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Some Geometric Properties of Riesz-Musielak-Orlicz Sequence Spaces

V. A. Khan

Abstract : The aim of this paper is to calculate Maluta's coefficient for Riesz-Musielak-Orlicz sequence spaces with the orlicz norm. Furthermore show that Riesz-Musielak-Orlicz sequence space has the Schur's property.

Keywords : Maluta's coefficient; Schur's property; Asymptotic equidistant sequence; Reflexivity; Weak convergence; The δ_2-condition; Riesz-Musielak-Orlicz sequence space; Luxemburg norm.
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1 Introduction

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a modular if

(*i*) $\varrho(x) = 0$ if and only if $x = \theta$;

(*ii*) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(*iii*) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called convex if

(*iv*) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, the space

$$X_\varrho = \{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0\}$$

and

$$X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \ \text{ for some } \lambda > 0\}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, for $x \in X_\varrho$,

$$||x|| = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\}. \tag{1.1}$$

It is known that if ϱ is a convex modular on X, then $X_\varrho = X_\varrho^*$ and $||.||$ is a norm on X_ϱ under which it is a Banach space. The norm $||.||$ defined as in (1.1) is called the Luxemburg norm.

A map $\phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if ϕ vanishes only at 0, and ϕ is even and convex (see [10, 12]).

A sequence $M = (M_k)$ of Orlicz functions is called a Musielak - Orlicz function (see [3, 12]). In addition, a Musielak - Orlicz function $N = (N_k)$ is called a complementary function of a Musielak - Orlicz function M if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots$$

For a given Musielak - Orlicz function M, the Musielak - Orlicz sequence space l_M and its subspace h_M are defined as follows :

$$l_M := \{x \in l^0 : I_M(cx) < \infty \text{ for some } c > 0\},$$

$$h_M := \{x \in l^0 : I_M(cx) < \infty \text{ for all } c > 0\},$$

where I_M is a convex modular defined by

$$I_M = \sum_{k=1}^{\infty} M_k(x(k)), \quad x = (x(k)) \in l_M.$$

We consider l_M equipped with the Luxemburg norm

$$||x|| = \inf\left\{k > 0 : I_M\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz - Amemiya norm

$$||x||^0 = \inf\left\{\frac{1}{k}(1 + I_M(kx)) : k > 0\right\}.$$

A Musielak - Orlicz function $\mathcal{M}$ satisfies the δ_2 condition ($\mathcal{M} \in \delta_2$ for short) if there exist constants $K \geq 2, u_0 > 0$ and a sequence (c_k) of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$ and the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

holds for every $k \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $M_k(u) \leq u_0$.

If $\mathcal{M} \in \delta_2$ and $N \in \delta_2$, then we write $\mathcal{M} \in \delta_2 \cap \delta_2^*$. It is known that $l_{\mathcal{M}} = h_{\mathcal{M}}$ if and only if $\mathcal{M} \in \delta_2$ (see [12]).

The Riesz sequence space introduced in [1] is :

$$r^q(p) = \left\{x = (x_k) \in l^0 : \sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} q_j x(j)\right|^{p_k} < \infty\right\},$$

where l^0 is the space of all real sequences, l_∞ the space of all real bounded sequences $x = (x_k)$, and $(p_k) \in l_\infty$.

Let $M = (M_k)$ be a Musielak - Orlicz function and $q = (q_k)$ be a bounded sequence of real numbers. Vakeel A. Khan [8] defined Riesz-Musielak-Orlicz sequence space $r_p^q(M)$ as follows :

$$r_p^q(M) := \{x \in l^0 : \varrho_M(cx) < \infty \text{ for some } c > 0\},$$

where ϱ_M is a convex modular defined by

$$\varrho_M(x) = \sum_{k=1}^{\infty} M_k \left(\left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x(j) \right| \right),$$

and $Q_k = \sum_{i=1}^{k} q_i$. We consider $r_p^q(M)$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

under which it is a Banach space.

Vakeel A. Khan [8] defined the subspace $Sr_p^q(M)$ of $r_p^q(M)$ by

$$Sr_p^q(M) := \{x \in l^0 : \varrho_M(cx) < \infty \text{ for all } c > 0\}.$$

Let X be a reflexive infinite dimensional Banach space (which automatically does not have the schur property) and let $S(X)$ denote its unit sphere. For a sequence $(x_n) \subset X$, Y. Cui, H. Hudzik and H. Zhu [4] defined:

$$A(x_n) = \lim_{n\to\infty} \sup\{||x_i - x_j|| : i, j \geq n, i \neq j\},$$

$$A_1(x_n) = \lim_{n\to\infty} \inf\{||x_i - x_j|| : i, j \geq n, i \neq j\}.$$

The weak uniform normal structure coefficient of X is defined by (see [2])

$$WSC(X) = \sup\{k > 0 : \text{for each weakly convergent sequence } (x_n) \in S(X),$$

$$\text{some } y \in conv(x_n) \text{ such that } k \lim_n \sup ||x_n - y|| \leq A((x_n))\}.$$

A sequence $(x_n) \in X$ is said to be an asymptotic equidistant sequence if $A((x_n)) = A_1((x_n))$. This definition was introduced in [13], where it was proved that

$$WSC(X) = \inf\{A((x_n)) : (x_n) \text{ is an asymptotic equidistant sequence in } S(X)$$

$$\text{and } x_n \to 0 \text{ weakly}\}.$$

Maluta's coefficient is connected with normal structure, which is very important property of Banach spaces that gurantees the fixed point property for them (see [5, 6, 7]). Maluta's coefficient $\text{Ł}(X)$ of a Banach space X is defined by (see [11])

$$\text{Ł}(X) = \sup \left\{ \frac{\limsup_n d(x_{n+1}, conv(x_j)_{j=1}^{\infty})}{A((x_n))} : \begin{array}{ll} (x_n) & \text{is a bounded nonconstant} \\ & \text{sequence in } X \end{array} \right\}.$$

We have

$$\text{Ł}(X) = \frac{1}{WCS(X)}, \quad \text{for each reflexive Banach space } \; X$$

and

$$\text{Ł}(X) = 1 \quad \text{for each nonreflexive Banach space } X.$$

For every $m, n \in \mathbb{N}$, $k > 1$, Y. Cui, H. Hudzik and H. Zhu [4] defined :

$$c(k, m, n) = \inf \left\{ c_{k,x} > 0 : I_M \left(\frac{kx}{c_{k,x}} \right) = \frac{k-1}{2} \; \text{ and } \; x = \sum_{i=m}^{m+n} x(i) e_i \in S(r_p^q(M)) \right\},$$

and

$$d(M) = \inf \{ d_k : k > 1 \}.$$

2 Main Results

Theorem 2.1. *Suppose that $M = (M_i)$ is a Musielak - Orlicz function such that all M_i $(i = 1, 2, \ldots)$ are finitely valued and*

$$\frac{M_i(u)}{u} \to +\infty \quad as \;\; u \to +\infty \;\; \forall \;\; i \in \mathbb{N}.$$

Then

(a) *When $r_p^q(M)$ is nonreflexive, then $\text{Ł}(X) = 1$;*

(b) *When $r_p^q(M)$ is reflexive, then $\text{Ł}(X) = \frac{1}{d(M)}$.*

Proof. (a) Follows immediately from the fact that $\text{Ł}(X) = 1$ for every nonreflexive Banach space X. Now we need to show that $WCS(r_p^q(M)) = d(M)$ whenever $r_p^q(M)$ is reflexive. It is well known that the reflexivity of $r_p^q(M)$ is equivalent to the fact that both M and N satisfy the δ_2 - condition.

First of all show that $WCS(r_p^q(M)) \leq d(M)$. For each $\epsilon > 0$, by the definition of $d(M)$, there is $k > 1$ such that $d(M) > d_k - \epsilon$. We know that $d_k \geq d(k, m)$ $\forall$ $k > 1$ and $m \in \mathbb{N}$. By the definition of $d(k, m)$ there is $n(m) \in \mathbb{N}$ such that

$$d(k, m) > c(k, m, n) - \epsilon, \quad \text{whenever} \;\; n > n(m).$$

Finally, by the definition of $c(k, m, n)$ there exists $x_{m,n} \in S(r_p^q(M))$ such that

(i) $c_{k,x_{m,n}} - \epsilon < c(k,m,n)$, $x_{m,n} = \sum_{i=m}^{m+n} x_{m,n}(i)e_i \in S(r_p^q(M))$,

(ii) $I_M\left(\frac{kx_{m,n}}{c_{k,x_{m,n}}}\right) = \frac{k-1}{2}$.

Take $m_1 = 1$. Then there exists $n-1 \in \mathbb{N}$, $n_1 > n(m_1)$ and x_{m_1,n_1} satisfying (i) and (ii) with m_1 and n_1 in place of m and n. Take $m_2 = m_1 + n_1 + 1$. There exists x_{m_2,n_2} satisfying (i) and (ii) with m_2, n_2, $n_2 > n(m_2)$, in place of m, n. By induction, we can construct a sequence $(x_{m_i,n_i})_{i=1}^{\infty}$ in $S(r_p^q(M))$ with pairwise disjoint supports and satisfying (i) and (ii) with m_i and n_i, $n_i > nm_i)$ in place of m and n for $i = 1, 2, \ldots$

Define $y_k = x_{m_k,n_k}$. Then we have $y_n \in S(r_p^q(M))$ for every $n \in \mathbb{N}$. Moreover, $y_n \to 0$ weakly and for every $j, k \in \mathbb{N}$,

$$\begin{aligned}
\left\|\frac{y_j - y_k}{d(M)+2\epsilon}\right\| &\leq \frac{1}{k}\left(1 + I_M\left(k\frac{y_v - y_l}{d(M)+2\epsilon}\right)\right) \\
&= \frac{1}{k}\left(1 + I_M\left(k\frac{y_v}{d(M)+2\epsilon} + \frac{y_l}{d(M)+2\epsilon}\right)\right) \\
&\leq \frac{1}{k}\left(1 + I_M\left(\frac{kx_{m_v,n_v}}{c_{k,x_{m_v,n_v}}}\right) + I_M\left(\frac{kx_{m_l,n_l}}{c_{k,x_{m_l,n_l}}}\right)\right) \\
&= \frac{1}{k}\left(1 + \frac{k-1}{2} + \frac{k-1}{2}\right) = 1.
\end{aligned}$$

$A((y_n)) \leq d(M) + 3\epsilon$. Since ϵ is arbitrary, we have $A((y_n)) \leq d(M)$. We know that for each weakly convergent sequence on the unit sphere of a Banach space X there exists an asymptotic equidistant subsequence (see Proposition 2 in [13]). Thus $WCS(r_p^q(M)) \leq d(M)$. Next, prove that $WCS(r_p^q(M)) \geq d(M)$. First of all we will show the equality

$$\begin{aligned}
WCS(r_p^q(M)) &= \inf\left\{A((x_n)) : \begin{array}{l} x_n = \sum_{i=l_{n-1}+1}^{l_n} x_{n(i)}e_i \text{ and } (x_n) \text{ is an} \\ \text{asymptotic equidistant sequence in } S(r_p^q(M)) \end{array}\right\}. \\
&= d.
\end{aligned}$$

It is obvious that $WCS(r_p^q(M)) \leq d$, so we need to show that $WCS(r_p^q(M)) \geq d$. For any $\epsilon > 0$, by the definition of $WCS(r_p^q(M))$, there exists a sequence $(x_n) \in S(r_p^q(M))$ being an asymptotic equidistant sequence, weakly convergent to 0 and such that

$$A((x_n)) < WCS(r_p^q(M)) + \epsilon.$$

Take $v_1 = x_1$. Then there exists $i_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=i_1+1}^{\infty} v_1(i)e_i\right\| < \epsilon.$$

A number l_1 exists since by the reflexivity of $r_p^q(M)$, the generating function $M = (M_i)$ satisfies the δ_2 - condition. By $x_n(i) \to 0$ as $n \to \infty$, $(i = 1, 2, \dots, l_1)$ there is $n_0 \in I\!\!N$ such that

$$\left\|\sum_{i=1}^{l_1} x_n(i)e_i\right\| < \epsilon, \quad \text{whenever } n > n_0.$$

Fix $N_1 > n_0$ and set $v_2 = x_{x_{N_1}}$. Then

$$\left\|\sum_{i=1}^{l_1} v_2(i)e_i\right\| < \epsilon.$$

Take $l_2 > l_1$ such that $||\sum_{i=i_2+1}^{\infty} v_2(i)e_i|| < \epsilon$. By $x_n(i) \to 0$ as $n \to \infty$, for $i = 1, 2, \dots,$ we can find $N_2 > N_1$ such that

$$\left\|\sum_{i=1}^{l_2} x_n(i)e_i\right\| < \epsilon \ \text{ whenever } n > N_2.$$

Let choose $N_3 > N_2$ and set $v_3 = xN_3$. Then

$$\left\|\sum_{i=1}^{l_2} v_3(i)e_i\right\| < \epsilon.$$

Take $l_3 > l_2$ such that

$$\left\|\sum_{i=i_3+1}^{\infty} v_3(i)e_i\right\| < \epsilon.$$

In thus way we can construct by induction a sequence (l_n) of natural numbers with $l_1 < l_2 < \cdots$ and a subsequence (v_n) of (x_n) satisfying $A((v_n)) = A((x_n))$ and

$$\left\|\sum_{i=1}^{l_{n-1}} v_n(i)e_i\right\| < \epsilon,$$

$$\left\|\sum_{i=l_n+1}^{\infty} v_n(i)e_i\right\| < \epsilon,$$

where $l_0 = 0$ by definition.

Let us take

$$u_n = \frac{\sum\limits_{i=l_{n-1}+1}^{l_n} v_n(i)e_i}{\left\|\sum\limits_{i=l_{n-1}+1}^{l_n} v_n(i)e_i\right\|} \quad (n = 1, 2, \dots).$$

Then $u_n \in S(r_p^q(M))$ for each $n \in \mathbb{N}$. Moreover, for every $m, n \in \mathbb{N}$, $n < m$, we have

$$\begin{aligned} ||v_n - v_m|| &= \Bigg\| \sum_{i=1}^{l_{n-1}} (v_n(i) - v_m(i))e_i + \sum_{i=l_{n-1}+1}^{l_n} (v_n(i) - v_m(i))e_i \\ &\quad + \sum_{i=l_n+1}^{l_{m-1}} (v_n(i) - v_m(i))e_i + \sum_{i=l_{m-1}+1}^{l_m} (v_n(i) - v_m(i))e_i \\ &\quad + \sum_{i=l_m+1}^{\infty} (v_n(i) - v_m(i))e_i \Bigg\| \\ &\geq \left\| \sum_{i=l_{n-1}+1}^{l_n} v_n(i)e_i - \sum_{i=l_{m-1}+1}^{l_m} v_m(i)e_i \right\| - 4\epsilon \\ &\geq \|(u_n - u_m)(1 - 2\epsilon)\| - 4\epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} A((u_n)) &\leq \frac{A((u_n))}{1 - 2\epsilon} + \frac{4\epsilon}{1 - 2\epsilon} \\ &= \frac{A((x_n)) + 4\epsilon}{1 - 2\epsilon} \\ &\leq \frac{WSC(r_p^q(M)) + 5\epsilon}{1 - 2\epsilon}. \end{aligned}$$

Since ϵ is arbitrary, we have

$$d \leq WSC(r_p^q(M)).$$

Finally show that $d \geq d(M)$. For any equidistant sequence

$$x_n = \sum_{i=l_{n-1}+1}^{l_n} x_n(i)e_i \in S(r_p^q(M)) \quad (n = 1, 2, \ldots),$$

there $k_{m,n} > 0$ such that

$$\left\| \frac{(x_m - x_n)}{d(M)} \right\| = \frac{1}{k_{m,n}} \left(1 + I_M \left(k_{m,n} \frac{x_m - x_n}{d(M)} \right) \right)$$

for all $m, n \in \mathbb{N}$, $m \neq n$.

Suppose that $m, n \in \mathbb{N}$, $m \neq n$. We will consider two cases.

(i) If $k_{m,n} \leq 1$, then $||x_m - x_n|| \geq d(M)$.

(ii) If $k_{m,n} > 1$, then

$$\begin{aligned} \|\frac{(x_m - x_n)}{d(M)}\| &= \frac{1}{k_{m,n}}\left(1 + I_M\left(\frac{k_{m,n}, x_m}{d(M)}\right) + I_M\left(\frac{k_{m,n}, x_n}{d(M)}\right)\right) \\ &\geq \frac{1}{k_{m,n}}\left(1 + \frac{k_{m,n}-1}{2} + \frac{k_{m,n}-1}{2}\right) \\ &= 1, \end{aligned}$$

hence we get $\|x_m - x_n\| \geq d(M)$. Consequently $A((x_n)) \geq d(M)$. Since (x_n) is an asymptotic equidistant sequence in $S(r_p^q(M))$. Therefore $WSC(r_p^q(M)) \geq d(M)$. □

Theorem 2.2. *In Riesz - Musielak - Orlicz Sequence Space, if the Musielak - Orlicz function $M = (M_k)$ then $d(M) = 2^{1/p}$.*

Proof. It is clear that $r_p^q(M) = l^p$. Moreover,

$$\|x\|p^{1/p}q^{1/q}\|x\|_p \text{ for any } x \in r_p^q(M), \text{ where } 1/p + 1/q = 1$$

and

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \quad (see[9]).$$

Take arbitrary $k > 1$ and $x \in S(r_p^q(M))$ with finite support. It is easy to see that the number $c = c(k, x) > 0$ satisfying the equality $I_M(\frac{kx}{c}) = \frac{k-1}{2}$ is equal to $2^{1/p}k(k-1)^{-1/p}p^{-1/p}q^{-1/q}$. Therefore,

$$d(M) = \inf\left\{2^{1/p}k(k-1)^{-1/p}p^{-1/p}q^{-1/q} : k > 1\right\}.$$

To find this infimum it is sufficient to calculate

$$\inf\{k(k-1)^{-1/p} : k > 1\}.$$

This infimum is attained at $k_0 = q$. Since $k_0 - 1 = q/p$, we get $d(M) = 2^{1/p}$. □

Theorem 2.3. *If $\sum_{i=1}^{\infty} N_i(a_i) \leq 1$ then $r_1^q(M)$ has the Schur Schur property, i.e. every weakly convergent sequence is norm convergent in $r_1^q(M)$.*

Proof. Let $x_n = (x_n(i)) \in S(r_p^q(M))$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ weakly. By $\sum_{i=1}^{\infty} N_i(a_i) \leq 1$ we have

$$\|x_n\| = \sum_{i=1}^{\infty} a_i|x_n(i)| \quad (n = 1, 2, \ldots)$$

Define

$$z_n = (a_1x_n(1), a_2x_n(2), \ldots)$$

and

$$z_0 = (a_1 x_0(1), a_2 x_0(2), \ldots).$$

Then $z_n \in l^1$ for $n = 0, 1, 2, \ldots$ and $z_n \to z_0$ weakly in l^1 (since the weak convergence in $r_p^q(M)$ implies the weak convergence in $l^1((a_i))$). Since l^1 has the schur property, we get $||z_n - z_0||_{l^1} \to 0$. Hence in view of the equality

$$||x_n - x_0|| = \sum_{i=1}^{\infty} a_i |x_n(i) - x_0(i)| = ||z_n - z_0||_{l^1},$$

we get $\lim_n ||x_n - x_0|| = 0$, i.e. $r_1^q(M)$ has the schur property. □

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Vakeel A. Khan
Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002, INDIA.
e-mail : `vakhan@math.com`