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On Self-graphoidal Graphs

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Abstract : A graph H is called a graphoidal graph if there exists a graph G and a graphoidal cover ψ of G such that $H \cong \Omega(G, \psi)$. A graph G is self-graphoidal if $G \cong \Omega(G, \psi)$ for some graphoidal cover ψ of G. In this paper we study self-graphoidal graph for different families of graphs.

Keywords : Graphoidal cover; Graphoidal covering number; Graphoidal graph; Self-graphoidal graph.
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1 Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only finite undirected graph with neither loops nor multiple edges. Also all graphs are connected and nontrivial. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by Acharya and Sampathkumar [1]. The reader may refer [3] for the terms not defined here.

Definition 1.1. *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$ or η.

Definition 1.2. *If $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_n\}$ be a family of distinct nonempty subsets of a set S whose union is S then the intersection graph of $\mathcal{F}$, denoted by $\Omega(\mathcal{F})$, is the graph whose vertex - and edge - sets are given by*

$$V_{\Omega(\mathcal{F})} = \{S_1, S_2, S_3, \ldots, S_n\}$$

$$\text{and } E_{\Omega(\mathcal{F})} = \{S_iS_j : i \neq j \text{ and } S_i \cap S_j \neq \emptyset\}.$$

For a graph G and $\psi \in \mathcal{G}_G$ (the set of all graphoidal covers of G), the intersection graph on ψ is denoted by $\Omega(G, \psi)$.

Definition 1.3. *A graph H is called a graphoidal graph if there exists a graph G and $\psi \in \mathcal{G}_G$ such that $H \cong \Omega(G, \psi)$.*

Let us denote $\Theta(G) = \{graph\ \ H : G \cong \Omega(H, \psi),\ \ for\ \ some\ \ \psi \in \mathcal{G}_H\}$. Then G is graphoidal iff $\Theta(G) \neq \phi$.

Definition 1.4. *A graphoidal graph G is called self-graphoidal if $G \in \Theta(G)$, i.e., $G \cong \Omega(G, \psi)$ for some $\psi \in \mathcal{G}_G$.*

The following problem has been proposed in [4]

Problem 1. *Which graphoidal graphs G satisfy $G \in \Theta(G)$?*

The present work is a partial solution to this problem.

2 Main Results

Theorem 2.1. *A graph G is self-graphoidal if the number of paths in a graphoidal cover of G is equal to the number of vertices in G but the converse is not necessarily true. (Example $K_{3,6}$)*

Theorem 2.2. (Harary [3, Theorem 8.2]) *A connected graph is isomorphic to its line graph iff it is a cycle.*

Corollary 2.3. *Every cycle is self-graphoidal.*

Theorem 2.4. *A complete graph K_p is self-graphoidal iff $3 \leq p \leq 5$.*

Proof. Let us assume that $p > 5$. Then we have to take a graphoidal cover ψ of K_p such that the number of paths in a graphoidal cover of K_p is equal to p. If $|\psi| = \eta$, where η is the minimum graphoidal cover of K_p, then we know from [2] that $\eta > p \Rightarrow |\psi| \geq \eta > p$. Hence K_p is not self-graphoidal if $p > 5$.
Next, if $p = 5$ then label its vertices as $\{v_0, v_1, v_2, v_3, v_4\}$ and construct the path

$$P_i = \{v_{(i+4)\ (\text{mod } 5)},\ v_i,\ v_{(i+3)\ (\text{mod } 5)}\} \text{ for } i = 0, 1, 2, 3, 4.$$

Then $\psi = \{P_0, P_1, P_2, P_3, P_4\}$ is a graphoidal cover of K_5 and $\Omega(K_5, \psi) \cong K_5$.
If $p = 4$ then label its vertices as $\{v_1, v_2, v_3, v_4\}$ and construct the path

$$P_1 = (v_1, v_2, v_4),\ P_2 = (v_1, v_3, v_2),\ P_3 = (v_3, v_4),\ P_4 = (v_1, v_4).$$

Then $\Omega(K_4, \psi) \cong K_4$.
If $p = 3$ then the result follows from §2.3. □

Theorem 2.5. *$K_{2,2}$ is the only complete bipartite graph which is self-graphoidal.*

Proof. Let us assume that for some m and n, $K_{m,n}$ is self-graphoidal, i.e. $K_{m,n} \cong \Omega(K_{m,n}, \psi)$, $|\psi|$ is the number of paths in a graphoidal cover of $K_{m,n}$.

Consider a bipartition of the vertex set of $K_{m,n}$ to be $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ such that $d(v_i) = n$ and $d(w_j) = m$ for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Since $K_{m,n}$ is self-graphoidal, $\Omega(K_{m,n}, \psi)$ is also a complete bipartite graph with the bipartition $X^* = \{P_1, P_2, \ldots, P_m\}$ and $Y^* = \{Q_1, Q_2, \ldots, Q_n\}$ such that each v_i corresponds to P_i and each w_j corresponds to Q_j for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Also, $d(P_i) = n$ and $d(Q_j) = m$. Then for any vertex $v_1 \in P_1, d_{P_1}(v_1) \leq 2$ and $v_1 \in Q_j$ for exactly one $j = 1, 2, 3, \ldots, n$, otherwise it will contradict the bipartition of X^* and Y^*. Thus $d_{\Omega(K_{m,n},\psi)}(v_1) \leq 2$ and hence $d(v_i) \leq 2$, $d(w_j) \leq 2$ in $\Omega(K_{m,n})$ for all i and j. This implies $K_{m,n}$ is self-graphoidal when $m, n \leq 2$, i.e., the only possible such $K_{m,n}$ are $K_{1,1}, K_{1,2}, K_{2,2}$. But $K_{1,1}$ and $K_{1,2}$ are trees and so by §2.1 they are not self-graphoidal. Also, by §2.3, $K_{2,2} = C_4$ which is self-graphoidal. □

Theorem 2.6. *Wheel W_p is self-graphoidal iff $p \leq 5$.*

Proof. If $p = 4$ then $W_4 = K_4$ so $\Omega(W_4, \psi) \cong W_4$. If $p = 5$ then label its vertices as $\{v_1, v_2, v_3, v_4, v_5\}$. Now, Construct the path $P_1 = (v_5, v_1, v_3, v_4)$, $P_2 = (v_2, v_5)$, $P_3 = (v_5, v_4)$, $P_4 = (v_1, v_4)$, $P_5 = (v_1, v_2, v_3)$. Hence $\Omega(W_5, \psi) \cong W_5$.

Suppose $p \geq 6$ then $\Delta \geq 5$. Take a vertex v in W_p such that $deg\ v = \Delta$. Then the number of paths in a graphoidal cover containing v is at least $deg\ v - 1$ and hence the intersection graph $\Omega(W_p, \psi)$ of graphoidal cover of W_p will form a complete graph $K_{deg\ v-1}$ which contradicts the property that no wheel contains a complete subgraph of $p \geq 4$. □

Theorem 2.7. *There exists a 3-regular self-graphoidal graph on $p \equiv 0(mod 4)$ vertices.*

Proof. Let us take a cycle C_p, where $p \equiv 0(mod 4)$, and label its vertices in cyclic order as $0, 1, 2, \ldots, p-1$. Now join the vertices $\{0, 2\}$, $\{1, 3\}$, $\{4, 6\}$, $\{5, 7\}$ and so on. Continuing in this process, we get a $3-$regular graph. Now, Construct the paths as

$$\psi = \{(i, i+2, i+1), (i+2, i+3, i+1), (i+3, i+4), (i+4, i+5)\}$$

where i is a multiple of 4 and $0 \leq i \leq p-1$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$. □

We refer to [3] for the definition of the square graph G^2, total graph $T(G)$, subdivision graph $S(G)$ and line graph $L(G)$ of a graph G.

Theorem 2.8. *There exists a 4-regular self-graphoidal graph.*

Proof. Let us take a cycle C_p and label its vertices in cyclic order as $0, 1, 2, \ldots, p-1$. Then $(C_p)^2$ is 4−regular graph. Now, Construct the paths as

$$P_i = \{(i+p-1) \pmod{p},\ i,\ (i+p-2) \pmod{p}\} \text{ for } i = 0, 1, 2, \ldots, p-1.$$

Then $\psi = \{P_0, P_1, P_2, \ldots, P_{p-1}\}$ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$. □

Remark 2.9. *Combining §2.3, §2.7 and §2.8, we observe that there exists a d-regular self-graphoidal graph if* $2 \leq d \leq 4$.

Corollary 2.10. $T(C_p)$ *is self-graphoidal for all* $p \geq 3$.

Proof. Let C_p be a cycle of length greater than 2. Then the total graph of C_p is 4−regular and hence from §2.8, we have $T(C_p)$ is self-graphoidal for all $p \geq 3$. □

Remark 2.11. *Since* $T(C_p) \cong S^2(C_p)$. *So,* $S^2(C_p)$ *is self-graphoidal for all* $p \geq 3$.

Corollary 2.12. $L(K_p)$ *is self-graphoidal iff* $p = 3, 4$.

Proof. Let $L(K_p)$ is self-graphoidal. Then $L(K_p)$ is $2(p-2)$−regular and so from §2.3, we have $2(p-2) = 2 \Rightarrow p = 3$. Again, from §2.8, we have $2(p-2) = 4 \Rightarrow p = 4$. Conversely, Consider the graph $L(K_3)$. then from §2.2, we get the desired result. Again, Consider the graph $L(K_4)$. We know that $T(K_p) \cong L(K_{p+1})$, i.e., $T(K_3) \cong L(K_{3+1})$. But $K_3 = C_3$. So, by §2.10, we have $T(K_3) = T(C_3)$ is self-graphoidal, i.e., $L(K_{3+1}) = L(K_4)$ is self-graphoidal. □

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