Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 555–563
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A Note on Continuous Exponential Families

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Abstract : In this paper, we produce characterization of continuous exponential families through a representation for a survival function in terms of hazard measure and covariance identities. These results subsume previous results given by Hudson [7] and Prakasa Rao [14] among others.

Keywords : Characterization; Exponential families; Hazard measure; Covariance identity; Chernoff-type inequalities.
2000 Mathematics Subject Classification : 60E05; 62C15.

1 Introduction

There is an extensive literature in the theory of exponential families is that discussed in many books and monographs such as Barndorff-Nielsen [2], Brown [3] and Letac [9] and many papers are published during earlier years.

Hudson [7] found a natural identity for an exponential family in the discrete and continuous cases, later Prakasa Rao [14] characterized the exponential family through some identities and Chou [6] obtained an identity for multi-dimensional continuous exponential families. Papathanasiou [13], Prakasa Rao and Sreehari [15] and Srivastava and Sreehari [16] characterized results related to this matters.

In this paper, we produce characterization of the continuous exponential families through version of the hazard measure.

2 Characterizations via Hazard Measure

Let F be a distribution function on $\Re$. Then, the measure m_F on the Borel σ-field of $\Re$ such that $m_F(B) = \int_B \frac{dF(x)}{1-F(x-)}$ for every set B is referred to as the *hazard measure* relative to F. Kotz and Shanbhag [8] introduced this measure and established a representation for a distribution function in terms of the corresponding hazard measure as follows :

Theorem 2.1. (Kotz and Shanbhag [8]) *Let m_F be as defined above and m_F^c be the continuous (non-atomic) part of m_F and let $H_c(x) = m_F^c\big((-\infty, x]\big)$. Then the survivor function $S(x) = 1 - F(x-)$ is given by*

$$S(x) = \left[\prod_{x_r \in D_x} (1 - m_F(\{x_r\}))\right] e^{-H_c(x)},\ x < b, \tag{2.1}$$

where D_x is the set of all points $y \in (-\infty, x)$ such that $m_F(\{x\}) > 0$ and b denotes the right extremity of F.

Corollary 2.2. *In Theorem 2.1, if m_F is a non-atomic (continuous) measure, then, we have $1 - F(x) = \exp\{-H(x)\}$ for all x, in place of (2.1) such that $S(x) = 1 - F(x)$ and $H(x) = m_F\big((-\infty, x]\big)$.*

Remark 2.3. Theorem 2.1 implies that the hazard measure m_F relative to F uniquely determines the df F.

Following Alharbi and Shanbhag [1] and Mohtashami Borzadaran and Shanbhag [11, 12], we extend and unify the existing literature on exponential families. In particular, we arrive at here, new representations via the hazard measure and characterizations of families of distributions that are continuous without necessarily being absolutely continuous (w.r.t. Lebesgue measure). Analogous results corresponding to the discrete case are also addressed in Mohtashami Borzadaran [10]. Before we discuss about the generalized continuous exponential families, we mention two lemmas and a theorem, giving certain representations for distributions in terms of distributions that are mixture of continuous and discrete. We can see the proof of the discrete version them in Mohtashami Borzadaran [10].

Lemma 2.4. *With $w(.) > 0$ for almost all $(a.a.)[\nu_{F^*}]x \in \Re$, F is a df that absolutely continuous w.r.t. ν_{F^*} and $t(X) \geq \theta$ almost surely $(a.s.)$ (with $t(X) > \theta$ almost surely $(a.s.)$ for $X \geq x_0$ for some x_0 with $P\{X \geq x_0\} > 0$) where X is an r.v. with df F, we have*

$$\left(\int_{[x,\infty)} (t(y) - \theta) dF(y)\right) d\nu_{F^*}(x) = w(x) dF(x),\ x \in \Re, \tag{2.2}$$

if and only if, for some $c \in (0, \infty)$,

$$dF(x) = c\left\{\frac{1}{w(x)}\left\{\prod_{x_r \in D_x^{(1)}} \Big(1 - m^{(1)}\left(\{x_r\}\right)\Big)\right\} e^{-H_c^{(1)}(x)}\right\} d\nu_{F^*}(x),\ x \in \Re, \tag{2.3}$$

where

$$m^{(1)}(\bullet) = \int_{[x_0,\infty)\bigcap\bullet} (t(y) - \theta)(w(y))^{-1} d\nu_{F^*}(y), \quad H_c^{(1)}(x) = m_c^{(1)}((-\infty, x]),$$

and $m_c^{(1)}$ *is the continuous part of* $m^{(1)}$ *and* $D_x^{(1)}$ *is the set of all points* $y \in (-\infty, x)$ *such that* $m^{(1)}\big(\{y\}\big) > 0$. *(Here* F^* *be a non-constant Lebesgue-Stieltjes measure function on* $\Re$ *and* ν_{F^*} *be the measure determined by it.)*

Lemma 2.5. *With* $0 < w^*(x) = w(x) + (\theta - t(x))\nu_{F^*}\big(\{x\}\big)$ *for almost all* (a.a.)$[\nu_{F^*}]x \in \Re$, F *is a df that absolutely continuous w.r.t.* ν_{F^*} *and* $t(X) < \theta$ *almost surely* (a.s.) *(with* $t(X) < \theta$ *almost surely* (a.s.) *for* $X < x_0$ *for some* x_0 *with* $P\{X < x_0\} > 0$*) where* X *is an r.v. with df* F, *we have*

$$\left(\int_{(-\infty,x)} (\theta - t(y))dF(y)\right) d\nu_{F^*}(x) = w(x)dF(x),\ \ x \in \Re, \tag{2.4}$$

if and only if, for some $c \in (0, \infty)$,

$$dF(x) = c\left\{\frac{1}{w^*(x)}\left\{\prod_{x_r \in D_x^{(2)}}\left(1 - m^{(2)}\left(\{x_r\}\right)\right)\right\}e^{-H_c^{(2)}(x)}\right\}d\nu_{F^*}(x),\ \ x \in \Re, \tag{2.5}$$

where

$$m^{(2)}(\bullet) = \int_{(-\infty,x_0)\bigcap\bullet} (\theta - t(y))(w^*(y))^{-1}d\nu_{F^*}(y), \quad H_c^{(2)}(x) = m_c^{(2)}([x,\infty)),$$

and $m_c^{(2)}$ *be the continuous part of* $m^{(2)}$ *and* $D_x^{(2)}$ *is the set of all points* $y \in (x, \infty)$ *such that* $m^{(2)}\big(\{y\}\big) > 0$.

Theorem 2.6. *With* $w(x) > 0$ *for a.a.*$[\nu_{F^*}]x \in \Re$, F *is a df that absolutely continuous w.r.t.* ν_{F^*} *and there exists a point* x_0 *such that* $t(x) > \theta$ *for a.a.*$[\nu_{F^*}]x \in \Re$, *lying in* (x_0, ∞) *and* $t(x) < \theta$ *for a.a.*$[\nu_{F^*}]x \in \Re$, *lying in* $(-\infty, x_0)$ *with* $E(t(X)) = \theta$ *and* $t(x_0) \geq \theta$ *where* X *is an r.v. with df* F. *Then* F *satisfies (2.2), if and only if for some* $c \in (0, \infty)$,

$$dF(x) = \begin{cases} c\{\frac{1}{w(x)}\{\prod_{x_r \in D_x^{(1)}}\left(1 - m^{(1)}\big(\{x_r\}\big)\right)\}e^{-H_c^{(1)}(x)}\}d\nu_{F^*}(x) & \text{if } x \geq x_0 \\ c\{\frac{1}{w^*(x)}\{\prod_{x_r \in D_x^{(2)}}\left(1 - m^{(2)}\big(\{x_r\}\big)\right)\}e^{-H_c^{(2)}(x)}\}d\nu_{F^*}(x) & \text{if } x < x_0 \end{cases} \tag{2.6}$$

where

$$m^{(1)}(\bullet) = \int_{[x_0,\infty)\bigcap\bullet} (t(y) - \theta)(w(y))^{-1}d\nu_{F^*}(y), \quad H_c^{(1)}(x) = m_c^{(1)}((-\infty, x]),$$

and

$$m^{(2)}(\bullet) = \int_{(-\infty,x_0)\bigcap\bullet} (\theta - t(y))(w^*(y))^{-1}d\nu_{F^*}(y), \quad H_c^{(2)}(x) = m_c^{(2)}([x,\infty)),$$

with $0 < w^*(x) = w(x) + (\theta - t(x))\nu_{F^*}\big(\{x\}\big)$, $m_c^{(1)}$ *and* $m_c^{(2)}$ *as continuous parts of* $m^{(1)}$ *and* $m^{(2)}$ *respectively, and* $D_x^{(1)}$ *and* $D_x^{(2)}$ *as the sets of discontinuity points of* $m^{(1)}$ *that lie in* $(-\infty, x)$ *and of* $m^{(2)}$ *that lie in* (x, ∞) *respectively.*

The following corollary implies a version of the continuous exponential families:

Corollary 2.7. *If we have the assumptions in Theorem 2.6 met with F^* continuous, then the conclusion of the theorem holds with the following in place of (2.6) :*

$$dF_\theta(x) = \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)-\theta}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x) \tag{2.7}$$

where x_0 is as the statement of Theorem 2.6.

Let F_θ satisfies (2.7), then the fact that $0 \leq F_\theta(x_2) - F_\theta(x_1) (\leq 1) < \infty$ for every $x_2 > x_1$ and $c(\theta)$ is a normalizing constant (lying in $(0,\infty)$) implies that for each x with $|\int_{(x_0,x]}(w(y))^{-1} d\nu_{F^*}(y)| = \infty$, we have $|\int_{(x_0,x]} \frac{t(y)-\theta}{w(y)} d\nu_{F^*}(y)| = \infty$. (Note that $|F_\theta(x) - F_\theta(x_0)| \geq c(\theta)|\int_{(x_0,x]}(w(y))^{-1} d\nu_{F^*}(y)| \exp\{-|\int_{(x_0,x]} \frac{t(y)-\theta}{w(y)} d\nu_{F^*}(y)|\}$.) This, in turn, implies that if F_θ satisfies (2.7), then the distribution is concentrated on $\{x \in \Re \;:\; |\int_{(x_0,x]}(w(y))^{-1} d\nu_{F^*}(y)| < \infty\}$; we shall denote this set by $\mathcal{D}$.

In here, we assume throughout that ν_{F^*} is a non-atomic measure (i.e. F^* is continuous).

Corollary 2.8. *If we have the assumptions in Theorem 2.6 met with F^* continuous, then for any distribution F_θ (that is absolutely continuous w.r.t. ν_{F^*}) the conclusion of the theorem holds on taking, in place of (2.7), that F_θ is concentrated on $\mathcal{D}$ and it satisfies the following :*

$$\begin{aligned} dF_\theta(x) &= \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} d\nu_{F^*}(y)\right\} \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x) \\ &= c(\theta) k_1(x) \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x), \;\; x \in \mathcal{D}, \end{aligned} \tag{2.8}$$

where

$$k_1(x) = \frac{1}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} d\nu_{F^*}(y)\right\}. \tag{2.9}$$

3 Continuous Exponential Families via Covariance Identities

In view of extension of the Chernoff-types [5] inequality, Cacoullos and Papathanasiou [4] for random variable X with mean μ, variance $\sigma^2 < \infty$ and density f, proved that

$$Cov[h(X), g(X)] = E[Z(X)g'(X)],$$

where g is absolutely continuous function with $|E[Z(X)g'(X)]| < \infty$ and $h(x)$ is a given function leads to

$$Z(x) = \frac{1}{f(x)} \int_a^x [E(h(X)) - h(t)] f(t) dt.$$

This idea extended by Mohtashami Borzadaran and Shanbhag [11] to a general case. So, in what follow, special case of family is obtained with the form of (2.8), that is derived some characterization in view of the continuous exponential families.

Theorem 3.1. *Let X be an r.v. with distribution (2.8) and $E_\theta(t(X)) = \theta$. Also, let g be a real-valued function that is absolutely continuous w.r.t. ν_{F^*} with Radon-Nikodym derivative g' satisfying $E_\theta\{| w(X)g'(X) |\} < \infty$. Then*

$$Cov_\theta\{t(X), g(X)\} = E_\theta(w(X)g'(X)). \tag{3.1}$$

Proof. Applying the argument used to prove the " if " part of the Theorem 2.2 in Mohtashami Borzadaran and Shanbhag [11] with $t(X)$ in place of $h^*(X)$ and $w(X)$ in place of $Z(X)$, we obtain the theorem. □

Theorem 3.2. *The assumptions in Theorem 2.6 met with F^* continuous and $E_\theta(t(X)) = \theta$. Also, let τ be the class of real-valued function g such that $g'(x) \equiv \cos(ux)$ or $g'(x) \equiv \sin(ux)$ where g' is the Radon-Nikodym derivative of g w.r.t. ν_{F^*}. Suppose (3.1) holds for each $g \in \tau$. Then the distribution of X is as in Corollary 2.8.*

Proof. For any real u, let $g_u(x)$ be such that $g'_u(x) = \cos(ux)$ is the Radon-Nikodym derivative of g_u w.r.t. ν_{F^*}. Then, for any real a,

$$E_\theta\left\{(t(X)-\theta)\int_{(a,X]}\cos(uy)d\nu_{F^*}(y)\right\} = E_\theta\{w(X)\cos(uX)\}.$$

This implies that, for any real a,

$$\begin{aligned}\int_{\Re}(t(x)-\theta)&\left\{\int_{(a,x]}\cos(uy)d\nu_{F^*}(y)\right\}dF_\theta(x)\\ &= \int_{\Re}\cos(uy)\left\{\int_{[y,\infty)}(t(x)-\theta)dF_\theta(x)\right\}d\nu_{F^*}(y)\\ &= \int_{\Re} w(y)\cos(uy)dF_\theta(y).\end{aligned} \tag{3.2}$$

From (3.2) and the analogue of (3.2) with $\sin(uy)$ in place of $\cos(uy)$, we get that

$$\int_{\Re} e^{i(uy)}\left\{\int_{[y,\infty)}(t(x)-\theta)dF_\theta(x)\right\}d\nu_{F^*}(y) = \int_{\Re} w(t)e^{i(uy)}dF_\theta(y),\ u \in \Re.$$

From the uniqueness of Fourier transforms,

$$\left\{\int_{[y,\infty)}(t(x)-\theta)dF_\theta(x)\right\}d\nu_{F^*}(y) = w(y)dF_\theta(y).$$

By using Theorem 2.6 with ν_{F^*} as non-atomic measure, $dF_\theta(x)$ is seen to be that given by (2.8). □

Remark 3.3. In Corollary 2.8, $F^*(x) = x,\ x \in \Re$, implies that we have

$$\begin{aligned} dF_\theta(x) &= \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} dy\right\} \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} dy\right\} dx \\ &= c(\theta) k_1(x) \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} dy\right\} dx, \qquad (3.3) \end{aligned}$$

in place of (2.8) and

$$k_1(x) = \frac{1}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} dy\right\}, \qquad (3.4)$$

in place of (2.9). Also, in this case, Theorems 3.1 and 3.2 are valid.

Based on formula (2.8), when $F^*(x) = x$, we have the characterization that is derived in Table 1 as examples.

Table 1: Characterizations Based on $w(x)$ and $t(x)$ in Continuous case

$w(x)$	$t(x)$	Domain of the r.v. X	Name of Distribution
1	x	$x \in \Re$	Normal
x	$cx - 1$	$x \in (0, \infty)$	Gamma
x	$\frac{cx-1}{1-x}$	$x \in (0, 1)$	Beta
1	e^x	$x \in (0, \infty)$	Standard Log Gamma

Let us now define an exponential family that is a special case of (2.8). Let X be an r.v. with df F_θ that is absolutely continuous w.r.t. ν_{F^*} with density $f_\theta(x)$ of the form:

$$f_\theta(x) = c(\theta) k(x) e^{\theta \mu^*(x)},\ x \in \Re,\ \theta \in \Re, \qquad (3.5)$$

where $0 < k(x) = \exp\{-\int_{(x_0,x]} t(y) d\nu_{F^*}(y)\}$, and $\mu^*(x) = F^*(x) - F^*(a)$ and $E_\theta[t(X)] = \theta$.

Remark 3.4. In Theorem 3.2, if $w(x) \equiv 1$, then we have the family (3.5) in place of (2.8).

Corollary 3.5. *Let X be an r.v. with distribution (3.5) and $E_\theta(t(X)) = \theta$. Also, let g be a real-valued absolutely continuous function such that g' be the Radon-Nikodym derivative of g w.r.t. ν_{F^*} and $E_\theta\{\mid g'(X) \mid\} < \infty$. Then*

$$E_\theta\{(t(X) - \theta)g(X)\} = E_\theta(g'(X)). \quad (3.6)$$

Proof. The result follows on using the same argument as in the proof of the Theorem 3.1, but with $w(x) \equiv 1$. □

Corollary 3.6. *The assumptions in Theorem 2.6 met with F^* continuous and $E_\theta(t(X)) = \theta$. Also, let τ be the class of real-valued functions g such that $g'_u(x) \equiv \cos(ux)$ or $g'_u(x) \equiv \sin(ux)$ where g'_u is the Radon-Nikodym derivative of g w.r.t. ν_{F^*}. Suppose (3.6) holds for each $g \in \tau$. Then the distribution of X is of the form (3.5).*

Proof. The result follows on using the same argument as in the proof of Theorem 3.2, with $w(x) \equiv 1$. □

In (3.5) if we take, $F^*(x) = x,\ x \in \Re$, then (3.5) simplifies to

$$f_\theta(x) \propto k(x)e^{\theta x},\ x \in \Re,\ \theta \in \Re. \quad (3.7)$$

In this case we arrive at the following characterizations as consequences of Corollaries 3.5 and 3.6 :

Remark 3.7. Let X be an r.v. with density (3.7) where $k(x) > 0$ for all $x \in \Re$ and is differentiable and $E_\theta(t(X)) = \theta$. Also g be any differentiable function with $E_\theta\{\mid g'(X) \mid\} < \infty$. Then, Hudson [7] proved that

$$E_\theta\{(t(X) - \theta)g(X)\} = E_\theta(g'(X)). \quad (3.8)$$

Also, let $t(x) = -\frac{k'(x)}{k(x)}$ such that $E_\theta(t(X)) = \theta$. Suppose (3.8) holds for all g_u such that $g_u(x) \equiv e^{iux},\ u \in \Re$. Then, Prakasa Rao [14] obtained that X has density w.r.t. Lebesgue measure and it is given by (3.7).

4 Conclusion

In this paper, we derive characterization of continuous exponential families via hazard measure. This characterization is an extended version of the results of Hudson [7] and Prakasa Rao [14] related to a representation of the exponential families and covariance identity.

Acknowledgements : I would like to express my sincere thanks to Prof. D. N. Shanbhag for his evaluable suggestion. Also, I am grateful to the editor and the referees for their helpful comments which have greatly improved the paper.

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(Received 12 July 2009)

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