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Viscosity Approximation Methods for Generalized Equilibrium Problems and Fixed Point Problems of Finite Family of Nonexpansive Mappings in Hilbert Spaces

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Abstract : In this paper, we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space and we prove a strong convergence theorem in a Hilbert space which connected with Kangtunyakarn and Suantai [A. Kangtunyakarn and S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Anal., 3 (2009), 296–309.] and Takahashi and Takahashi's results [S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl., 331 (2007), 506–515]. Our results extend and improve some recent corresponding results in the literature.

Keywords : Fixed point; Generalized equilibrium problem; Nonexpansive mapping; Inverse strongly monotone mapping.
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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H and let P_C be the projection of H onto C. A mapping T of H into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T. Goebel and Kirk [7] showed that $F(T)$ is always closed convex and nonempty provided T has a bounded trajectory. Recall that a mapping $A : C \to H$ is called *monotone* if for all $x, y \in C$,

$$\langle x - y, Ax - Ay\rangle \geq 0.$$

The mapping A is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$; see [4, 11, 14]. We know that if $T : C \to C$ is nonexpansive, then $A = I - T$ is $\frac{1}{2}$- inverse strongly monotone; see [15, 16, 17] for more details.

The classical variational inequality problem is to find a $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. The set of solutions of variational inequality is denoted by $VI(C, A)$. The variational inequality has been extensively studied in the literature; see [3, 5, 14].

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and a nonlinear mapping $A : C \to H$. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such $z \in C$ is denoted by EP i.e.,

$$EP = \big\{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C\big\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is denoted by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see [2, 12].

For $r > 0$, let $T_r : H \to C$ be defined by

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}. \tag{1.2}$$

Combettes and Hirstoaga [6] showed that under some suitable conditions of F, T_r is single-valued and firmly nonexpansive and satisfies $F(T_r) = EP(F)$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. In 1999, Atsushiba and Takahashi [1] defined the mapping W_n as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I,\\
U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I,\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,\\
W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I,
\end{aligned}$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called *the W-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [18] proved that if X is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \ldots, N$.

In 2007, Takahashi and Takahashi [19] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space H. They defined the iterative sequences $\{x_n\}$ and $\{u_n\}$ as follows: $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $f : H \to H$ is a contraction mapping and $\{\alpha_n\} \subset [0,1], \{r_n\} \subset (0,\infty)$. They proved under some suitable conditions on the sequences $\{\alpha_n\}, \{r_n\}$ and bifunction F, that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(F)$, where $z = P_{F(T)\cap EP(F)} f(z)$.

In 2008, Takahashi and Takahashi [20] introduced a hybrid iterative method for finding a common element of EP and $F(T)$. They defined $\{x_n\}$ as follows: $u, x_1 \in C$ and

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n + (1-\beta_n)T(a_n u + (1-a_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{a_n\} \in [0,1], \{\beta_n\} \subset [0,1], \{\lambda_n\} \subset [0, 2\alpha]$ and proved strong convergence of the scheme (1.4) to $z \in \bigcap_{i=1}^N F(T_i) \cap EP$, where $z = P_{\cap_{i=1}^N F(T_i)\cap EP} u$.

In 2009, Kangtunyakarn and Suantai [10] defined the mappings S_n as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I \\ S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I \end{aligned}$$

where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. The mapping S_n is called the *S-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$.

If $\alpha_2^{n,j} = 0, j = 1, 2, \ldots, N-1$, then the mapping S reduces to the W-mapping defined by Atsushiba and Takahashi [1] and if $\alpha_3^{n,j} = 0, j = 1, 2, \ldots, N$, then the mapping S reduces to the K-mapping defined by Kangtunyakarn and Suantai [9].

In this paper, we introduce the iterative scheme as follows. For given $x_1 \in C$, let $\{z_n\}$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_n z_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$. Using the viscosity approximation method we will find a common element of the set of solutions of the equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space. Then, we shall prove a strong convergence theorem which is connected with Kangtunyakarn and Suantai [10] and Takahashi and Takahashi's results [19].

2 Preliminaries

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C.

Lemma 2.1. [15] *Let $x \in H$ and $z \in C$. Then $P_C x = z$ if and only if $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Lemma 2.2. [21] *Let $\{a_n\} \subset [0,\infty), \{b_n\} \subset [0,\infty)$ and $\{c_n\} \subset [0,1)$ be sequences of real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n \quad \text{for all } n \in \mathbb{N},$$
$$\sum_{n=1}^{\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.$$

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.3. [2] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4. [6] *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:
(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $F(T_r) = EP(F)$;
(4) $EP(F)$ *is closed and convex.*

Definition 2.5. *Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Kangtunyakarn and Suantai [10] defined the mapping $S : C \to C$ as follows:*

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned}$$

This mapping is called S-mapping *generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.*

Lemma 2.6. [10] *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j), j = 1, 2, 3, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0,1]$, $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.7. [10] *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and for each $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1], \alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Suppose $\alpha_i^{n,j} \to \alpha_i^j$ as $n \to \infty$ for $i = 1, 2, 3$ and $j = 1, 2, 3, \ldots, N$. Let S and S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty} \|S_n x - Sx\| = 0$ for all $x \in C$.*

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let A*

be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \cap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let f be a contraction of H into itself and let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{z_n\}, \{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq b < 2\alpha,\ 0 < c \leq \beta_n \leq d < 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$, *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.
Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap EP$, where $z = P_{\cap_{i=1}^N F(T_i)\cap EP} f(z)$.

Proof. Let $Q = P_{\cap_{i=1}^N F(T_i)\cap EP}$. Note that f is a contraction with coefficient $k \in [0, 1)$. Then

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\|$$

for all $x, y \in H$. Thus Qf is a contraction of H into itself. Since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of C.

Next, we show that $(I - \lambda_n A)$ is nonexpansive. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda_n < 2\alpha$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|x - y - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda_n\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 \end{aligned}$$

Therefore $\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \leq \|x - y\|^2$ for all $x, y \in C$. Thus $(I - \lambda_n A)$ is nonexpansive. Since

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

we obtain

$$F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (I - \lambda_n A)x_n\rangle \geq 0, \quad \forall y \in C.$$

By Lemma 2.4, we have $z_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$ for all $n \in \mathbb{N}$.

Let $z \in \cap_{i=1}^N F(T_i) \cap EP$. Then $F(z, y) + \langle y - z, Az\rangle \geq 0$ for all $y \in C$. So $F(z, y) + \frac{1}{\lambda_n}\langle y - z, z - z + \lambda_n Az\rangle \geq 0$ for all $y \in C$. Again by Lemma 2.4, we obtain $z = T_{\lambda_n}(z - \lambda_n Az)$. Since $I - \lambda_n A$ and T_{λ_n} are nonexpansive, we have

$$\begin{aligned} \|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n A x_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\ &\leq \|x_n - z\|^2 \end{aligned}$$

and hence $\|z_n - z\| \leq \|x_n - z\|$. This implies that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) + (1 - \alpha_n - \beta_n)(S_n z_n - z)\| \\ &\leq \alpha_n\|f(x_n) - z\| + \beta_n\|x_n - z\| + (1 - \alpha_n - \beta_n)\|z_n - z\| \\ &\leq \alpha_n k\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\|x_n - z\| \end{aligned} \tag{3.2}$$

Putting $K = \max\{\|x_1 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$. By (3.2), we can show by induction that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded. Hence $\{Ax_n\}, \{S_n z_n\}, \{f(x_n)\}$ are also bounded.

Next, we shall show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Putting $u_n = x_n - \lambda_n A x_n$. Then, we have $z_{n+1} = T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1} A x_{n+1}) = T_{\lambda_{n+1}} u_{n+1}$, and $z_n = T_{\lambda_n}(x_n - \lambda_n A x_n) = T_{\lambda_n} u_n$. Thus

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|T_{\lambda_{n+1}} u_{n+1} - T_{\lambda_n} u_n\| \\ &\leq \|T_{\lambda_{n+1}} u_{n+1} - T_{\lambda_{n+1}} u_n\| + \|T_{\lambda_{n+1}} u_n - T_{\lambda_n} u_n\| \\ &\leq \|u_{n+1} - u_n\| + \|T_{\lambda_{n+1}} u_n - T_{\lambda_n} u_n\|. \end{aligned} \tag{3.3}$$

Since $I - \lambda_{n+1} A$ is nonexpansive, we obtain

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|x_{n+1} - \lambda_{n+1} A x_{n+1} - x_n + \lambda_n A x_n\| \\ &= \|(I - \lambda_{n+1} A) x_{n+1} - (I - \lambda_{n+1} A) x_n + (\lambda_n - \lambda_{n+1}) A x_n\| \\ &\leq \|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}| A x_n\|. \end{aligned} \tag{3.4}$$

By Lemma 2.4, we obtain

$$F(T_{\lambda_n} u_n, y) + \frac{1}{\lambda_n}\langle y - T_{\lambda_n} u_n, T_{\lambda_n} u_n - u_n\rangle \geq 0, \qquad \forall y \in C$$

and

$$F(T_{\lambda_{n+1}} u_n, y) + \frac{1}{\lambda_{n+1}}\langle y - T_{\lambda_{n+1}} u_n, T_{\lambda_{n+1}} u_n - u_n\rangle \geq 0, \qquad \forall y \in C.$$

In particular, we have

$$F(T_{\lambda_n} u_n, T_{\lambda_{n+1}} u_n) + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}} u_n - T_{\lambda_n} u_n, T_{\lambda_n} u_n - u_n\rangle \geq 0 \tag{3.5}$$

and

$$F(T_{\lambda_{n+1}}u_n, T_{\lambda_n}u_n) + \frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle \geq 0. \quad (3.6)$$

Summing up (3.5) and (3.6) and using (A2), we obtain

$$\frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0,$$

for all $y \in C$. It follows that

$$\left\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, \frac{T_{\lambda_{n+1}}u_n - u_n}{\lambda_{n+1}} - \frac{T_{\lambda_n}u_n - u_n}{\lambda_n}\right\rangle \geq 0$$

This implies

$$\begin{aligned} 0 &\leq \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n - \frac{\lambda_n}{\lambda_{n+1}}(T_{\lambda_{n+1}}u_n - u_n)\right\rangle \\ &= \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(T_{\lambda_{n+1}}u_n - u_n)\right\rangle. \end{aligned}$$

It follows that

$$\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|(\|T_{\lambda_{n+1}}u_n\| + \|u_n\|).$$

Hence, we obtain

$$\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L, \quad (3.7)$$

where $L = \sup\{\|u_n\| + \|T_{\lambda_{n+1}}u_n\| : n \in \mathbb{N}\}$. By (3.3), (3.4) and (3.7), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \|u_{n+1} - u_n\| + \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| \\ &\leq \|x_{n+1} - x_n + b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| Ax_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L \quad (3.8) \end{aligned}$$

Next, we show that

$$\lim_{n\to\infty} \|S_{n+1}z_n - S_nz_n\| = 0.$$

For $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}
\|U_{n+1,k}z_n - U_{n,k}z_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}z_n + \alpha_2^{n+1,k}U_{n+1,k-1}z_n + \alpha_3^{n+1,k}z_n \\
&\quad - \alpha_1^{n,k}T_kU_{n,k-1}z_n - \alpha_2^{n,k}U_{n,k-1}z_n - \alpha_3^{n,k}z_n\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}z_n - T_kU_{n,k-1}z_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}z_n \\
&\quad + (\alpha_3^{n+1,k} - \alpha_3^{n,k})z_n + \alpha_2^{n+1,k}(U_{n+1,k-1}z_n - U_{n,k-1}z_n) \\
&\quad + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}z_n\| \\
&\le \alpha_1^{n+1,k}\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| \\
&\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}z_n\| \\
&= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| \\
&\quad + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| \\
&\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}z_n\| \\
&\le \|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| + (\alpha_1^{n,k} - \alpha_1^{n+1,k}) \\
&\quad + (\alpha_3^{n,k} - \alpha_3^{n+1,k})\|U_{n,k-1}z_n\| \\
&\le \|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}z_n\| \\
&\quad + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}z_n\| \\
&= \|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_kU_{n,k-1}z_n\| \\
&\quad + \|U_{n,k-1}z_n\|) + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|z_n\| + \|U_{n,k-1}z_n\|).
\end{aligned} \tag{3.9}$$

By (3.9), we obtain that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|S_{n+1}z_n - S_nz_n\| &= \|U_{n+1,N}z_n - U_{n,N}z_n\| \\
&\le \|U_{n+1,1}z_n - U_{n,1}z_n\| + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}z_n\| \\
&\quad + \|U_{n,j-1}z_n\|) + \sum_{j=2}^{N}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|z_n\| + \|U_{n,j-1}z_n\|) \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1z_n - z_n\| + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}z_n\| \\
&\quad + \|U_{n,j-1}z_n\|) + \sum_{j=2}^{N}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|z_n\| + \|U_{n,j-1}z_n\|).
\end{aligned}$$

This together with condition (iv), we obtain

$$\lim_{n\to\infty} \|S_{n+1}z_n - S_n z_n\| = 0. \tag{3.10}$$

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

By (3.1), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_n z_n - \alpha_{n-1} f(x_{n-1}) - \beta_{n-1}x_{n-1} \\
&\quad - (1-\alpha_{n-1}-\beta_{n-1})S_{n-1}z_{n-1}\| \\
&= \|\alpha_n\,(f(x_n) - f(x_{n-1})) + \beta_n(x_n - x_{n-1}) + (1-\alpha_n-\beta_n) \\
&\quad (S_n z_n - S_{n-1}z_{n-1}) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) + (\alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n) \\
&\quad S_{n-1}z_{n-1} + (\beta_n - \beta_{n-1})x_{n-1}\| \\
&\le \alpha_n\|f(x_n) - f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| + |1-\alpha_n-\beta_n| \\
&\quad \|S_n z_n - S_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + |\alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n| \\
&\quad \|S_{n-1}z_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&\le (\alpha_n k + \beta_n)\|x_n - x_{n-1}\| + |1-\alpha_n-\beta_n|\|S_n z_n - S_{n-1}z_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|K + |\alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n|K \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\|,
\end{aligned} \tag{3.11}$$

where $K = \sup\{\|f(x_n)\| + \|S_n z_n\|, n \in \mathbb{N}\}$. By (3.11) and since

$$\|S_n z_n - S_{n-1}z_{n-1}\| \le \|z_{n-1} - z_n\| + \|S_n z_{n-1} - S_{n-1}z_{n-1}\|$$

and (3.8), (3.10), Lemma 2.2, conditions (ii) and (iii), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (\alpha_n k + \beta_n)\|x_n - x_{n-1}\| + |1-\alpha_n-\beta_n|(\|x_{n-1} - x_n\| \\
&\quad + b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|Ax_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + \|S_n z_{n-1} - S_{n-1}z_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|K + |\alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n|K + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&= (1 - \alpha_n(1-k))\|x_{n-1} - x_n\| + (|1-\alpha_n-\beta_n| b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|Ax_n\| \\
&\quad + |1-\alpha_n-\beta_n|\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |1-\alpha_n-\beta_n|\|S_n z_{n-1} - S_{n-1}z_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|K + |\alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n|K + |\beta_n - \beta_{n-1}|\|x_{n-1}\|)
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Next, we shall show that

$$\lim_{n\to\infty} \|S_n z_n - x_n\| = 0.$$

By (3.1), we obtain

$$\begin{aligned}
\|S_n z_n - x_n\| &= \|S_n z_n - S_{n-1}z_{n-1} + S_{n-1}z_{n-1} - \alpha_{n-1}f(x_{n-1}) \\
&\quad - \beta_{n-1}x_{n-1} - S_{n-1}z_{n-1} + \alpha_{n-1}S_{n-1}z_{n-1} + \beta_{n-1}S_{n-1}z_{n-1}\| \\
&= \|S_n z_n - S_{n-1}z_{n-1} + \alpha_{n-1}(S_{n-1}z_{n-1} - f(x_{n-1})) \\
&\quad + \beta_{n-1}(S_{n-1}z_{n-1} - x_{n-1})\| \\
&\le \|S_n z_n - S_{n-1}z_{n-1}\| + \alpha_{n-1}\|S_{n-1}z_{n-1} - f(x_{n-1})\| \\
&\quad + \beta_{n-1}\|S_{n-1}z_{n-1} - x_{n-1}\|
\end{aligned}$$

By (3.10), we obtain

$$\lim_{n\to\infty} \|S_n z_n - x_n\| = 0. \tag{3.13}$$

Next, we want to show

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

By monotonicity of A and nonexpansiveness of T_{λ_n}, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_n z_n - z\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 + (1-\alpha_n-\beta_n)\|S_n z_n - z\|^2 \\
&\le \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\|z_n - z\|^2 \\
&\le \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 \\
&= \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 \\
&= \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 + (1-\alpha_n-\beta_n) \\
&\quad (\|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2) \\
&\le \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 + (1-\alpha_n-\beta_n) \\
&\quad (\|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2) \\
&= \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \beta_n\|x_n - z\|^2 + (1-\alpha_n-\beta_n) \\
&\quad (\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2) \\
&= \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2 \\
&\le \alpha_n k^2\|x_n - z\|^2 + \alpha_n\|f(z) - z\|^2 + \|x_n - z\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2.
\end{aligned} \tag{3.14}$$

By (3.14), we obtain

$$(1-\alpha_n-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Az\|^2 \le \alpha_n k^2\|x_n-z\|^2+\alpha_n\|f(z)-z\|^2 + \|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Since $0<a\le\lambda_n\le b<2\alpha$ and $0<c\le\beta_n\le d<1$, we obtain

$$(1-\alpha_n-\beta_n)a(2\alpha-\lambda_n)\|Ax_n-Az\|^2 \le (1-\alpha_n-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Az\|^2.$$

Thus

$$\begin{aligned}(1-\alpha_n-\beta_n)a(2\alpha-\lambda_n)\|Ax_n-Az\|^2 &\le \alpha_n k^2\|x_n-z\|^2+\alpha_n\|f(z)-z\|^2 \\ &\quad + \|x_n-z\|^2-\|x_{n+1}-z\|^2 \\ &\le \alpha_n k^2\|x_n-z\|^2+\alpha_n\|f(z)-z\|^2 \\ &\quad + \|x_{n+1}-x_n\|(\|x_n-z\|+\|x_{n+1}-z\|).\end{aligned}$$

This implies, by (3.12) and condition (i), that

$$\lim_{n\to\infty}\|Ax_n-Az\|=0. \tag{3.15}$$

Since T_{λ_n} is a firmly nonexpansive, we obtain

$$\begin{aligned}\|z_n-z\|^2 &= \|T_{\lambda_n}(x_n-\lambda_nAx_n)-T_{\lambda_n}(z-\lambda_nAz)\|^2 \\ &\le \langle(x_n-\lambda_nAx_n)-(z-\lambda_nAz), z_n-z\rangle \\ &= \frac{1}{2}(\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\|^2+\|z_n-z\|^2-\|(x_n-\lambda_nAx_n) \\ &\quad -(z-\lambda_nAz)-(z_n-z)\|^2) \\ &\le \frac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|(x_n-z_n)-\lambda_n(Ax_n-Az)\|^2) \\ &= \frac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\langle x_n-z_n, Ax_n-Az\rangle \\ &\quad -\lambda_n^2\|Ax_n-Az\|^2).\end{aligned}$$

It follows that

$$\|z_n-z\|^2 \le \|x_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\|x_n-z_n\|\|Ax_n-Az\| \tag{3.16}$$

Since

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n k^2\|x_n-z\|^2+\alpha_n\|f(z)-z\|^2+\beta_n\|x_n-z\|^2 \\ &\quad +(1-\alpha_n-\beta_n)\|z_n-z\|^2\end{aligned}$$

and by (3.16), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 + \beta_n \|x_n - z\|^2 \\ &\quad + (1 - \alpha_n - \beta_n)(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|) \\ &\le \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 + \|x_n - z\|^2 - (1 - \beta_n)\|x_n - z_n\|^2 \\ &\quad + \alpha_n \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\| \\ &= \alpha_n (k^2 \|x_n - z\|^2 + \|f(z) - z\|^2 + \|x_n - z_n\|^2) + \|x_n - z\|^2 \\ &\quad - (1 - \beta_n)\|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|\end{aligned}$$

This implies

$$\begin{aligned}(1 - \beta_n)\|x_n - z_n\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n (k^2 \|x_n - z\|^2 \\ &\quad + \|f(z) - z\|^2 + \|x_n - z_n\|^2) + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|\end{aligned}$$

and by condition (i), we obtain

$$\begin{aligned}(1 - d)\|x_n - z_n\|^2 &\le \|x_{n+1} - x_n\|(\|x_n - z\| + \|x_{n+1} - z\|) + \alpha_n (k^2 \|x_n - z\|^2 \\ &\quad + \|f(z) - z\|^2 + \|x_n - z_n\|^2) + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.17}$$

From (3.13) and (3.17), we obtain

$$\begin{aligned}\|S_n z_n - z_n\| &= \|S_n z_n - x_n + x_n - z_n\| \\ &\le \|S_n z_n - x_n\| + \|x_n - z_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned} \tag{3.18}$$

We shall show that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, z_n - z_0 \rangle \le 0,$$

where $z_0 = P_{\cap_{i=1}^N F(T_i) \cap EP} f(z_0)$. To show this inequality, we choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, z_n - z_0 \rangle = \limsup_{k\to\infty} \langle f(z_0) - z_0, z_{n_k} - z_0 \rangle.$$

Without loss of generality, we may assume that $z_{n_k} \rightharpoonup w$ as $k \to \infty$ where $w \in C$. We first show $w \in EP$. Since $z_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$, we obtain

$$F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \ge 0, \quad \forall y \in C.$$

From (A2), we have

$$\langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \ge F(y, z_n).$$

Thus

$$\langle Ax_{n_k}, y - z_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle y - z_{n_k}, z_{n_k} - x_{n_k}\rangle \geq F(y, z_{n_k}), \quad \forall y \in C. \tag{3.19}$$

Putting $z_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.19), we obtain

$$\begin{aligned}\langle z_t - z_{n_k}, Az_t\rangle &\geq \langle z_t - z_{n_k}, Az_t\rangle - \langle z_t - z_{n_k}, Ax_{n_k}\rangle - \langle z_t - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle + F(z_t, z_{n_k})\\ &= \langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle\\ &\quad - \langle z_t - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle + F(z_t, z_{n_k}).\end{aligned}$$

Since $\|z_{n_k} - x_{n_k}\| \to 0$ as $k \to \infty$, we obtain $\|Az_{n_k} - Ax_{n_k}\| \to 0$ as $k \to \infty$. Further, from the monotonicity of A, we have $\langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle \geq 0$. So, from (A4), we obtain

$$\langle z_t - w, Az_t\rangle \geq F(z_t, w). \tag{3.20}$$

From (A1), (A4) and (3.20), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, w)\\ &\leq tF(z_t, y) + (1-t)\langle z_t - w, Az_t\rangle\\ &= tF(z_t, y) + (1-t)t\langle y - w, Az_t\rangle\end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - w, Az_t\rangle$$

Letting $t \to 0$, we obtain

$$0 \leq F(w, y) + \langle y - w, Aw\rangle, \quad \forall y \in C.$$

Therefore $w \in EP$. Next, we show that $w \in \cap_{i=1}^N F(T_i)$. We assume that

$$\alpha_1^{n_k,j} \to \alpha_1^j \in (0,1) \quad \text{and} \quad \alpha_1^{n_k,N} \to \alpha_1^N \in (0,1] \quad \text{as} \quad k \to \infty$$

for $j = 1, 2, \ldots, N-1$ and $\alpha_3^{n_k,j} \to \alpha_3^j \in [0,1)$ as $k \to \infty$ for $j = 1, 2, \ldots, N$. Let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\beta_1, \beta_2, \ldots, \beta_N$, where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ for $j = 1, 2, \ldots, N$. By Lemma 2.7, we have

$$\lim_{k\to\infty} \|S_{n_k}x - Sx\| = 0, \quad \forall x \in C. \tag{3.21}$$

By Lemma 2.6, we have $\cap_{i=1}^N F(T_i) = F(S)$. Assume that $Sw \neq w$. By using the Opial property and (3.18) and (3.21), we obtain

$$\begin{aligned}\liminf_{k\to\infty} \|z_{n_k} - w\| &< \liminf_{k\to\infty} \|z_{n_k} - Sw\|\\ &\leq \liminf_{k\to\infty} \big(\|z_{n_k} - S_{n_k}z_{n_k}\| + \|S_{n_k}z_{n_k} - S_{n_k}w\| + \|S_{n_k}w - Sw\|\big)\\ &\leq \liminf_{k\to\infty} \|z_{n_k} - w\|\end{aligned}$$

which is a contradiction. Thus $Sw = w$ and $w \in F(S) = \cap_{i=1}^N F(T_i)$. Therefore $w \in \cap_{i=1}^N F(T_i) \cap EP$. Since $z_{n_k} \rightharpoonup w$ and $w \in \cap_{i=1}^N F(T_i) \cap EP$, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z_0) - z_0, z_n - z_0\rangle &= \limsup_{k\to\infty}\langle f(z_0) - z_0, z_{n_k} - z_0\rangle \\ &= \langle f(z_0) - z_0, w - z_0\rangle \le 0. \qquad (3.22)\end{aligned}$$

From $x_{n+1} - z_0 = \alpha_n(f(x_n) - z_0) + \beta_n(x_n - z_0) + (1 - \alpha_n - \beta_n)(S_n z_n - z_0)$, we obtain

$$\begin{aligned}(1-\alpha_n-\beta_n)^2\|S_n z_n - z_0\|^2 \ge \|x_{n+1} - z_0\|^2 &- 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &- \beta_n\|x_n - z_0\|^2.\end{aligned}$$

So we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &\le (1-\alpha_n-\beta_n)^2\|z_n - z_0\|^2 + \beta_n\|x_n - z_0\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n-\beta_n)^2\|z_n - z_0\|^2 + \beta_n\|x_n - z_0\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n-\beta_n)^2\|x_n - z_0\|^2 + \beta_n\|x_n - z_0\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n-\beta_n)^2\|x_n - z_0\|^2 + \beta_n\|x_n - z_0\|^2 \\ &\quad + 2\alpha_n k\|x_n - z_0\|\|x_{n+1} - z_0\| + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le \Big((1-\alpha_n-\beta_n)^2 + \beta_n\Big)\|x_n - z_0\|^2 + \alpha_n k(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.\end{aligned}$$

So

$$\begin{aligned}(1-\alpha_n k)\|x_{n+1} - z_0\|^2 \le \Big((1-\alpha_n-\beta_n)^2 + \beta_n + \alpha_n k\Big)&\|x_n - z_0\|^2 \\ + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle&.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &\le \frac{(1-\alpha_n-\beta_n)^2 + \beta_n + \alpha_n k}{1-\alpha_n k}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n k}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le \frac{1 - 2\alpha_n + \alpha_n k}{1-\alpha_n k}\|x_n - z_0\|^2 + \frac{\alpha_n^2}{1-\alpha_n k}\|x_n - z_0\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n k}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le \Big(1 - \frac{2(1-k)\alpha_n}{1-\alpha_n k}\Big)\|x_n - z_0\|^2 \\ &\quad + \frac{2(1-k)\alpha_n}{1-\alpha_n k}\Big(\frac{\alpha_n M}{2(1-k)} + \frac{1}{1-k}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\Big),\end{aligned}$$

where $M = \sup\{\|x_n - z_0\|^2 : n \in \mathbb{N}\}$. Put $\beta_n = \frac{2(1-k)\alpha_n}{1-\alpha_n k}$. Then $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Let $\epsilon > 0$. From (3.22), there exists $m \in \mathbb{N}$ such that

$$\frac{\alpha_n M}{2(1-k)} \le \frac{\epsilon}{2} \qquad \text{and} \qquad \frac{1}{1-k}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \le \frac{\epsilon}{2}$$

for all $n \ge m$. Then

$$\|x_{n+1} - z_0\|^2 \le (1-\beta_n)\|x_n - z_0\|^2 + \big(1 - (1-\beta_n)\big)\epsilon.$$

Similarly, we have

$$\|x_{m+n} - z_0\|^2 \le \prod_{k=m}^{m+n-1}(1-\beta_k)\|x_m - z_0\|^2 + \Big(1 - \prod_{k=m}^{m+n-1}(1-\beta_k)\Big)\epsilon.$$

From $\sum_{k=m}^{\infty} \beta_k = \infty$, we have $\prod_{k=m}^{\infty}(1-\beta_k) = 0$. Therefore

$$\limsup_{n\to\infty} \|x_n - z_0\|^2 = \limsup_{n\to\infty} \|x_{m+n} - z_0\|^2 \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty} \|x_n - z_0\|^2 \le 0.$$

Thus $\{x_n\}$ converges strongly to $z_0 \in \cap_{i=1}^N F(T_i) \cap EP$, where
$z_0 = P_{\cap_{i=1}^N F(T_i)\cap EP} f(z_0)$. □

Using our main theorem (Theorem 3.1), we obtain strong convergence in a Hilbert space.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \cap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \le \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \le \theta_3 < 1$. Let f be a contraction of H into itself and let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{z_n\}, \{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) S_n z_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the following conditions:
(i) $0 < a \le \lambda_n \le b < \infty,\ 0 < c \le \beta_n \le d < 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.
Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap EP(F)$, where
$z = P_{\cap_{i=1}^N F(T_i)\cap EP(F)} f(z)$.

Proof. In Theorem 3.1, put $A \equiv 0$. Then, for all $\alpha \in (0, \infty)$, we have

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. So, taking $a, b \in (0, \infty)$ with $0 < a \leq b < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $a \leq \lambda_n \leq b$, we obtain the result from Theorem 3.1. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \cap VI(C, A) \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let f be a contraction of H into itself and let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{x_n\} \subset C$ be sequences generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n P_C(x_n - \lambda_n A x_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq b < 2\alpha,\ 0 < c \leq \beta_n \leq d < 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.
Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap VI(C, A)$, where
$z = P_{\cap_{i=1}^N F(T_i) \cap VI(C,A)} f(z)$.

Proof. In Theorem 3.1, put $F \equiv 0$. Then, we obtain

$$\langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C,\ \forall n \in \mathbb{N}.$$

This implies that

$$\langle y - z_n, x_n - \lambda_n A x_n - z_n \rangle \leq 0, \quad \forall y \in C.$$

So, we obtain $P_C(x - \lambda_n A x_n) = z_n$ for all $n \in \mathbb{N}$. Then, we obtain the result from Theorem 3.1. □

A mapping $G : C \to C$ is called strictly pseudocontractive if there exists g with $0 \leq g < 1$ such that

$$\|Gx - Gy\|^2 \leq \|x - y\|^2 + g\|(I - G)x - (I - G)y\|^2, \quad \forall x, y \in C.$$

Such a mapping G is called strictly g-pseudocontractive. Putting $A = I - G$, we know that

$$\langle x - y, Ax - Ay \rangle \geq \frac{1 - g}{2} \|Ax - Ay\|^2, \quad \forall x, y \in C;$$

see [8]. So, we have the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let G be a strictly g-pseudocontractive mapping of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \cap EP \neq \emptyset$, where $A = I - G$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let f be a contraction of H into itself and let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{z_n\}, \{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle (I-G)x_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 1-g]$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq b < 1 - g,\ 0 < c \leq \beta_n \leq d < 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty,$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.
Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap EP$, where $z = P_{\cap_{i=1}^N F(T_i)\cap EP} f(z)$.

Proof. A strictly g-pseudocontractive mapping is $\frac{1-g}{2}$-inverse-strongly monotone. So, from Theorem 3.1, we obtain the desired result. □

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