^2 .$$

A basis of $\{x_n, n \in Z^+\} \subset$ H is called *unconditional* if there exists a $C > 1$ such that for any two finite sequences of scalars $\{a_k, 1 \leq k \leq n\}$ and $\{b_k, 1 \leq k \leq n\}$ if $| a_k | \leq | b_k |, 1 \leq k \leq n$, then

$$\| \sum_{k=1}^{n} a_k x_k \| \leq C \| \sum_{k=1}^{n} b_k x_k \| .$$

Theorem 2.5 *A sequence* $\{x_n, n \in \mathbb{Z}^+\}$ *in a Hilbert space* H *is an exact frame for* H *if and only if it is bounded unconditional basis for* H.

Now we will enlist some basic definitions which will be used oftenly in this paper.

Definition 2.6 ([7]) Let X and Y be two non empty subsets of the space of all real or complex numbers. Let $A = (a_{nk}), (n, k = 1, 2, 3, \ldots)$, be an infinite matrix of real or complex numbers. By (X, Y) we denote the class of all such matrices A such that the series

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

converges for all $x \in X$ and the sequence $Ax = (A_n(x))$ will be called A-*transform* of x. Also

$$\lim Ax = \lim_{n\to\infty} A_n(x),$$

whenever it exists.

Definition 2.7 ([7]) A matrix $A = (a_{nk})$ is said to be *regular* if the matrix transformation $A : X \to Y$ is defined on a convergent sequence to a convergent sequence and limit is preserved i.e.,

$$\lim_n A_n(x) = \lim_n x_n.$$

Definition 2.8 ([7]) A necessary and sufficient condition for matrix $A = (a_{nk})$ is said to be regular if and only if

(i) $\sup_n \sum_k \mid a_{nk} \mid < \infty$

(ii) $\lim_{n\to\infty} a_{nk} = 0$

(iii) $\lim_{n\to\infty} \sum a_{nk} = 1$.

Definition 2.9 An A-frame for an infinite non-negative regular matrix A=(a_{nk}) is an infinite sequence $\{A_n(\phi)\}$ such that for any $f \in L^2(R)$

$$C_1 \parallel f \parallel^2 \leq \sum_n \mid \langle f, A_n(\phi)\rangle \mid^2 \leq C_2 \parallel f \parallel^2 \tag{2.4}$$

where $A_n(\phi) = \sum a_{nk}\phi_k$ and $C_1, C_2 > 0$ are constants and are called *bounds* of the frame. The sequence $\phi = \{\phi_k\}$ is a sequence of vectors. The number $\beta_n = \langle f, A_n(\phi)\rangle$ are called A-*moment sequence* of $f \in L^2(R)$ relative to the frame.

3 Main Results

Theorem 3.1 *If $\{A_n(\phi)\}$ is an exact A-frame in a Hilbert space H, then it is bounded in norm.*

Proof. Assume $\{A_n(\phi)\}$ is an exact A-frame with $C_1, C_2 > 0$, then $\{A_n(\phi)\}$ and $\{S^{-1}A_n(\phi)\}$ are biorthogonal, so for fixed m, we have

$$\begin{aligned} C_1 \parallel S^{-1}A_m(\phi) \parallel^2 &\leq \sum_n \mid S^{-1}A_m(\phi), A_n(\phi) \mid^2 \\ &= \mid \langle S^{-1}A_m(\phi), A_m(\phi)\rangle \mid^2 \\ &= \parallel S^{-1}A_m(\phi) \parallel^2 \parallel A_m(\phi) \parallel^2 . \end{aligned}$$

Then we have

$$C_1 \leq \| A_m(\phi) \|^2$$

and

$$\begin{aligned}\| A_m(\phi) \|^4 &= | \langle A_m(\phi), A_m(\phi) \rangle |^2 \\ &\leq \sum_n | \langle A_m(\phi), A_n(\phi) \rangle |^2 \leq C_2 \| A_m(\phi) \|^2 .\end{aligned}$$

Thus we obtain

$$\| A_m(\phi) \|^2 \leq C_2.$$

Therefore, $\{A_n(\phi)\}$ is bounded in norm. This completes the proof. □

Theorem 3.2 *If $\{A_n(\phi)\}$ is A-frame and $\{c_n\}$ is a sequence of numbers such that $\{c_n\} \in \ell^2$, then $\sum_n c_n A_n(\phi)$ converges and*

$$\| \sum_{n=1}^{\infty} c_n A_n(\phi) \|^2 \leq C_2 \sum_{n=1}^{\infty} | c_n |^2 . \tag{3.1}$$

Proof. If we take

$$f_k = \sum_{n=1}^{k} c_n A_n(\phi)$$

then for any $k \geq j$, we have from Schwartz inequality and the frame condition (2.4),

$$\begin{aligned}\| f_k - f_j \|^2 &= \sum_{n=j+1}^{k} c_n \langle A_n(\phi), f_k - f_j \rangle \\ &\leq \Big\{ \sum_{n=j+1}^{k} | c_n |^2 \Big\}^{\frac{1}{2}} \Big\{ c_2 \| f_k - f_j \|^2 \Big\}^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\| f_k - f_j \|^2 \leq C_2 \sum_{n=j+1}^{k} | c_n |^2 .$$

This completes the proof. □

Theorem 3.3 *If $\{A_n(\phi)\}$ is A-frame and for any $f \in L^2(\mathbb{R})$. Then there exists a A-moment sequence $\{\beta_n\}$, such that*

$$\beta_n = \langle f, A_n(\phi) \rangle$$

and

$$f = \sum_{n=1}^{\infty} \beta_n A_n(\phi) \tag{3.2}$$

and

$$C_2^{-1} \parallel f \parallel^2 \leq \sum_{n=1}^{\infty} \mid \beta_n \mid^2 \leq C_1^{-1} \parallel f \parallel^2 . \quad (3.3)$$

If $\{b_n\}$ is any other sequence such that $f = \sum_{n=1}^{\infty} b_n A_n(\phi)$, then $\{b_n\}$ is not the A- moment sequence and

$$\sum_{n=1}^{\infty} \mid b_n \mid^2 = \sum_{n=1}^{\infty} \mid \beta_n \mid^2 + \sum_{n=1}^{\infty} \mid b_n - \beta_n \mid^2 .$$

Proof. For any $f \in L^2(\mathbb{R})$. Let us define a linear transform S by the relation

$$Sf = \sum_{n=1}^{\infty} \langle f, A_n(\phi) \rangle A_n(\phi).$$

The transformation is self adjoint, and if we use (2.4), it follows that

$$C_1 \parallel f \parallel^2 \leq \langle Sf, f \rangle \leq C_2 \parallel f \parallel^2 .$$

This shows that S is positive definite with positive upper and lower bounds. Hence S^{-1} exist as a self adjoint transformation and

$$C_2^{'} \parallel f \parallel^2 \leq \langle Sf, f \rangle \leq C_1^{'} \parallel f \parallel^2 .$$

The proof of this theorem follows by using (3.1). This completes the proof. □

Theorem 3.4 *If $\{A_n(\phi)\}$ is an exact A-frame then $\{A_n(\phi)\}$, where $B_n(\phi) = S^{-1} A_n(\phi)$ are biorthogonal. Any sequence of numbers $\{c_n\} \in \ell^2$ is a A-moment sequence of any function $f \in L^2(\mathbb{R})$ with respect to $\{A_n(\phi)\}$ and*

$$C_1 \sum_{n=1}^{\infty} \mid c_n \mid^2 \leq \parallel \sum_{n=1}^{\infty} c_n A_n(\phi) \parallel^2 \leq C_2 \sum_{n=1}^{\infty} \mid c_n \mid^2 . \quad (3.4)$$

Proof. If $\{A_n(\phi)\}$ is an exact A-frame then $\langle A_m(\phi), A_n(\phi) \rangle = \delta_{m,n}$ is true for all m and n, so $\{A_n(\phi)\}$ and $\{B_n(\phi)\}$ are biorthogonal. Given a sequence $\{c_n\} \in \ell^2$ and any $f \in L^2(\mathbb{R})$ such that $f = \sum c_n A_n(\phi)$ has a finite norm,then (3.4) follows from (3.3). This completes the proof. □

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