Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 581–588

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Some Extensions of Univalence Conditions for a General Integral Operator

S. Bulut

Abstract : In [4], Breaz and Güney considered the subclasses $\mathcal{T}_2, \mathcal{T}_{2,\mu}$ and $\mathcal{S}(p)$ of analytic functions f in the open unit disk $\mathbb{U}$ and proved some univalent conditions for an integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ of f belonging to the classes $\mathcal{T}_2, \mathcal{T}_{2,\mu}$ and $\mathcal{S}(p)$. In this note, we consider the subclasses $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)\,(j=2,3,\ldots)$ and generalize the results of Breaz and Güney.

Keywords : Analytic functions; Univalent functions; Integral operator; Schwarz lemma.
2000 Mathematics Subject Classification : 30C45.

1 Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

Let $\mathcal{A}_j$ be the subclass of $\mathcal{A}$ consisting of functions f given by

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N}_1^* := \mathbb{N}\setminus\{0,1\} = \{2,3,\ldots\}). \tag{1.1}$$

Let $\mathcal{T}$ be the univalent subclass of $\mathcal{A}$ consisting of functions f which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

Let $\mathcal{T}_j$ be the subclass of $\mathcal{T}$ for which $f^{(k)}(0) = 0$ $(k = 2, 3, \ldots, j)$. Let $\mathcal{T}_{j,\mu}$ be the subclass of $\mathcal{T}_j$ consisting of functions f of the form (1.1) which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some μ $(0 < \mu \leq 1)$, and let us denote by $\mathcal{T}_{j,1} \equiv \mathcal{T}_j$ when $\mu = 1$.

For some real number p with $0 < p \leq 2$, we define the subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all functions f which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \qquad (z \in \mathbb{U}). \tag{1.3}$$

Singh [9] has shown that if $f \in \mathcal{S}(p)$, then f satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}). \tag{1.4}$$

Let $\mathcal{S}_j(p)$ be the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}_j$ which satisfy (1.3) and

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^j \qquad (z \in \mathbb{U},\ j \in \mathbb{N}_1^*), \tag{1.5}$$

and let us denote by $\mathcal{S}_2(p) \equiv \mathcal{S}(p)$.

The subclasses $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$ are introduced by Seenivasagan [7].

To discuss our problems, we have to recall here the following results.

General Schwarz Lemma. ([5]) *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}\,|z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if $f(z) = e^{i\theta}(M/R^m)\,z^m$, where θ is constant.

Theorem A. ([1, 2]) *Let c be a complex number, $|c| \leq 1$, $c \neq -1$. If $f(z) = z + a_2 z^2 + \cdots$ is a regular function in $\mathbb{U}$ and*

$$\left|c\,|z|^2 + (1 - |z|^2)\frac{z f''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathbb{U}$, then the function f is regular and univalent in $\mathbb{U}$.

Theorem B. ([6]) *Let β be a complex number, $\Re(\beta) > 0$, c a complex number, $|c| \le 1$, $c \ne -1$, and $h(z) = z + a_2 z^2 + \cdots$ a regular function in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \le 1$$

for all $z \in \mathbb{U}$, then the function

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} h'(t) dt \right\}^{\frac{1}{\beta}} = z + \cdots$$

is regular and univalent in $\mathbb{U}$.

In [8], the authors considered the integral operator

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}} \tag{1.6}$$

for $f_i \in \mathcal{A}_2$ $(i = 1, 2, \ldots, n)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$.

For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ becomes the integral operator $F_{\alpha,\beta}$ considered in [3].

The purpose of this paper is to generalize the main results of Breaz and Güney [4].

In the sequel, by $\mathbb{N}^*$ we denote the set of strictly positive integers.

2 Main Results

Theorem 2.1. *Let $M_i \ge 1$, let the functions $f_i \in \mathcal{S}_j(p_i)$ $(i = 1, 2, \ldots, n\,;\, n \in \mathbb{N}^*\,;\, j \in \mathbb{N}_1^*)$, let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \ge \sum_{i=1}^{n} \frac{(1+p_i)M_i+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If*

$$|c| \le 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1+p_i)M_i + 1}{|\alpha_i|},$$

$$|f_i(z)| \le M_i,$$

for all $z \in \mathbb{U}$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is in the class $\mathcal{S}$.*

Proof. Let us define the function h by

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt$$

for $f_i \in \mathcal{S}_j(p_i)$ $(i = 1, 2, \ldots, n\,;\, n \in \mathbb{N}^*\,;\, j \in \mathbb{N}_1^*)$. Since

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\frac{1}{\alpha_i}},$$

we see that $h(0) = 0$ and $h'(0) = 1$. Also, after some calculation, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right). \tag{2.1}$$

It follows from (2.1) that

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{(f_i(z))^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right). \tag{2.2}$$

Since $|f_i(z)| \leq M_i$ $(i = 1, 2, \ldots, n\,;\, z \in \mathbb{U})$, applying the general Schwarz lemma, we know that

$$|f_i(z)| \leq M_i\, |z| \quad (i = 1, 2, \ldots, n\,;\, z \in \mathbb{U}).$$

Using this inequality in (2.2), we find that

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{(f_i(z))^2} \right| M_i + 1 \right) \\ &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{(f_i(z))^2} - 1 \right| M_i + M_i + 1 \right) \\ &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(p_i M_i\, |z|^j + M_i + 1 \right) \\ &< \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|}. \end{aligned} \tag{2.3}$$

Thus, it follows from (2.3) that

$$\begin{aligned} \left| c\, |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| &< |c| + \frac{1}{|\beta|} \left| \frac{zh''(z)}{h'(z)} \right| \\ &\leq |c| + \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|} \leq 1 \end{aligned}$$

because $|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1+p_i)M_i+1}{|\alpha_i|}$. Finally, applying Theorem B for the function h, we prove that $F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta} \in \mathcal{S}$. □

Corollary 2.2. *Let* $M_i \geq 1$, *let the functions* $f_i \in \mathcal{S}_j(p_i)$ $(i = 1, 2, \ldots, n\,;\, n \in \mathbb{N}^*\,;\, j \in \mathbb{N}_1^*)$, *let* $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{(1+p_i)M_i+1}{|\alpha|}$, *and let* $c \in \mathbb{C}$. *If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in \mathbb{U}$, *then the function*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \tag{2.4}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. □

Corollary 2.3. *Let* $M \geq 1$, *let the functions* $f_i \in \mathcal{S}_j(p)$ $(i = 1, 2, \ldots, n\,; n \in \mathbb{N}^*\,; j \in \mathbb{N}_1^*)$, *let* $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{(1+p)M+1}{|\alpha_i|}$, *and let* $c \in \mathbb{C}$. *If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1+p)M+1}{|\alpha_i|},$$

$$|f_i(z)| \leq M,$$

for all $z \in \mathbb{U}$, *then* $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined in* (1.6) *is in the class* $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $p_1 = p_2 = \cdots = p_n = p$ and $M_1 = M_2 = \cdots = M_n = M$. □

Remark 2.4. *If we set* $j = 2$ *in Corollary* 2.3, *then we have Theorem* 2.1 *in [4].*

Corollary 2.5. *Let* $M \geq 1$, *let the functions* $f \in \mathcal{S}_j(p)$ $(j \in \mathbb{N}_1^*)$, *let* $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \frac{(1+p)M+1}{|\alpha|}$, *and let* $c \in \mathbb{C}$. *If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \frac{(1+p)\,M+1}{|\alpha|},$$

$$|f(z)| \leq M,$$

for all $z \in \mathbb{U}$, *then the function*

$$G_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \tag{2.5}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $n = 1$. □

Theorem 2.6. *Let* $M_i \geq 1$, *let the functions* $f_i \in \mathcal{T}_{j,\mu_i}$ $(i = 1, 2, \ldots, n\,; n \in \mathbb{N}^*\,; j \in \mathbb{N}_1^*)$, *let* $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{(1+\mu_i)M_i+1}{|\alpha_i|}$, *and let* $c \in \mathbb{C}$. *If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1+\mu_i)M_i+1}{|\alpha_i|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in \mathbb{U}$, *then* $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined in* (1.6) *is in the class* $\mathcal{S}$.

Proof. Defining the function h by

$$h(z)=\int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt,$$

we take the same steps as in the proof of Theorem 2.1. Then, we obtain that

$$\begin{aligned}\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| &< |c|+\frac{1}{|\beta|}\left|\frac{zh''(z)}{h'(z)}\right| \\ &\leq |c|+\frac{1}{\Re(\beta)}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}-1\right|M_i+M_i+1\right) \\ &\leq |c|+\frac{1}{\Re(\beta)}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\mu_i M_i+M_i+1\right) \\ &= |c|+\frac{1}{\Re(\beta)}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}((1+\mu_i)M_i+1)\end{aligned}$$

for $f_i\in\mathcal{T}_{j,\mu_i}$ $(i=1,2,\ldots,n\,;\,n\in\mathbb{N}^*\,;\,j\in\mathbb{N}_1^*)$. In view of Theorem B, we have $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}\in\mathcal{S}$. □

Corollary 2.7. *Let $M_i\geq 1$, let the functions $f_i\in\mathcal{T}_{j,\mu_i}$ $(i=1,2,\ldots,n\,;\,n\in\mathbb{N}^*\,;\,j\in\mathbb{N}_1^*)$, let $\alpha,\beta\in\mathbb{C}$, $\Re(\beta)\geq\sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha|}$, and let $c\in\mathbb{C}$. If*

$$|c|\leq 1-\frac{1}{\Re(\beta)}\sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha|},$$

$$|f_i(z)|\leq M_i,$$

for all $z\in\mathbb{U}$, then $F_{\alpha,\beta}$ defined in (2.4) *is in the class $\mathcal{S}$.*

Proof. In Theorem 2.6, we consider $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$. □

Corollary 2.8. *Let $M\geq 1$, let the functions $f_i\in\mathcal{T}_{j,\mu_i}$ $(i=1,2,\ldots,n\,;\,n\in\mathbb{N}^*\,;\,j\in\mathbb{N}_1^*)$, let $\alpha_i,\beta\in\mathbb{C}$, $\Re(\beta)\geq\sum_{i=1}^{n}\frac{(1+\mu_i)M+1}{|\alpha_i|}$, and let $c\in\mathbb{C}$. If*

$$|c|\leq 1-\frac{1}{\Re(\beta)}\sum_{i=1}^{n}\frac{(1+\mu_i)M+1}{|\alpha_i|},$$

$$|f_i(z)|\leq M,$$

for all $z\in\mathbb{U}$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is in the class $\mathcal{S}$.*

Proof. In Theorem 2.6, we consider $M_1=M_2=\cdots=M_n=M$. □

Remark 2.9. *If we set $j=2$ in Corollary* 2.8, *then we have Theorem* 2.6 *in [4].*

Corollary 2.10. *Let $M \geq 1$, let the functions $f \in \mathcal{T}_{j,\mu}$ $(j \in \mathbb{N}_1^*)$, let $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \frac{(1+\mu)M+1}{|\alpha|}$, and let $c \in \mathbb{C}$. If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \frac{(1+\mu)M+1}{|\alpha|},$$

$$|f(z)| \leq M,$$

for all $z \in \mathbb{U}$, then $G_{\alpha,\beta}$ defined in (2.5) is in the class $\mathcal{S}$.

Proof. In Theorem 2.6, we consider $n = 1$. □

Theorem 2.11. *Let $M_i \geq 1$, let the functions $f_i \in \mathcal{T}_j$ $(i = 1, 2, \ldots, n;\, n \in \mathbb{N}^*;\, j \in \mathbb{N}_1^*)$, let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M_i + 1}{|\alpha_i|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in \mathbb{U}$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) is in the class $\mathcal{S}$.

Proof. In Theorem 2.6, we consider $\mu_1 = \mu_2 = \cdots = \mu_n = 1$. □

Corollary 2.12. *Let $M_i \geq 1$, let the functions $f_i \in \mathcal{T}_j$ $(i = 1, 2, \ldots, n;\, n \in \mathbb{N}^*;\, j \in \mathbb{N}_1^*)$, let $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha|}$, and let $c \in \mathbb{C}$. If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M_i + 1}{|\alpha|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in \mathbb{U}$, then $F_{\alpha,\beta}$ defined in (2.4) is in the class $\mathcal{S}$.

Proof. In Theorem 2.11, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. □

Corollary 2.13. *Let $M \geq 1$, let the functions $f_i \in \mathcal{T}_j$ $(i = 1, 2, \ldots, n;\, n \in \mathbb{N}^*;\, j \in \mathbb{N}_1^*)$, let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M + 1}{|\alpha_i|},$$

$$|f_i(z)| \leq M,$$

for all $z \in \mathbb{U}$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) is in the class $\mathcal{S}$.

Proof. In Theorem 2.6, we consider $M_1 = M_2 = \cdots = M_n = M$. □

Corollary 2.14. *Let $M \geq 1$, let the functions $f \in \mathcal{T}_j$ $(j \in \mathbb{N}_1^*)$, let $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \frac{2M+1}{|\alpha|}$, and let $c \in \mathbb{C}$. If*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \frac{2M+1}{|\alpha|},$$

$$|f(z)| \leq M,$$

for all $z \in \mathbb{U}$, then $G_{\alpha,\beta}$ defined in (2.5) is in the class $\mathcal{S}$.

Proof. In Theorem 2.11, we consider $n = 1$. □

References

[1] L. V. Ahlfors, Sufficient conditions for quasiconformal extension, Annals of Math. Stud., 79 (1973), 23–29.

[2] J. Becker, Löwnersche differentialgleichung und schlichtheitskriterien, Math. Ann., 202 (4) (1973), 321–335.

[3] D. Breaz and N. Breaz, The univalent conditions for an integral operator on the classes $\mathcal{S}(p)$ and $\mathcal{T}_2$, J. Approx. Theory Appl., 1 (2005), 93–98.

[4] D. Breaz and H. Ö. Güney, On the univalence criterion of a general integral operator, J. Inequal. Appl. Volume 2008 (2008), Article ID 702715, 8 pages.

[5] Z. Nehari, Conformal Mapping, Dover, New York, NY, USA, 1975.

[6] V. Pescar, A new generalization of Ahlfors's and Becker's criterion of univalence, Bull. Malaysian Math. Soc., 19 (2) (1996), 53–54.

[7] N. Seenivasagan, Sufficient conditions for univalence, Appl. Math. E-Notes, 8 (2008), 30–35.

[8] N. Seenivasagan and D. Breaz, Certain sufficient conditions for univalence, Gen. Math., 15 (4) (2007), 7–15.

[9] V. Singh, On class of univalent functions, Internat. J. Math. Math. Sci., 23 (2000), 855–857.

(Received 12 January 2010)

Serap BULUT
Kocaeli University,
Civil Aviation College,
Arslanbey Campus,
41285 İzmit-Kocaeli, TURKEY
e-mail : serap.bulut@kocaeli.edu.tr