Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 627–632
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

The Necessary and Sufficient Condition for the Convergence of a New Fixed Point Approximation Method for Continuous Functions on an Arbitrary Interval

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Abstract : In this paper, we consider a new two step iterations for approximating a fixed point of continuous functions on an arbitrary interval. Then, necessary and sufficient condition for the convergence of the proposed iterative process of continuous functions on an arbitrary interval is given. Our results extend and generalized the results of Borwein and Borwein [D. Borwein and J. Borwein, Fixed point iterations for real functions, J. Math. Anal. Appl., 157 (1991), 112-126]. Finally, we show the numerical examples for the proposed iterative process to compare with Mann, Ishikawa iterations.

Keywords : Approximation method; Arbitrary interval; Continuous function; Fixed point.
2000 Mathematics Subject Classification : 26A18; 47H10; 54C05.

1 Introduction

Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. A point $p \in E$ is a fixed point of f if $f(p) = p$. The Mann iteration (see [1]) is defined by $x_1 \in E$ and

$$x_{n+1} = (1 - \beta_n)\, x_n + \beta_n f(x_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\beta_n\}$ is a sequence in $[0, 1]$. The Ishikawa iteration (see [2]) is defined by $x_1 \in E$ and

$$y_n = (1 - \alpha_n)\, x_n + \alpha_n f(x_n)\,,\ x_{n+1} = (1 - \beta_n)\, x_n + \beta_n f(y_n)\,, \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. Clearly, Mann iteration is special cases of Ishikawa iteration.

In 1991, David Borwein and Jonathan Borwein [3] studied the convergence of the Mann iteration on a closed bounded interval in their paper.

Theorem 1.1. ([3]) *Let $f : [a,b] \to [a,b]$ be a continuous real function. Define the iteration as follows:*

$$x_1 \in [a,b], \;\; x_{n+1} = (1-\beta_n)x_n + \beta_n f(x_n),$$

where $\beta_n \in [0,1]$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, then the iteration $(x_n)_{n=1}^{\infty}$ converges to a fixed point of f.

In 2006, Qing and Qihou [4] extended their results to an arbitrary interval and to the Ishikawa iteration and gave some control conditions for the convergence of Ishikawa iteration on an arbitrary interval.

In this paper, we will extend the result obtained by David Borwein and Jonathan Borwein [3] to the modified Ishikawa iteration and proved the following results:

Theorem 1.2. *Let E be a closed interval on the real line (can be unbounded), $f : E \to E$ a continuous function. Let $0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,*

$$x_1 \in E, \;\; y_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \;\; x_{n+1} = (1-\beta_n)y_n + \beta_n f(y_n).$$

If $(x_n)_{n=1}^{\infty}$ is bounded, then $(x_n)_{n=1}^{\infty}$ converges to a fixed point of $f(x)$.

Theorem 1.3. *Let E be a closed interval on the real line (can be unbounded), $f : E \to E$ a continuous function. Let $0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,*

$$x_1 \in E, \;\; y_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \;\; x_{n+1} = (1-\beta_n)y_n + \beta_n f(y_n).$$

Then $(x_n)_{n=1}^{\infty}$ converges to a fixed point of $f(x)$ if and only if $(x_n)_{n=1}^{\infty}$ is bounded.

Theorem 1.4. *Let E be a closed interval on the real line (can be unbounded), $f : E \to E$ a continuous function. Let $0 \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,*

$$x_1 \in E, \;\; x_{n+1} = (1-\beta_n)x_n + \beta_n f(x_n).$$

Then Mann iteration $(x_n)_{n=1}^{\infty}$ converges to a fixed point of $f(x)$ if and only if $(x_n)_{n=1}^{\infty}$ is bounded.

Corollary 1.5. ([3]) *Let $f : [a,b] \to [a,b]$ be a continuous function. Let $0 \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,*

$$x_1 \in [a,b], \;\; x_{n+1} = (1-\beta_n)x_n + \beta_n f(x_n).$$

Then Mann iteration $(x_n)_{n=1}^{\infty}$ converges to a fixed point of $f(x)$.

We need to prove the following 2 lemmas in order to prove the theorems above.

Lemma 1.6. *Let E be a closed interval on the real line (can be unbounded), $f : E \to E$ a continuous function. Let $0 \le \alpha_n, \beta_n \le 1$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,*

$$x_1 \in E,\ y_n = (1-\alpha_n)x_n + \alpha_n f(x_n),\ x_{n+1} = (1-\beta_n)y_n + \beta_n f(y_n).$$

If $x_n \to a$, then a is a fixed point of $f(x)$.

Proof. Suppose $f(a) \ne a$. Since $x_n \to a$ and $f(x)$ is continuous, we have $f(x_n)$ is bounded. Since $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$ and $\alpha_n \to 0$, we obtain $y_n \to a$. Let $p_k = f(y_k) - y_k,\ q_k = f(x_k) - x_k$. Then

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty} [f(y_k) - y_k] = f(a) - a \ne 0,$$

$$\lim_{k\to\infty} q_k = \lim_{k\to\infty} [f(x_k) - x_k] = f(a) - a \ne 0.$$

Let $\lim_{k\to\infty} p_k = p$, $\lim_{k\to\infty} q_k = q$, then $p \ne 0$ and $q \ne 0$. Using $x_{n+1} = (1-\beta_n)y_n + \beta_n f(y_n)$, we get

$$\begin{aligned} x_{n+1} &= y_n - \beta_n y_n + \beta_n f(y_n) \\ &= y_n + \beta_n (f(y_n) - y_n) \\ &= (1-\alpha_n)x_n + \alpha_n f(x_n) + \beta_n (f(y_n) - y_n) \\ &= x_n - \alpha_n x_n + \alpha_n f(x_n) + \beta_n (f(y_n) - y_n) \\ &= x_n + \beta_n (f(y_n) - y_n) + \alpha_n (f(x_n) - x_n), \end{aligned}$$

which implies

$$\begin{aligned} x_n &= x_1 + \sum_{k=1}^{n} \beta_k (f(y_k) - y_k) + \sum_{k=1}^{n} \alpha_k (f(x_k) - x_k) \\ &= x_1 + \sum_{k=1}^{n} \beta_k p_k + \sum_{k=1}^{n} \alpha_k q_k. \end{aligned}$$

Then $\{x_n\}$ must diverge, since $p_k \to p \ne 0$ and $\sum_{k=1}^{\infty} \beta_k = \infty$, $q_k \to q \ne 0$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$, which is a contradiction with $x_n \to a$. Thus $f(a) = a$. □

Lemma 1.7. *If the sequence $(x_n)_{n=1}^{\infty}$ satisfying the conditions of Theorem 1.2 is bounded, then it is convergent.*

Proof. Suppose $(x_n)_{n=1}^{\infty}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$.

First, we prove that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \ne m$. Without loss of generality, we suppose $f(m) > 0$. Because $f(x)$ is continuous function, there exists δ, $0 < \delta < b - a$, such that :

$$f(x) - x > 0, \text{ for } |x - m| \le \delta. \tag{1.3}$$

Since $(x_n)_{n=1}^{\infty}$ is bounded, $(x_n)_{n=1}^{\infty}$ belong to a bounded closed interval. Since $f(x)$ is continuous, $f(x_n)$ belongs to another bounded closed interval, so $f(x_n)$ is bounded. Since $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$, we obtain $(y_n)_{n=1}^{\infty}$ is bounded, and thus $f(y_n)$ is bounded. Using $x_{n+1} - x_n = \beta_n(f(y_n) - y_n) + \alpha_n(f(x_n) - x_n)$, $y_n - x_n = \alpha_n(f(x_n) - x_n)$, and $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, we get:

$$|x_{n+1} - x_n| \to 0, \ \ |y_n - x_n| \to 0.$$

Thus there exists a positive integer N such that:

$$|x_{n+1} - x_n| < \frac{\delta}{2}, \ |y_n - x_n| < \frac{\delta}{2} \ \text{for all } n > N. \tag{1.4}$$

Since $b = \limsup_n x_n > m$, there exists $k_1 >$ N such that $x_{n_{k_1}} > m$. Let $n_{k_1} = k$, then $x_k > m$. For x_k, there exists only two cases:

(i) $x_k > m + \frac{\delta}{2}$, then $x_{k+1} > x_k - \frac{\delta}{2} \geq m$ using (1.4). So $x_{k+1} > m$.

(ii) $m < x_k < m + \frac{\delta}{2}$, then $m - \frac{\delta}{2} < y_k < m + \delta$ using (1.4). So we have: $|x_k - m| < \frac{\delta}{2} < \delta$, $|y_k - m| < \delta$. Using (1.3), we get:

$$f(x_k) - x_k > 0, \ f(y_k) - y_k > 0. \tag{1.5}$$

Since

$$\begin{aligned} x_{k+1} &= y_k + \beta_k(f(y_k) - y_k) \\ &= (1-\alpha_k)x_k + \alpha_k f(x_k) + \beta_k(f(y_k) - y_k) \\ &= x_k - \alpha_k x_k - \alpha_k f(x_k) + \beta_k(f(y_k) - y_k) \\ &= x_k - \alpha_k(f(x_k) - x_k) + \beta_k(f(y_k) - y_k) > x_k, \end{aligned}$$

we obtain $x_{k+1} > x_k > m$.

In conclusion by (i), (ii), we have $x_{k+1} > m$. Analogously, we have $x_{k+2} > m$, $x_{k+3} > m, \ldots$. Thus we get $x_n > m$, for all $n > k = n_{k_1}$. So $a = \lim_{k\to\infty} x_{n_k} \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$. Now, we consider the following two cases:

(I) There exists x_M such that $a < x_M < b$. From above proof, we obtain $f(x_M) = x_M$. It follows that

$$y_M = (1-\alpha_M)x_M + \alpha_M f(x_M) = x_M,$$

$$x_{M+1} = (1-\beta_M)y_M + \beta_M f(y_M) = (1-\beta_M)x_M + \beta_M f(x_M) = x_M.$$

Analogously, we have $x_M = x_{M+1} = x_{M+2} = \cdots$, so $x_n \to x_M$. And because there exists $x_{n_k} \to a$, so $x_M = a$ and $x_n \to a$ which is a contradiction with the assumption.

(II) For all n, $x_n \leq a$ or $x_n \geq b$. Because $b - a > 0$, and $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, so there exists $\bar{N}$ such that $|x_{n+1} - x_n| < \frac{(b-a)}{2}$ for $n > \bar{N}$. So it is always that $x_n \leq a$ for $n > \bar{N}$, then $b = \lim_{\ell\to\infty} x_{n_\ell} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \bar{N}$, so we have $a = \lim_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. So the assumption is not true. Then $x_n \to a$ $(n \to \infty)$. □

Proof of Theorem 1.2. Using Lemma 1.7 we get $(x_n)_{n=1}^{\infty}$ is convergent. Let $x_n \to a$ $(n \to \infty)$, then we get $f(a) = a$ by Lemma 1.6. That is $(x_n)_{n=1}^{\infty}$ converges to a fixed point of $f(x)$.

Proof of Theorem 1.3. If $(x_n)_{n=1}^{\infty}$ is bounded, then $(x_n)_{n=1}^{\infty}$ converges to fixed point of $f(x)$ by Lemma 1.6. If $(x_n)_{n=1}^{\infty}$ is bounded, then $(x_n)_{n=1}^{\infty}$ is not convergent.

Proof of Theorem 1.4. It follows directly from Theorem 1.3 by setting $\alpha_n = 0$.

Proof of Corollary 1.5. It follows directly by Theorem 1.4.

2 Numerical Examples

In this section, we will present two numerical examples.

Example 2.1. *Let* $f : [0, 2.5] \to [0, 2.5]$ *be defined by* $f(x) = \frac{x^2+4}{5}$. *Then* f *is a continuous function with fixed point* $p = 1$. *Use the initial point* $x_1 = 2$ *and control condition* $\beta_n = \frac{1}{n^{0.5}}$ *and* $\alpha_n = \frac{1}{n^{1.5}}$.

	Mann	Ishikawa	modified Ishikawa	
n	x_n	x_n	x_n	$\lvert f(x_n) - x_n\rvert$
1	2.000000	2.000000	2.000000	0.400000
5	1.201718	1.076968	1.060524	0.035582
10	1.058014	1.020448	1.013900	0.008302
15	1.022608	1.007810	1.004972	0.002978
20	1.010324	1.003537	1.002166	0.001298
25	1.005211	1.002017	1.001060	0.000636
30	1.002822	1.000961	1.000561	0.000337
35	1.001611	1.000547	1.000315	0.000189
40	1.000958	1.000325	1.000185	0.000111
45	1.000590	1.000200	1.000112	0.000067
50	1.000373	1.000126	1.000070	0.000042

Table 1:

Example 2.2. *Let* $f : [5, \infty) \to [5, \infty)$ *be defined by* $f(x) = (0.5 + \ln 14x)^{1.5}$. *Then* f *is a continuous function. Use the initial point* $x_1 = 20$ *and control condition* $\beta_n = \frac{1}{n^{0.5}}$ *and* $\alpha_n = \frac{1}{n^{1.5}}$.

	Mann	Ishikawa	modified Ishikawa	
n	x_n	x_n	x_n	$\lvert f(x_n) - x_n \rvert$
1	20.000000	20.000000	20.000000	4.805042
5	14.076448	13.899326	13.883531	0.024377
10	13.892732	13.859423	13.855531	0.003651
15	13.863165	13.853207	13.851939	0.000993
20	13.855257	13.851561	13.851069	0.000349
25	13.852569	13.851004	13.850789	0.000142
30	13.851511	13.850785	13.850684	0.000064
35	13.851050	13.850690	13.850639	0.000031
40	13.850834	13.850646	13.850619	0.000016
45	13.850726	13.850623	13.850609	0.000009
50	13.850670	13.850612	13.850603	0.000005

Table 2:

Remark 2.3. *From two examples above, we observe that the modified Ishikawa iteration is better than Mann and Ishikawa iterations.*

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(Received 14 June 2010)

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