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Global Stability Analysis of Predator-Prey Model with Harvesting and Delay

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Abstract : We study the global stability of a Michaelis-Menten type predator - prey model with harvesting and delay. Sufficient conditions on the system parameters are derived which guarantee that the equilibrium points of the system are globally asymptotically stable while the delay which has an effect on the stability of this system satisfies certain conditions. Numerical simulations are shown to confirm our theoretical results.

Keywords : Predator-prey model; Harvest management; Global stability; Lyaponov function.
2000 Mathematics Subject Classification : 93A30; 34D23.

1 Introduction

Population is complex therefore, biologists should have information on survival by age class, number of breeding by age, frequency distribution of ages, and its density. These are difficult data to obtain for most populations and are often not available because of limited time and resources. Thus, it is important to be able to estimate the parameters when defining populations.

Harvest management has been used to control increasing population and to meet the public demands for recreation, animal damage control or commercial harvesting. Many populations are managed under the assumption the population will continue to increase until it approaches the limits of the available resources to support it.

Many authors, such as Beretta and Kuang [1] studied predator - prey model by carrying out the global stability analysis on the delayed ratio-dependent predator

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- prey system. They proved global stability results for delayed Michaelis-Menten type ratio-dependent predator-prey system and convergence results for the delayed Holling-Tanner type (semi) ratio-dependent predator-prey system. Later, Hsu et.al., [2] used a change of variables and transformed the Michaelis-Menten type model to a Gause-type predator-prey system. They gave a complete classification of the asymptotic behavior of the solutions of the Michaelis-Menten type ratio-dependent model. In the work of Nindjinm et.al., [3], a sufficient condition for global stability of the positive equilibrium of two-dimensional delayed continuous time dynamical system was established. That model is a predator-prey food chain in a modified Leslie-Gower model of Holling type-II scheme. Investigation was done by constructing a Liapunov function. In [4], Aziz-Alaoui and Daher analyzed a predator - prey model in terms of boundedness of solutions, existence of an attracting set and global stability of the coexisting interior equilibrium.

Now, the conservative resources are important such that the focused attention on management of harvesting in predator - prey system has become an interesting topic in mathematical bio-economic research. The population with harvesting is related to the renewable resource management. The exploitation of harvesting population species is used in fishery, forestry and wildlife management. Kar and Pahari [5] studied the effect of harvesting and time delay on the dynamics of the generalized Gause type predator-prey models. Hoeckstra and Bergh [6] focused on optimal harvesting of prey in a predator-prey ecosystem. They found the conditions for the existence of the predators when the predators and humans compete for prey. In this work, we generalized the model to incorporate the delay and a harvesting term when the time delay represents an immature period or reaction time of predator. The population dynamics with harvesting are related to the optimal management of renewable resources.

We study some equilibriums properties for the referenced model system and give preliminaries on boundedness and a persistent result. Then we analyze the global stability of the system. It made for a boundary solution and sufficient conditions are provided for the positive equilibrium of both instantaneous system (nondelayed) and system with delay to be globally asymptotically stable.

2 Mathematical Model

Let $x = x(t)$ represent the prey density in time t. The model rests upon the logistic equation of population dynamics. There is a natural rate of increase, $\dot{x} = rx\left(1 - \frac{x}{K}\right)$ where r is prey intrinsic growth rate and K is carrying capacity. The functional response is how predator hunt prey. They pay the time to search, capture, handle, and consume at maximum rate c and with the half capturing saturation constant m. Let $y = y(t)$ represent the predator density in time t. We obtain $\frac{cxy}{my+x}$, the predator functional response to prey density which refers to the change in the density of prey killed per unit time per predator as the prey density changes. The functional response is diterminded by searching patterns, disire and handling time of the predator, searching efficiency or number of prey density [8].

How long does the population persist ? It is influenced by death rate (d) and the predator grows by numerical response on consuming prey, where α is the conversion rate when prey is consumed for predator growth. Therefore, the ratio-dependent predator-prey model [2] takes the form

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{cxy}{my+x} \\ \dot{y} &= y\left(-d+\frac{\alpha x}{my+x}\right) \end{aligned}$$

with the initial conditions $x(0) > 0$, and $y(0) > 0$.

The model with time delay is a more realistic approach to the understanding of the predator-prey dynamics. The delay in the model means when predator consumes prey, they use the time to reproduce the next generation.

The population dynamics with harvesting is related to the optimal management of renewable resources. The goals of management are to make the population increase or decrease to harvest the population for a continuing yield. The harvesting process as fishery and hunting includes searching for food or sport. Thus, the harvesting is a controller of the density of population. The harvesting depends on the effort of hunter (E) or the efficiency of the technique used to catch. The capability coefficient of the harvesting are q_1, and q_2. If we put in a good effort (labors, tools) then the harvesting rate increases. However, the rate of harvesting is limited by a carrying capacity. So, to manage the population dynamics in ecology, we will consider the system that has the delay and non-selective harvesting of both species as follow,

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)-\frac{cx(t)y(t)}{my(t)+x(t)}-q_1E_1x(t) \\ \dot{y}(t) &= y(t)\left(-d+\frac{\alpha x(t-\tau)}{my(t-\tau)+x(t-\tau)}\right)-q_2E_2y(t) \end{aligned} \tag{2.1}$$

with the initial conditions for the delayed system: $x_0(\theta)=\phi_1(\theta)\geq 0$, $y_0(\theta)=\phi_2(\theta)\geq 0$, $\theta\in[-\tau,0]$, $x(0)>0$, $y(0)>0$, where $x_t(\theta)=x(t+\theta)$, for $\theta\in[-\tau,0]$, and $(\phi_1,\phi_2)\in\mathcal{C}([-\tau,0],\mathbb{R}^2_+)$, $\mathbb{R}^2_+=\{(x,y)|x\geq 0, y\geq 0\}$.

An equilibrium point $(\bar{x},\bar{y})$ is determined analytically by solving $\dot{x}=\dot{y}=0$. The equilibrium of the above system is globally asymptotically stable if it attracts all positive solutions of that system. Our goal is to show that the equilibrium point of system (2.1) is globally asymptotically stable.

Proposition 2.1. *The point* $(\frac{1}{b}(r-q_1E_1),0)$ *is an equilibrium point of the system (2.1) and if the conditions (i)* $r-q_1E_1>\frac{c}{\alpha m}(\alpha-d-q_2E_2)$ *and (ii)* $\alpha>d+q_2E_2$ *hold, then there exists a positive equilibrium point* $E^*=(x^*,y^*)$, *where* $x^*=\frac{1}{b}\left(r-\frac{c}{\alpha m}(\alpha-d-q_2E_2)-q_1E_1\right)$ *and* $y^*=\frac{x^*}{m}\left(\frac{\alpha}{d+q_2E_2}-1\right)$.

Proof. By equation $\dot{y} = 0$ in system (2.1), we get $\bar{y} = 0$ or $\bar{y} = \frac{\bar{x}}{m}\left(\frac{\alpha}{d+q_2E_2} - 1\right)$. If $\bar{y} = 0$ and from $\dot{x} = 0$ in system (2.1), we have $\bar{x} = 0$ or $\bar{x} = \frac{1}{b}(r - q_1E_1)$, but in the biological model, the case that the equilibrium point $(\bar{x}, \bar{y}) = (0, 0)$ is uninteresting. Hence, we get $\bar{y} = 0$, $\bar{x} = \frac{1}{b}(r - q_1E_1)$. By the conditions (i) and (ii), we obtain that the point (x^*, y^*), where $x^* = \frac{1}{b}\left(r - \frac{c}{\alpha m}(\alpha - d - q_2E_2) - q_1E_1\right)$ and $y^* = \frac{x^*}{m}\left(\frac{\alpha}{d+q_2E_2} - 1\right)$, is a positive equilibrium point of the system (2.1). □

3 Boundedness and Permanence

In this section we study the boundedness of the solutions of system (2.1) defined on $[-\tau, A)$ where $A \in (0, \infty)$.

Lemma 3.1. *The positive quadrant int*$(\mathbb{R}^2_+)$ *is invariant for system* (2.1).

Proof. To show that for all $t \in [0, A)$, $x(t) > 0$ and $y(t) > 0$, suppose that it is not true. Then, there exists a T, $0 < T < A$, such that for all $t \in [0, T)$, $x(t) > 0$ and $y(t) > 0$ and either $x(T) = 0$ or $y(T) = 0$.

For all $t \in [-\tau, T)$ and from system (2.1), we have

$$x(t) = x(0)\exp\int_0^t \left(r\left(1 - \frac{x(s)}{K}\right) - q_1E_1 - \frac{cy(s)}{my(s) + x(s)}\right) ds, \tag{3.1}$$

$$y(t) = y(0)\exp\int_0^t \left(-d + \frac{\alpha x(s-\tau)}{my(s-\tau) + x(s-\tau)} - q_2E_2\right) ds. \tag{3.2}$$

As (x, y) is defined and continuous on $[-\tau, T)$, there is an $M \geq 0$ such that for all $t \in [-\tau, T)$,

$$x(t) = x(0)\exp\int_0^t \left(r\left(1 - \frac{x(s)}{K}\right) - q_1E_1 - \frac{cy(s)}{my(s) + x(s)}\right) ds \geq x(0)\exp(-TM)$$

$$y(t) = y(0)\exp\int_0^t \left(-d + \frac{\alpha x(s-\tau)}{my(s-\tau) + x(s-\tau)} - q_2E_2\right) ds \geq y(0)\exp(-TM).$$

Taking the limit as $t \to T$ and with initial conditions $x(0) > 0, y(0) > 0$, we get $x(T) \geq x(0)\exp(-TM) > 0$ and $y(T) \geq y(0)\exp(-TM) > 0$ which contradicts the fact that either $x(T) = 0$ or $y(T) = 0$. Therefore, $x(t) > 0$, $y(t) > 0$ $\forall t \in [0, A)$. □

Lemma 3.2. *Let* $(x(t), y(t))$ *be the solution of the system (2.1). Then* $\limsup_{t\to+\infty} x(t) \leq K$, *and if* $\alpha > d + q_2E_2$, *then*

$$\limsup_{t\to+\infty} y(t) \leq \frac{(\alpha - d - q_2E_2)Ke^{(\alpha-d-q_2E_2)\tau}}{m(d + q_2E_2)} \tag{3.3}$$

Proof. From the system (2.1), $\dot{x} \leq rx\left(1 - \frac{x}{K}\right)$. By a standard comparison argument, we have that for all $t \in [0, \infty)$, $x(t) \leq \tilde{x}(t)$, where $\tilde{x}(t)$ is the solution of the following ordinary differential equation

$$\begin{aligned} \dot{\tilde{x}}(t) &= r\tilde{x}(t)(1 - \frac{\tilde{x}(t)}{K}), \\ \tilde{x}(0) &= x(0) > 0. \end{aligned}$$

As $\lim_{t\to+\infty} \tilde{x}(t) = K$, then $\tilde{x}(t)$ and therefore $x(t)$ is bounded on $[0, \infty)$. Thus, as for all $t \geq 0, x(t) \leq \tilde{x}(t)$, then $\limsup_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} \tilde{x}(t) = K$. So, $\limsup_{t\to+\infty} x(t) \leq K$. From the predator equation, we have

$$\begin{aligned} \dot{y}(t) &\leq y(t)(-d + \alpha - q_2E_2) \qquad (3.4) \\ y(t) &\leq y(0)e^{(\alpha-d-q_2E_2)t} \end{aligned}$$

Thus, for $t > \tau$, integrating (3.4) on the interval $[t - \tau, t]$, one obtains,

$$y(t-\tau) \geq y(t)e^{-(\alpha-d-q_2E_2)\tau}. \qquad (3.5)$$

Observe that there exists a $T > 0$ such that, for $t > T, x(t) < K$. Using (3.5) then for $t > T + \tau$,

$$\dot{y}(t) \quad \leq \quad y(t)\left[-d + \frac{\alpha K}{my(t)e^{-(\alpha-d-q_2E_2)\tau} + K} - q_2E_2\right]$$

By integrating,

$$y \leq \frac{(\alpha - d - q_2E_2)Ke^{(\alpha-d-q_2E_2)t}}{1 + (d + q_2E_2)me^{(\alpha-d-q_2E_2)(t-\tau)}}.$$

Therefore,

$$\limsup_{t\to+\infty} y(t) \leq \frac{(\alpha - d - q_2E_2)Ke^{(\alpha-d-q_2E_2)\tau}}{(d + q_2E_2)m}.$$

□

Definition 3.3. *The system is called permanent* [1] *if there exist δ and β, such that $0 < \delta < \beta$, independent of the initial conditions, such that for all solutions of this system,*

$$\begin{aligned} \min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} &\geq \delta, \\ \max\left\{\limsup_{t\to+\infty} x(t), \limsup_{t\to+\infty} y(t)\right\} &\leq \beta. \end{aligned}$$

Theorem 3.4. *If (i) $r - q_1E_1 > \frac{c}{m}$ and (ii) $\alpha > d + q_2E_2$, then system (2.1) is permanent.*

Proof. By Lemma (3.2), there is a $\beta = \max\{K, L\}$, where

$$L = \frac{(\alpha - d - q_2E_2)Ke^{(\alpha-d-q_2E_2)\tau}}{(d + q_2E_2)m},$$

independent of the initial conditions so that

$$\max\left\{\limsup_{t\to+\infty} x(t), \limsup_{t\to+\infty} y(t)\right\} \leq \beta.$$

We only need to show that there is a $\delta > 0$ independent of the initial conditions such that

$$\min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} \geq \delta.$$

From the prey population of the system (2.1), we have

$$\dot{x} = x\left(r - \frac{rx}{K} - \frac{c}{m + \frac{x}{y}} - q_1E_1\right).$$

From Lemma 3.1, $x(t) > 0, y(t) > 0$ then $-\frac{1}{m+\frac{x}{y}} > -\frac{1}{m}$.

Therefore, $\dot{x}(t) \geq x\left(r - \frac{r}{K}x - \frac{c}{m} - q_1E_1\right)$. By solving the diffential equation, then $x \geq (r - \frac{c}{m} - q_1E_1)\frac{K}{r}$. Since $r > \frac{c}{m} + q_1E_1$, taking the lim inf we get,

$$\liminf_{t\to+\infty} x(t) \geq \frac{K}{r}\left(r - \frac{c}{m} - q_1E_1\right) > 0. \tag{3.6}$$

Let's denote $\underline{x} \equiv \frac{K}{r}\left(r - \frac{c}{m} - q_1E_1\right)$.

Now, we consider $\liminf_{t\to+\infty} y(t)$. From (3.6) and Lemma 3.2, there exists a T such that for $t > T + \tau$, $x(t) > \underline{x}/2$, where $\underline{x} = \liminf_{t\to+\infty} x(t)$. Thus,

$$\dot{y}(t) \geq y(t)\left(-d + \frac{\alpha\underline{x}/2}{my(t-\tau) + \underline{x}/2} - q_2E_2\right) \tag{3.7}$$

Next, for a large t, $-y(t-\tau) \geq -y(t)e^{(d+q_2E_2)\tau}$. Then equation (3.7) becomes,

$$\dot{y}(t) \geq y(t)\left(\frac{-y(t)m(d + q_2E_2)e^{(d+q_2E_2)\tau} + (\alpha - d - q_2E_2)\underline{x}/2}{my(t)e^{(d+q_2E_2)\tau} + \underline{x}/2}\right)$$

Therefore,

$$\liminf_{t\to+\infty} y(t) \geq \frac{(\alpha - d - q_2E_2)\underline{x}e^{-(d+q_2E_2)\tau}}{2m(d + q_2E_2)} = \underline{y} > 0. \tag{3.8}$$

Since $\delta = \min(\underline{x}, \underline{y}) > 0$, then we have shown that the system (2.1) is permanent. □

As prey density is bounded by the carrying capacity K, to prevent the population overflow, the death rate d and harvesting rate q_2E_2 of predator, $y(t)$, should be less than the growth rate α. Growth rate of the predators depends on the population of prey that is bounded by a carrying capacity such that predator density depends on the carrying capacity K also. In the case that the density of prey and predator are very small, the growth rate should be bigger than the death rate and harvesting rate of both populations to keep the population from extinction.

Theorem 3.5. *If* $\frac{c}{m} > r - q_1E_1 + d + q_2E_2$*, then system* (2.1) *is not persistent.*

Proof. System (2.1) is said not to be persistent if

$$\min\left(\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right) = 0$$

for some positive solutions $x(t), y(t)$. Let $\frac{c}{m} > r - q_1E_1 + d + q_2E_2$ in system (2.1). Then there exists an $\varepsilon > 0$, such that $\frac{c}{m+\varepsilon} = r - q_1E_1 + d + q_2E_2$ or $r - \frac{c}{m+\varepsilon} - q_1E_1 = -(d + q_2E_2)$. We let $\delta = \frac{x(0)}{y(0)} < \varepsilon$ and claim that for all $t > 0$, $\frac{x(t)}{y(t)} < \varepsilon$. Then, $\lim_{t\to+\infty} x(t) = 0$. Otherwise, there is a time t_1, such that $\frac{x(t_1)}{y(t_1)} = \varepsilon$ and for $t \in [0, t_1)$, $\frac{x(t)}{y(t)} < \varepsilon$. Then, for $t \in [0, t_1]$, we have

$$\dot{x} \leq x\left(r - \frac{c}{m+\varepsilon} - q_1E_1\right) = -(d + q_2E_2)x.$$

Since $\dot{x} \leq -(d+q_2E_2)x(t)$, it implies that $x(t) \leq x(0)e^{-(d+q_2E_2)t}$. In a similar manner, for all $t \geq 0$, $\dot{y}(t) \geq -y(t)(d+q_2E_2)$ which implies that $y(t) \geq y(0)e^{-(d+q_2E_2)t}$. This shows that for $t \in [0, t_1]$

$$\frac{x(t)}{y(t)} \leq \frac{x(0)}{y(0)} = \delta < \varepsilon$$

a contradiction to the existence of t_1, proving the claim. Now, consider $x(t) \leq x(0)e^{-(d+q_2E_2)t}$ for all $t \geq 0$, which implies $\lim_{t\to+\infty} x(t) = 0$. Hence, the system (2.1) is not persistent. □

Theorem 3.6. *If (i)* $\frac{c}{m} > r - q_1E_1 + d + q_2E_2$*, (ii)* $\alpha < (d + q_2E_2)(1 + \frac{m}{\varepsilon})$*, where* $\varepsilon = \frac{c}{(r-q_1E_1+d+q_2E_2)} - m$*, then there exist positive solutions* $(x(t), y(t))$ *of system* (2.1) *such that* $\lim_{t\to+\infty}(x(t), y(t)) = (0, 0)$.

Proof. By condition (i) and Theorem 3.5 we have $\lim_{t\to+\infty} x(t) = 0$ and for $t \geq 0$, then $x(t)/y(t) \leq \varepsilon$, provided that $\delta = x(0)/y(0) < \varepsilon$. By condition (ii) implies that for $t \geq \tau$,

$$\dot{y}(t) \leq y(t)\left(-d + \frac{\alpha}{\frac{m}{\varepsilon} + 1} - q_2E_2\right).$$

Therefore, $\dot{y}(t) \leq -sy(t)$ where $s = (-d + \alpha/(\frac{m}{\varepsilon} + 1) - q_2E_2)$, which implies $y(t) \leq y(0)e^{-st}$ such that $\lim_{t\to+\infty} y(t) = 0$. We can therefore conclude that $\lim_{t\to+\infty}(x(t), y(t)) = (0, 0)$. □

4 Global Stability Analysis

Theorem 4.1. *If (i)* $\frac{c}{m} < r - q_1E_1$ *and (ii)* $\alpha < d + q_2E_2$*, then* $(\frac{r-q_1E_1}{b}, 0)$ *is globally asymptotically stable for system* (2.1).

Proof. If $\alpha < d + q_2E_2$, clearly $\lim_{t\to+\infty} y(t) = 0$, and $\liminf_{t\to+\infty} x(t) \geq \underline{x}$. By Theorem 3.4, $\underline{x} = \frac{K}{r}\left(r - \frac{c}{m} - q_1E_1\right)$. Then, for any $\epsilon \in (0, r)$, there exists $T = T(\epsilon)$, such that for $t > T$,

$$x(t)\left(r - \epsilon - bx(t) - q_1E_1\right) \leq \dot{x} \leq x(t)\left(r - bx(t) - q_1E_1\right)$$

so that $x(t) = \frac{r-q_1E_1}{e^{-(r-q_1E_1)t}+b}$.

Hence, $\lim_{t\to+\infty} x(t) = \frac{r-q_1E_1}{b}$, proving the theorem. □

In the following section, we will show global stability by using a Lyapunov function [7]. Thus, we start by considering the autonomous system of delay differential equations , from

$$\dot{X} = f(X_t), \quad X_t(\theta) = X(t+\theta) \tag{4.1}$$

where $f : \mathcal{C} \to \mathbb{R}$ is a continuous function. Let X^* be an equilibrium point of f. If $V : \mathcal{C} \to \mathbb{R}$ is a continuous function, we define the derivative of V relative of equation (4.1) as

$$\dot{V}(\phi) = \dot{V}_{4.1} = \lim_{h\to 0^+} \frac{1}{h}\left[V(X_h(\phi)) - V(\phi)\right].$$

Theorem 4.2. *Suppose* $V : \mathcal{C} \to \mathbb{R}$ *is continuous and there exist non-negative functions* $\mu_1(a)$ *and* $\mu_2(a)$ *such that* $\mu_1(a) \to \infty$ *as* $a \to \infty$

(i) $V(\phi) \geq \mu_1(|\phi(0)|)$ *and* *(ii)* $\dot{V}(\phi) \leq -\mu_2(|\phi(0)|)$.

Then, the equilibrium point X^* *of equation* (4.1) *is stable and every solution is bounded. If in addition* $\mu_2(a)$ *is positive definite, then* X^* *is globally asymptotically stable* [7].

Next, we give a result on the global asymptotic stability of the positive equilibrium. We rewrite the system (2.1), by letting $U(\zeta) = \frac{\zeta}{m+\zeta}, U^* = U(\frac{x^*}{y^*}), \overline{x} = x - x^*, \overline{y} = y - y^*$ and $\overline{U} = U - U^*$ where $m \in \mathbb{R}^+$ and we change the variables $(x, y) \to (x, u)$ where $u = x/y$ which is not singular in the interior of $\mathbb{R}^2_+$, therefore implying that if $(x, u) \to (x^*, u^*)$ then $(x, y) \to (x^*, y^*)$ then (2.1) becomes,

$$\begin{aligned} \dot{\bar{x}} &= (\bar{x} + x^*)\left(-b\bar{x} + c\left[\frac{U^*}{u^*} - \frac{U(u)}{u}\right]\right) \\ \dot{u} &= u\left(-b\bar{x} + c\left[\frac{U^*}{u^*} - \frac{U(u)}{u}\right] - \alpha\left[U(u(t-\tau)) - U^*\right]\right). \end{aligned} \tag{4.2}$$

In the following, we define the new variables as follows

$$v_1(t) = x - x^*, v_2(t) = u - u^*, v_2(t-\tau) = u(t-\tau) - u^*. \tag{4.3}$$

such that $v_1 \geq -x^*$, $v_2 \geq -u^*$ and the function:

$$f(v_2) = U(u) - U^* = \frac{mv_2}{(m+u)(m+u^*)} \quad (4.4)$$

$$f(v_2(t-\tau)) = U(u(t-\tau)) - U^* = \frac{mv_2(t-\tau)}{(m+u(t-\tau))(m+u^*)}. \quad (4.5)$$

Observe that:

$$f(v_2)v_2 \geq 0 \text{ and } f(v_2)v_2 = 0 \text{ iff } v_2 = 0$$

$$f'(v_2) = \frac{m}{(m+u)(m+u^*)}.$$

Since,

$$\frac{U^*}{u^*} - \frac{U(u)}{u} = \frac{v_2}{(m+u)(m+u^*)} = \frac{1}{m}f(v_2). \quad (4.6)$$

Thus, the system (4.2) becomes

$$\begin{aligned} \dot{v_1} &= (x^*+v_1)\left[-bv_1 + \frac{c}{m}f(v_2)\right] \quad (4.7)\\ \dot{v_2} &= (u^*+v_2)\left[-bv_1 + \frac{c}{m}f(v_2) - \alpha f(v_2(t-\tau))\right]. \end{aligned}$$

Consider $f(v_2(t-\tau)) = f(v_2(t)) - \int_{t-\tau}^{t} \dot{f}(v_2(\xi))d\xi$, with $\dot{f}(v_2(t)) = \frac{df(v_2(t))}{dt} = f'(v_2)\dot{v_2}(t)$, where $\dot{v_2}(t) = \frac{dv_2}{dt}$. Then,

$$f(v_2(t-\tau)) = f(v_2(t)) - \int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi. \quad (4.8)$$

Therefore, from (4.8) $\dot{v_2}$ of system (4.7) becomes

$$\dot{v_2} = (u^*+v_2)\left[-bv_1 - \left(\alpha - \frac{c}{m}\right)f(v_2) + \alpha\int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi\right]$$

such that, from (4.2), finally we get:

$$\begin{aligned} \dot{v_1} &= (x^*+v_1)\left[-bv_1 + \frac{c}{m}f(v_2)\right] \\ \dot{v_2} &= (u^*+v_2)\left[-bv_1 - \left(\alpha - \frac{c}{m}\right)f(v_2) + \alpha\int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi\right]. \end{aligned} \quad (4.9)$$

Changing the initial time by letting

$$v_t \equiv (v_1(t+\theta), v_2(t+\theta)), \qquad \theta \in [-2\tau, 0], \quad (4.10)$$

we have the following Lemma.

Lemma 4.3. *For the trivial solution of* (4.9)*, there exists the Lyapunov functional* $V_3 : \mathcal{C}([-2\tau, 0], \mathbb{R}) \to \mathbb{R}_+$

$$\begin{aligned} V_3(v_t) &= \frac{bc}{m}\left(v_1 - x^* \ln\left(\frac{x^*+v_1}{x^*}\right)\right) + \left(v_2 - u^* \ln\left(\frac{u^*+v_2}{u^*}\right)\right) \\ &+ \frac{\alpha m}{2}\int_{t-\tau}^{t}\int_{s}^{t}\left[bv_1^2(\xi) + \frac{c}{m}f^2(v_2(\xi)) + \alpha f^2(v_2(\xi-\tau))\right]d\xi ds \\ &+ \frac{\alpha^2 m\tau}{2}\int_{t-\tau}^{t} f^2(v_2(s))ds \end{aligned} \tag{4.11}$$

whose time derivative along the solutions of (4.9) is

$$\dot{V}_3(v_t) \le -\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right)mbv_1^2 - \frac{1}{m}\left[\left(\frac{\alpha m - c}{m}\right) - \alpha\tau\left(\frac{b}{2} + \alpha + \frac{c}{m}\right)\right]v_2^2. \tag{4.12}$$

Proof. Construct the Lyapunov function

$$V_1(v_t) = \omega\left(v_1 - x^* \ln\left(\frac{x^*+v_1}{x^*}\right)\right) + \left(v_2 - u^* \ln\left(\frac{u^*+v_2}{u^*}\right)\right) \tag{4.13}$$

where $\omega \in \mathbb{R}_+$ is an arbitrary constant to be chosen later.

Consider $V_1(v_t)$ where $v_t = 0$, we have

$$V_1(0) = \omega\left(0 - x^* \ln\left(\frac{x^*+0}{x^*}\right)\right) + \left(0 - u^* \ln\left(\frac{u^*+0}{u^*}\right)\right) = 0.$$

Next, consider the derivative of $\left(v_1 - x^* \ln\left(\frac{x^*+v_1}{x^*}\right)\right)$ and $\left(v_2 - u^* \ln\left(\frac{u^*+v_2}{u^*}\right)\right)$ with respect to v_t, we have $\left(1 - \frac{x^*}{x^*+v_1}\right)$ and $\left(1 - \frac{u^*}{u^*+v_2}\right)$, respectively, which are positive. Therefore, $V_1(v_t) > 0$ for $v_t > 0$.

The derivative of $V_1(v_t)$ along v_t is

$$\begin{aligned} \dot{V}_1(v_t) &= V_{v_1}\dot{v_1} + V_{v_2}\dot{v_2} \\ &= -b\omega v_1^2 + \frac{c\omega}{m}f(v_2)v_1 - bv_1v_2 - \left(\alpha - \frac{c}{m}\right)f(v_2)v_2 + \alpha v_2\int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi. \end{aligned}$$

By using the function (4.5), therefore,

$$\begin{aligned} \dot{V}_1(v_t) &= -\omega bv_1^2 + c\omega\frac{v_1v_2}{(m+u)(m+u^*)} - bv_1v_2 - \left(\alpha - \frac{c}{m}\right)f(v_2)v_2 \\ &+ \alpha v_2\int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi. \end{aligned}$$

Since $\frac{1}{(m+u)(m+u^*)} \le \frac{1}{m}$ where $m > 0$, we have

$$\dot{V}_1(v_t) \le -\omega bv_1^2 + \left(\frac{c\omega}{m} - b\right)v_1v_2 - \left(\alpha - \frac{c}{m}\right)f(v_2)v_2 + \alpha v_2\int_{t-\tau}^{t} f'(v_2)\dot{v_2}(\xi))d\xi.$$

Since $\dot{v_2} = u\left[-bv_1 + \frac{c}{m}f(v_2) - \alpha f(v_2(t-\tau))\right]$, then

$$\begin{aligned}\dot{V_1}(v_t) \quad &\le \quad -\omega b v_1^2 + \left(\frac{c\omega}{m} - b\right) v_1 v_2 - \left(\alpha - \frac{c}{m}\right) f(v_2) v_2 \\ &+\alpha \int_{t-\tau}^{t} f'(v_2)[-bv_2(t)u(\xi)v_1(\xi) + \frac{c}{m} v_2(t) u(\xi) f(v_2(\xi)) \\ &-\alpha v_2(t) u(\xi) f(v_2(\xi-\tau))] d\xi. \qquad (4.14)\end{aligned}$$

Consisder $v_2(t)u(\xi)f(v_2(\xi-\tau)) \le \frac{1}{2}\left(v_2^2 + u^2(\xi)f^2(v_2(\xi-\tau))\right)$.

Therefore, we get

$$\begin{aligned}\dot{V_1}(v_t) \quad &\le \quad -\omega b v_1^2 + \left(\frac{c\omega}{m} - b\right) v_1 v_2 - \left(\alpha - \frac{c}{m}\right) f(v_2) v_2 \\ &+\frac{1}{2}\alpha\left(b + \frac{c}{m} + \alpha\right) v_2^2(t) \int_{t-\tau}^{t} f'(v_2) d\xi \\ &+\frac{\alpha}{2}\int_{t-\tau}^{t} f'(v_2) u^2(\xi)\left[bv_1^2 + \frac{c}{m} f^2(v_2) + \alpha f^2(v_2(\xi-\tau))\right] d\xi (4.15)\end{aligned}$$

By choosing $\omega = \frac{bc}{m}$ and substituting in (4.15), we get

$$\begin{aligned}\dot{V_1}(v_t) \quad &\le \quad -\frac{b^2 m}{c} v_1^2 - \left(\alpha - \frac{c}{m}\right) f(v_2) v_2 + \frac{1}{2}\alpha\left(b + \frac{c}{m} + \alpha\right) v_2^2(t) \int_{t-\tau}^{t} f'(v_2) d\xi \\ &+\frac{\alpha}{2}\int_{t-\tau}^{t} f'(v_2) u^2(\xi)\left[bv_1^2 + \frac{c}{m} f^2(v_2) + \alpha f^2(v_2(\xi-\tau))\right] d\xi. \qquad (4.16)\end{aligned}$$

Observe that

$$f'(v_2) = \frac{m}{(m+u)(m+u^*)} < \frac{1}{m}, \qquad f'(v_2)u^2 = \frac{mu^2}{(m+u)(m+u^*)} < m. \quad (4.17)$$

We have

$$\begin{aligned}\dot{V_1}(v_t) \quad &\le \quad -\frac{b^2 m}{c} v_1^2 - \left(\alpha - \frac{c}{m}\right) f(v_2) v_2 + \frac{\alpha\tau}{2m}\left(b + \frac{c}{m} + \alpha\right) v_2^2(t) \\ &+\frac{\alpha m}{2}\int_{t-\tau}^{t}\left[bv_1^2 + \frac{c}{m} f^2(v_2) + \alpha f^2(v_2(\xi-\tau))\right] d\xi. \qquad (4.18)\end{aligned}$$

From the structure of (4.18), we construct a new function

$$\begin{aligned}V_2(v_t) \quad &= \quad V_1(v_t) \\ &+ \quad \frac{\alpha m}{2}\int_{t-\tau}^{t}\int_{s}^{t}\left[bv_1^2(\xi) + \frac{c}{m} f^2(v_2(\xi)) + \alpha f^2(v_2(\xi-\tau))\right] d\xi ds (4.19)\end{aligned}$$

Considering the case $v_t = 0$, we get $V_2(0) = V_1(0) + 0 = 0$. Since $V_1(v_t) > 0$ and $\frac{\alpha m}{2}\int_{t-\tau}^{t}\int_{s}^{t}\left[bv_1^2(\xi) + \frac{c}{m}f^2(v_2(\xi)) + \alpha f^2(v_2(\xi-\tau))\right] d\xi ds > 0$ for $v_t > 0$, we have $V_2(v_t) > 0$ for $v_t > 0$.

The derivative of $V_2(v_t)$ depends on v_1 and v_2 as

$$\begin{aligned}\dot{V}_2(v_t) &= \dot{V}_1(v_t) + \frac{\alpha m}{2}\int_{t-\tau}^{t}\left[bv_1^2(t) + \frac{c}{m}f^2(v_2(t)) + \alpha f^2(v_2(t-\tau))\right]ds \\ &\quad -\frac{\alpha m}{2}\int_{t-\tau}^{t}\left[bv_1^2(s) + \frac{c}{m}f^2(v_2(s)) + \alpha f^2(v_2(s-\tau))\right]ds \\ &\leq -\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right)mbv_1^2 - \left(\alpha - \frac{c}{m}\right)f(v_2)v_2 + \frac{\alpha\tau}{2m}\left(b + \frac{c}{m} + \alpha\right)v_2^2 \\ &\quad +\frac{\alpha c\tau}{2}f^2(v_2) + \frac{\alpha^2 m\tau}{2}f^2(v_2(t-\tau)). \end{aligned} \tag{4.20}$$

From the structure of (4.20), we construct the Lyapunov function again as

$$V_3(v_t) = V_2(v_t) + \frac{\alpha^2 m\tau}{2}\int_{t-\tau}^{t} f^2(v_2(s))ds. \tag{4.21}$$

Since $V_2(0) = 0$, $V_3(0) = 0$ and, since $V_2(v_t) > 0$, and $\frac{\alpha^2 m\tau}{2}\int_{t-\tau}^{t} f^2(v_2(s))ds > 0$ for $v_t > 0$, $V_3(v_t)$ is therefore positive definite.

The time derivative is given by

$$\begin{aligned}\dot{V}_3(v_t) &= \dot{V}_2(v_t) + \frac{\alpha^2 m\tau}{2}f^2(v_2(t)) - \frac{\alpha^2 m\tau}{2}f^2(v_2(t-\tau)) \\ &\leq -\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right)mbv_1^2 - \frac{1}{m}\left[\left(\frac{\alpha m - c}{m}\right) - \frac{\alpha\tau}{2}\left(b + 2\alpha + \frac{2c}{m}\right)\right]v_2^2\end{aligned}$$

which proves (4.12). Further, (4.13), (4.19) and (4.21) define the Lyapunov functional (4.11). □

Theorem 4.4. *If (i)* $\alpha > \frac{c}{m}$, *(ii)* $\tau < \tau^*$ *hold, where*

$$\tau^* = min\left\{\frac{2b}{\alpha c}, \frac{2(\alpha m - c)}{\alpha m(b + 2\alpha + \frac{2c}{m})}\right\} \tag{4.22}$$

then the (x^*, y^*) *of* (2.1) *is globally asymptotically stable in* $\mathbb{R}_+^2$.

Proof. To prove the global asymptotically stability of the positive equilibrium of (2.1) is equivalent to providing that of the trivial solution of (4.9). By Theorem 4.2, and Lemma 4.3, we have the Lyapunov functional $V_3(v_t)$. Let

$$\begin{aligned}\mu_1(v_t) = V_3(v_t) &= \frac{bc}{m}\left(v_1 - x^*\ln\left(\frac{x^* + v_1}{x^*}\right)\right) + \left(v_2 - u^*\ln\left(\frac{u^* + v_2}{u^*}\right)\right) \\ &\quad +\frac{\alpha m}{2}\int_{t-\tau}^{t}\int_{s}^{t}\left[bv_1^2(\xi) + \frac{c}{m}f^2(v_2(\xi)) + \alpha f^2(v_2(\xi-\tau))\right]d\xi ds \\ &\quad +\frac{\alpha^2 m\tau}{2}\int_{t-\tau}^{t} f^2(v_2(s))ds\end{aligned}$$

such that $V_3(v_t) \geq \mu_1(|v_t|)$, $\mu_1(\cdot)$ is a continuous positive definite function of v_t, $v_t \geq 0$, where $v_t = (v_1, v_2)$ such that $\mu_1(0) = 0$.

Consider the terms $\left(v_1 - x^* \ln\left(\frac{x^*+v_1}{x^*}\right)\right)$ and $\left(v_2 - u^* \ln\left(\frac{u^*+v_2}{u^*}\right)\right)$, we can see in Lemma 4.3 that $\mu_1(v_t) \to +\infty$ as $v_t \to +\infty$. Then, condition (i) of the Theorem 4.2 holds for any $(x,u) \in \mathbb{R}^2_+$. Furthermore, we can show that $\dot{V}_3(v_t)$ is negative definite for any $(x,u) \in \mathbb{R}^2_+$. Consider

$$\begin{aligned}
\dot{V}_3(v_t) &\leq -\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right) mbv_1^2 - \frac{1}{m}\left[\left(\frac{\alpha m - c}{m}\right) - \frac{\alpha\tau}{2}\left(b + 2\alpha + \frac{2c}{m}\right)\right] v_2^2 \\
&\leq -\left(\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right) mbv_1^2 + \frac{1}{m}\left[\left(\frac{\alpha m - c}{m}\right) - \frac{\alpha\tau}{2}\left(b + 2\alpha + \frac{2c}{m}\right)\right] v_2^2\right).
\end{aligned}$$

Let

$$\mu_2(v_t) = \left(\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right) mbv_1^2 + \frac{1}{m}\left[\left(\frac{\alpha m - c}{m}\right) - \frac{\alpha\tau}{2}\left(b + 2\alpha + \frac{2c}{m}\right)\right] v_2^2\right).$$

Since,

$$\left(\frac{b}{c} - \frac{\alpha\tau}{2}\right) > 0 \quad \text{and} \quad \left[\left(\frac{\alpha m - c}{m}\right) - \frac{\alpha\tau}{2}\left(b + 2\alpha + \frac{2c}{m}\right)\right] > 0$$

provided that $(\alpha m - c) > 0$, that is (i) $\alpha > \frac{c}{m}$ and (ii) $\tau < \frac{2b}{\alpha c}$ and $\tau < \frac{2(\alpha m - c)}{\alpha m(b + 2\alpha + \frac{2c}{m})}$ such that $\tau < \tau^*$, where τ^* is given by (4.22). Hence, $\dot{V}_3(v_t) \leq -\mu_2(|v_t|)$ where $\mu_2(\cdot)$ is a positive definite function of $v_t, v_t \geq 0$ such that $\lim_{v_t \to +\infty} \mu_2(v_t) = +\infty$. Therefore condition (ii) of Theorem 4.2 holds, which implies the globally asymptotically stable of the equilibrium of (2.1). □

5 Numerical Simulations

The numerical simulations are carried out by using DDE23 in Matlab in the following 3 cases.

Case I For the equilibrium point $(\bar{x}, 0)$, we choose the following parametric values : $r = 3$, $m = 1$, $K = 3$, $c = 2$, $d = 3$, $\alpha = 0.3$, $q_1 = 1$, $E_1 = 0.29$, $q_2 = 1$, $E_2 = 0.24$ and the initial conditions are $x(0) = 2$ and $y(0) = 3$. The conditions (i) and (ii) of Theorem 4.1 are satisfied. Then, the equilibrium point $(\bar{x}, 0)$ of systems (2.1) is globally asymptotically stable and are identical to (2.71,0) for any time delay $\tau \geq 0$. (See Fig. 1)

Case II For the following parametric values: $r = 3.05$, $m = 1$, $K = 3$, $c = 2.75$, $d = 0.3$, $\alpha = 3$, $q_1 = 1$, $E_1 = 0.29$, $q_2 = 1$, $E_2 = 0.24$ and the initial conditions $x(0) = 3$ and $y(0) = 2$, the conditions (i) and (ii) of the Theorem 4.4 are satisfied. Then, the positive equilibrium point (x^*, y^*) of system (2.1) where $\tau = 0.01$ is globally asymptotically stable and is equal to $(0.4967, 2.2628)$. (See Fig. 2)

Case III For the initial conditions $x(0) = 1$ and $y(0) = 2$ and the following parametric values : $r = 3.05$, $m = 1$, $K = 3$, $c = 2.75$, $d = 0.3$, $\alpha = 3$, $q_1 = 1$, $E_1 = 0.29$, $q_2 = 1$, $E_2 = 0.24$, the condition (i) of the Theorem 4.4 is satisfied but with $\tau = 10$ the condition (ii) of Theorem 4.4 is not true. Then, the persistence of a limit cycle is observe in our simulations. (See Fig. 3)

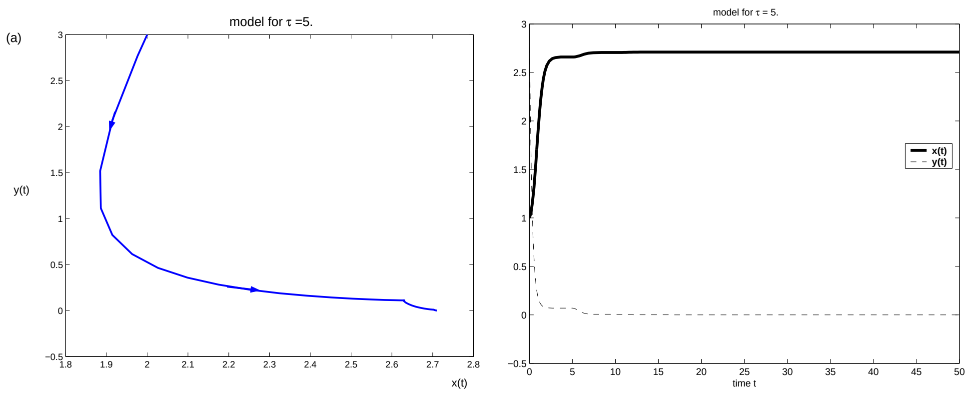

Figure 1: In case I, both populations converge to their values at equilibrium point $(\bar{x}, \bar{y}) = (2.71, 0)$. (a) Phase portrait in the $x - y$ plane. (b) Time series of solutions x, and y.

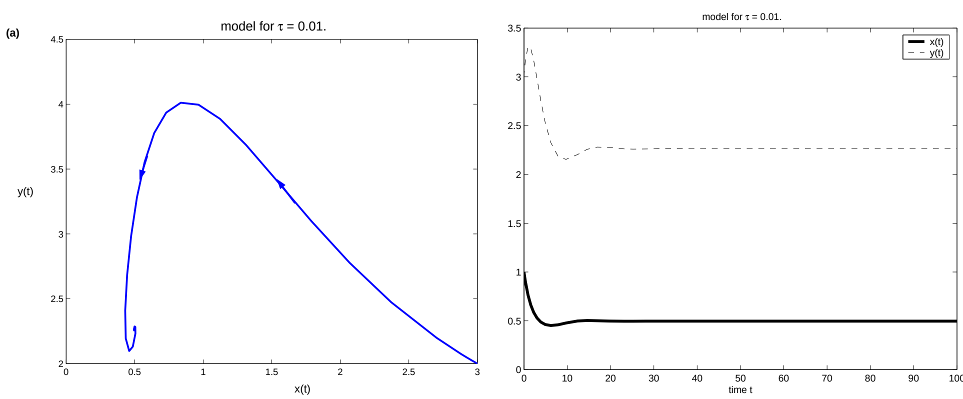

Figure 2: In case II, for $\tau = 0.01$, the populations converge to their equilibrium values $(x^*, y^*) = (0.4967, 2.2628)$. (a) Phase portrait in the $x - y$ plane. (b) Time series of solutions x, and y.

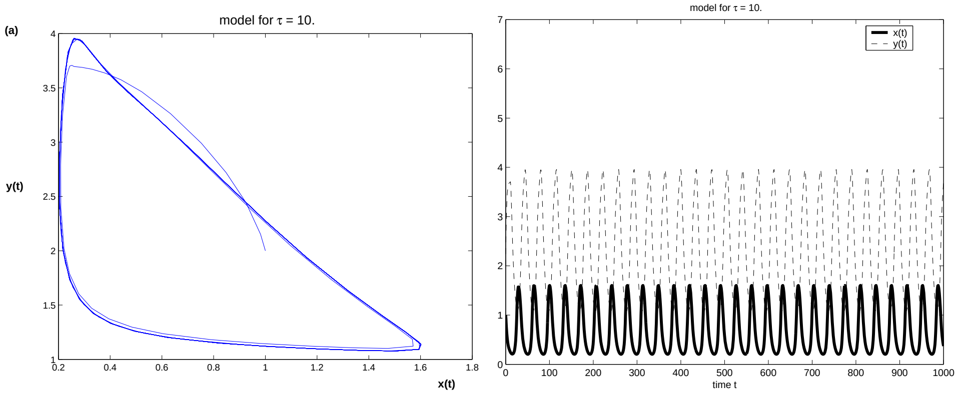

Figure 3: In case III, for $\tau = 10$, the system (2.1) has a solution that tends to a limit cycle. (a) Phase portrait in the $x - y$ plane. (b) Time series of solutions x, and y.

6 CONCLUSION

In this work, we have studied the stability of a predator - prey system with harvesting and time delay. It has two equilibrium points, which are $(\bar{x}, 0)$ where $\bar{x} = \frac{(r - q_1 E_1)}{b}$ and the positive equilibrium point (x^*, y^*), provided the conditions in Proposition 2.1 hold.

The equilibrium point $(\bar{x}, 0)$ is the globally asymptotically stable when $r - q_1 E_1 > \frac{c}{m}$ and $\alpha < d + q_2 E_2$. First condition means prey's intrinsic growth rate minus harvesting rate of prey is greater than the ratio of the capturing rate of prey and a haft capturing saturation constant. The other condition means the conversion rate when prey is consumed for conversion to predator density is less than the mix of death rate and the harvesting rate of predator. When both conditions hold, the extinction of the predator population will occur and the prey population converges to a constant value.

The positive equilibrium point (x^*, y^*) will exist if the conditions in the Proposition 2.1 hold. The first condition is that the growth rate of prey after harvesting is still more than the ratio of the capturing rate of prey and a haft capturing saturation constant. The second condition means the conversion rate is greater than the mixture of death rate and the harvesting rate.

We analyzed the stability of (x^*, y^*) by using the Lyapunov function. We consider both the model with delay and one without delays. We found that the time delay changes the system's stability behavior.

In Case II, we showed that when $\tau \neq 0$ the positive equilibrium point is globally asymptotically stable if $\alpha > \frac{c}{m}$ and $\tau < \tau^*$, which means the conversion rate must be greater than the ratio of the capturing rate of prey and a haft capturing saturation constant and the time delay τ has to be small enough. Since time delay represents on immature period or reaction time of predators, the small time delay means a short period is required for the immature predators to become adult predators so that it can start to prey faster. By choosing an appropriate time delay, both populations can persist.

In Case III, for the system with delay, if the condition (i) of the Theorem 4.4 is satisfied, that is $\alpha > \frac{c}{m}$, but the condition (ii) of Theorem 4.4 is not true, that is when $\tau > \tau^*$, which means the predators need a long time to become adults, then limit cycles occur. A limit cycle results in fluctuation in the animal populations and the resources. In nature, such limit cycle behavior has been observed, though rarely, so that our model can reflect real situations. A stable equilibrium point implies that populations and their consumable resources are locked into a fixed distribute.

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