Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 545–554

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Algorithm for Finding the Coefficients of Rook Polynomials

V. Longani

Abstract : In solving problems on finding the number of arrangements of n objects with restrictions, it is usually required to find the coefficients of certain polynomials called rook polynomials. See [1], [2], and [3] for topics on rook polynomials. It is not difficult to find these coefficients when n is not large. However, when n becomes larger the calculation becomes laboring and less practical. In this paper, we propose a simple algorithm for finding these coefficients. Also, this algorithm can be modified for more general problems on rook polynomials.

Keywords : Permutation; Arrangements with restrictions; Rook polynomials.
2000 Mathematics Subject Classification : 05A05.

1 Rook polynomials and permanent of matrices

Let B be any $m \times n$ $(m \leq n)$ board such as B_1 in Figure 1(a) and B_2 in Figure 1(b) with some squares are darkened , and other squares are white. Let $r_k(B)$ be the number of ways to choose k, $(k \leq m \leq n)$, darkened squares, no two of which lie in the same row and no two of which lie in the same column. We may think that B is part of generalized chessboard. A rook is a piece that can travel either horizontally or vertically on the board. A rook is said to be able to capture another if the two are in the same row or the same column. We want to place k rooks on darkened squares in such a way that no rook can capture another. Therefore, $r_k(B)$ is the number of ways noncapturing rooks can be placed in darkened squares of B.

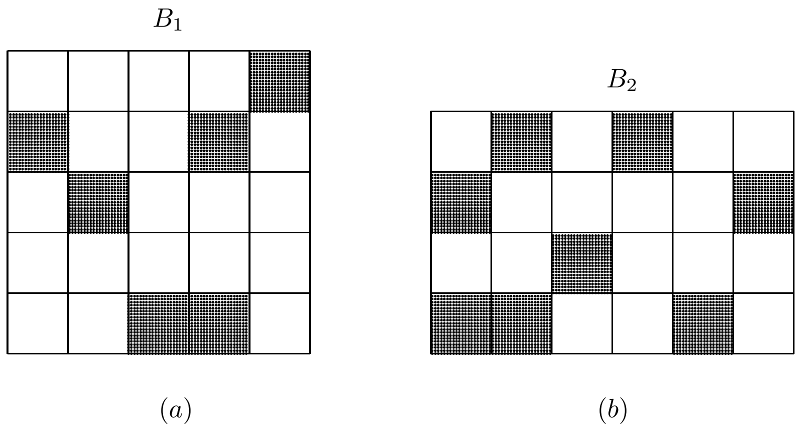


Figure 1

The rook polynomial for the board B is defined by

$$R(x,B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

The coefficients $r_k(B)$ of the rook polynomial is used in finding the number of arrangements with restrictions. It is quite laborious to find the values of $r_k(B)$ when k and the size of B are large. So, it is useful and desirable to find a systematic way for finding the value $r_k(B)$. We shall discuss a method for finding the values $r_k(B)$ from which an algorithm could be created. In section 2, we show how to apply the method in assisting the usual method used in combinatorics texts. In sections 3, we modify our method to provide shorter alternative way in finding the number of arrangements with restrictions.

Before proceeding further, we need to define some definitions. Let $A = [a_{ij}]$ be $m \times n$ matrix, where $m \leq n$. The permanent of A is defined as

$$Per(A) = \left| \overline{\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{array}} \right. = \sum a_{1i_1} a_{2i_2} a_{3i_3} \ldots a_{mi_m}$$

where the summation is over all m-permutations of the set $N = 1, 2, \ldots, n$.

We note that the value of $Per(A)$ does not change if we interchange any two rows or columns of A. For examples, consider matrices A , B, C, D and E:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 3 & 7 \\ 2 & 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

$$D = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \quad E = \begin{pmatrix} 4 & 1 & 2 & 0 \\ 6 & 2 & 0 & 7 \\ 1 & 0 & 3 & 2 \end{pmatrix}.$$

We then have

$$\begin{aligned}
Per(A) &= \overline{\left| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right.} = a_{11}a_{22} + a_{21}a_{12} \\
Per(B) &= \overline{\left| \begin{matrix} 2 & 3 & 1 \\ 5 & 3 & 7 \\ 2 & 0 & 4 \end{matrix} \right.} = 2\overline{\left| \begin{matrix} 3 & 7 \\ 0 & 4 \end{matrix} \right.} + 3\overline{\left| \begin{matrix} 5 & 7 \\ 2 & 4 \end{matrix} \right.} + 1\overline{\left| \begin{matrix} 5 & 3 \\ 2 & 0 \end{matrix} \right.} \\
&= (2)(3)(4) + (2)(0)(7) + (3)(5)(4) + (3)(2)(7) \\
&\quad +(1)(5)(0) + (1)(2)(3) \\
&= 132 \\
Per(C) &= \overline{\left| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \end{matrix} \right.} = a_{11} + a_{12} + a_{13} + a_{14} \\
Per(D) &= \overline{\left| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix} \right.} \\
&= a_{11}\overline{\left| \begin{matrix} a_{22} & a_{23} & a_{24} \end{matrix} \right.} + a_{12}\overline{\left| \begin{matrix} a_{21} & a_{23} & a_{24} \end{matrix} \right.} + a_{13}\overline{\left| \begin{matrix} a_{21} & a_{22} & a_{24} \end{matrix} \right.} \\
&\quad +a_{14}\overline{\left| \begin{matrix} a_{21} & a_{22} & a_{23} \end{matrix} \right.} \\
&= a_{11}\left(a_{22} + a_{23} + a_{24}\right) + a_{12}\left(a_{21} + a_{23} + a_{24}\right) + a_{13}\left(a_{21} + a_{22} + a_{24}\right) \\
&\quad +a_{14}\left(a_{21} + a_{22} + a_{23}\right) \\
&= a_{11}a_{22} + a_{11}a_{23} + a_{11}a_{24} + a_{12}a_{21} + a_{12}a_{23} + a_{12}a_{24} + a_{13}a_{21} \\
&\quad +a_{13}a_{22} + a_{13}a_{24} + a_{14}a_{21} + a_{14}a_{22} + a_{14}a_{23} \\
Per(E) &= \overline{\left| \begin{matrix} 4 & 1 & 2 & 0 \\ 6 & 2 & 0 & 7 \\ 1 & 0 & 3 & 2 \end{matrix} \right.} \\
&= 4\overline{\left| \begin{matrix} 2 & 0 & 7 \\ 0 & 3 & 2 \end{matrix} \right.} + 1\overline{\left| \begin{matrix} 6 & 0 & 7 \\ 1 & 3 & 2 \end{matrix} \right.} + 2\overline{\left| \begin{matrix} 6 & 2 & 7 \\ 1 & 0 & 2 \end{matrix} \right.} + 0\overline{\left| \begin{matrix} 6 & 2 & 0 \\ 1 & 0 & 3 \end{matrix} \right.} \\
&= 4[2(3+2) + 0(0+2) + 7(0+3)] + 1[6(3+2) + 0(1+2) + 7(1+3)] \\
&\quad +2[6(0+2) + 2(1+2) + 7(1+0)] \\
&= 232.
\end{aligned}$$

We note that when $m = n$ the permanent of the square matrix is made up of the same products that compose the determinant, but without the prefixed algebraic signs.

Next, for $k \leq m \leq n$, we define $Per_k(A)$, where $k = 1, 2, 3, ..., m$ as the sum of the permanents of all $k \times n$ submatrices of A. For example, if

$$D = \begin{pmatrix} 3 & 1 & 4 & 0 \\ 5 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

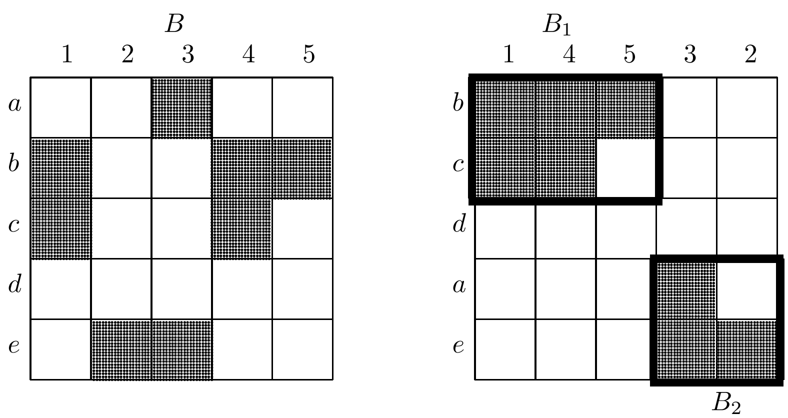
then

$$\begin{aligned} Per_2(D) &= \overline{\left| \begin{matrix} 3 & 1 & 4 & 0 \\ 5 & 0 & 0 & 2 \end{matrix} \right.} + \overline{\left| \begin{matrix} 3 & 1 & 4 & 0 \\ 2 & 0 & 1 & 0 \end{matrix} \right.} + \overline{\left| \begin{matrix} 5 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{matrix} \right.} \\ &= 41 + 14 + 11 = 66. \end{aligned}$$

Also note that, when $k = m$, $Per_k(A) = Per_m(A) = Per(A)$. In next section we show that we can use certain permanents of matrices in finding the coefficients of rook polynomials from which the number of arrangements with restrictions can be found.

2 Applications for arrangements with restrictions

We are interested in finding the number of arrangements of n objects with restrictions, that is some particular objects are not allowed in certain positions. For example, with $n = 5$, we want to find all arrangements of a, b, c, d, e with the restrictions indicated by darkened squares of the board B in Figure 2(a).



(a) (b)

Figure 2

Let N be the number of such arrangement. Using inclusion-exclusion principle, it can be shown that

$$N = 5! - r_1(B)4! + r_2(B)3! - r_3(B)2! + r_4(B)1! - r_5(B)0! \qquad (2.1)$$

In fact, for any arrangement with restrictions of n objects, we can construct $n \times n$ board B indicating the restricted positions by darkened squares and we shall call this board as restriction board B. Theorem 2.1 shows available general formula for finding the value of N. See [1], [2], and [3] for details.

Theorem 2.1. *The number of ways to arrange n distinct objects with restrictions is equal to*
$N = n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)! + \cdots + (-1)^n r_n(B)0!$

When n is large it is rather laborious to find correct values of each $r_k(B)$ so, in some cases, we can reduce this problem by rearrange elements of B so that we can obtain two or more disjoint subboards of B. For example, in Figure 2(b), the new form of B consists of two smaller boards B_1 and B_2. The following theorem is also available in combinatorics texts.

Theorem 2.2. *If B can be decomposed into two disjoint subboards B_1 and B_2 then*

$$r_k(B) = r_k(B_1)r_0(B_2) + r_{k-1}(B_1)r_1(B_2) + \cdots + r_0(B_1)r_k(B_2)$$

where we define $r_0(B_1) = r_0(B_2) = 1$.

Let an $m \times n$ board B are defined as in section 1. We define the $m \times n$ matrix $M_B = [a_{ij}]$ of a board B as a matrix of which the i,j elements are 1 if the corresponding i,j squares of B are darkened , and the i,j elements are 0 if otherwise. For example, the M_B of the board B in Figure 2(a) is

$$M_B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad (2.2)$$

The following theorem will help us systematically in finding the values of $r_k(B)$.

Theorem 2.3. *Let M_B be the corresponding matrix of a board B of size $m \times n$, where $m \leq n$. Then we have $r_k(B) = Per_k(M_B)$.*

Proof. When $k = m$, consider the definition

$$Per_k(M_B) = Per(M_B) = \sum a_{1i_1}a_{2i_2}a_{3i_3}...a_{mi_m}.$$

Consider a product $a_{1i_1}a_{2i_2}a_{3i_3}...a_{mi_m}$ of the summation. This product is equal to 1 if all a_{ij} of the product is equal to 1, otherwise the product will be zero. Each product that equal to 1 will represent a way of placing m rooks on B no two of which lie in the same row and no two of which lie in the same column. The summation will count exactly all the possible ways in placing m rooks. When $k < m$, i.e. $k = 1, 2, 3, ..., m-1$, $Per_k(M_B)$ is the sum of the permanents of all

$k \times n$ submatrices of A. Each of these permanents of the $k \times n$ submatrices of M_B will count the number of ways in placing k rooks on each of the submatrices of M_B with the required condition. The summation will then count all possible ways in placing k rooks on B. Hence, $r_k(B) = Per_k(M_B)$. □

To illustrate Theorem 2.3, we consider the problem of arrangement of 5 objects a, b, c, d, e with the restrictions board B are as in Figure 2(a). The required number N can be calculated from (2.1) with (2.2) as the M_B of B. For (2.1), we need to know the values of $r_k(B)$, where $k = 1, 2, ..., 5$. From Theorem 2.3,

$$
\begin{aligned}
r_1(B) &= \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&= 1 + 3 + 2 + 0 + 2 \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
r_2(B) &= \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&= 3 + 2 + 0 + 1 + 4 + 0 + 6 + 0 + 4 + 0 \\
&= 20
\end{aligned}
$$

$$
\begin{aligned}
r_3(B) &= \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&\quad + \overline{\left|\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right.} \\
&= 4 + 0 + 3 + 0 + 2 + 0 + 0 + 8 + 0 + 0 \\
&= 17
\end{aligned}
$$

$$
\begin{aligned}
r_4(B) &= \overline{\left|\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right.} + \overline{\left|\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{matrix}\right.} + \overline{\left|\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{matrix}\right.} \\
&\quad + \overline{\left|\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{matrix}\right.} + \overline{\left|\begin{matrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{matrix}\right.} \\
&= 0+4+0+0+0 \\
&= 4.
\end{aligned}
$$

$r_5(B) = Per_5(B) = Per(M_B) = 0$ (since all elements in the fourth row of M_B are zero).

Substitute these values in (2.1) we then obtain $N = 5! - 8(4!) + 20(3!) - 17(2!) + 4(1!) - 0(0!) = 18$.

Computer program for calculating permanents of matrices could be constructed without much difficulty and so we can obtain the values of each $Per_k(M_B)$. Now we have a systematic way to calculate the number of arrangements of the problem for the usual inclusion-exclusion method in Theorem 2.1. However, with the use of Theorem 3.1 in section 3, we propose an alternative direct way to solve the problem.

3 Short alternative applications

In this section we propose Theorem 3.1 and Theorem 3.2 which give an alternative way in finding the number of arrangements with restrictions. Let $m \times n$ board B be a board of restriction which defined as in section 1 and section 2. We define complementary board $\overline{B}$ of B as the board obtained from B by interchanging dark and white squares. We then have the following theorem.

Theorem 3.1. *The number of arrangements of n objects with restriction board B is equal to $N = r_n(\overline{B}) = Per(M_{\overline{B}})$.*

Proof. Since the element 1's in $M_{\overline{B}}$ correspond to the allowed positions of the arrangements, while the elements 0's in $M_{\overline{B}}$ correspond to the unallowed positions. As in Theorem 2.3 the value $r_n(\overline{B}) = Per(M_{\overline{B}})$ count all the number of ways in choosing n darkened squares, no two of which lie in the same row and no two of which lie in the same column. This number will be the required number of all allowed arrangements of n objects. □

To illustrate Theorem 3.1, consider the number of arrangements with restrictions according to Figure 2(a). Here, we have

$$M_{\overline{B}} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Since the value of the permanent does not change if we interchange any two rows or columns of the matrix, by interchanging the first and second rows of $M_{\overline{B}}$ we then have

$$\begin{aligned} Per(M_{\overline{B}}) &= \overline{\left| \begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array} \right.} \\ &= \overline{\left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array} \right.} + \overline{\left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array} \right.} \\ &= \overline{\left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right.} + \overline{\left| \begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right.} + \overline{\left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right.} + \overline{\left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right.} \\ &= 6 + 2 + 6 + 4 \\ &= 18. \end{aligned}$$

This agrees with the result in section 2, but with much less steps of calculation. From the particular case of Figure 2, we have shown that Theorem 2.3 can provide a systematic way in finding the values of $r_1(B), r_2(B), ..., r_5(B)$ which lead to the value of N. For this case, direct consideration of these values can still be practical enough, though mistakes can easily occur. The idea of decomposing B into B_1 and B_2 and then using Theorem 2.2 is useful but this need some consideration and sometimes it is not possible to decompose the board B. When the board B is larger it is time consuming and less practical for the process. However, Theorem 2.3 could be easily used in providing algorithm for computer programming in calculating these values of $r_k(B)$'s.

We generally note that the more darkened squares the board B has the more difficult it is in considering directly the values of $r_k(B)$'s from B. Also, if darkened squares in B are increased then the number of darkened squares in $\overline{B}$ would be decreased. For the board B of size $n \times n$, the idea of finding the number N from $r_n(\overline{B})$ is less known and rarely mentioned in combinatorics texts. The cases when n's are not large and the number of darkened squares in B's are relatively small, finding $r_1(B), r_2(B), ..., r_n(B)$ by direct consideration may possibly be easier than finding $r_n(\overline{B})$. For most other cases finding the values of N from $r_n(\overline{B})$ by using Theorem 3.1 could be simpler alternatives. For larger sizes of B's, Theorem 3.1

could provide an algorithm for computer program that lead to the required number N.

The number $Per(M_{\overline{B}})$ in Theorem 3.1 can be interpreted as the number of distributing n objects to n different boxes with the condition that some certain objects are not allowed to be in some boxes. For more general applications with $k \leq m \leq n$, suppose we have m different objects, n different boxes , and each box can contain at most one object. What is the number of all possible distributions of k objects, chosen from m objects, to the n different boxes. The problem can be described by $m \leq n$ restriction board B. Next theorem provide the answers for the problems.

Theorem 3.2. *For $k \leq m \leq n$, the number of all possible distributions of k objects, chosen from m objects, to n different boxes is equal to $Per_k(M_{\overline{B}})$.*

Proof. Using Theorem 2.3, here we have $r_k(\overline{B}) = Per_k(M_{\overline{B}})$. Also, we can verify that the number of all possible distributions of k objects, chosen from m, objects to n different boxes is equal to $r_k(\overline{B})$, and hence the theorem is proved. □

To illustrate Theorem 3.2, let there be 3 objects a,b, and c, 4 boxes, and the restriction board B be as in Figure 3.

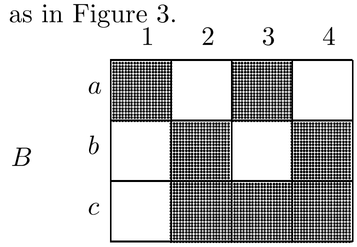


Figure 3

How many ways to distribute the 3 objects to the boxes, and how many ways to distribute 2 objects chosen from the 3 objects to the boxes. From Theorem 3.2, with $m = 3$, $n = 4$, $k = 2$, the answer for the first question is equal to $Per_3(M_{\overline{B}})$ $= Per(M_{\overline{B}})$, and the answer for the second question is equal to $Per_2(M_{\overline{B}})$, where

$$M_{\overline{B}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} Per(M_{\overline{B}}) &= \overline{\left| \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \right.} = 2 \\ Per_2(M_{\overline{B}}) &= \overline{\left| \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix} \right.} + \overline{\left| \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix} \right.} + \overline{\left| \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \right.} \\ &= 4 + 2 + 1 = 7. \end{aligned}$$

For 9 distributions in Figure 4, the first 2 distributions are for the first question, and the other 7 distributions are for the second question.

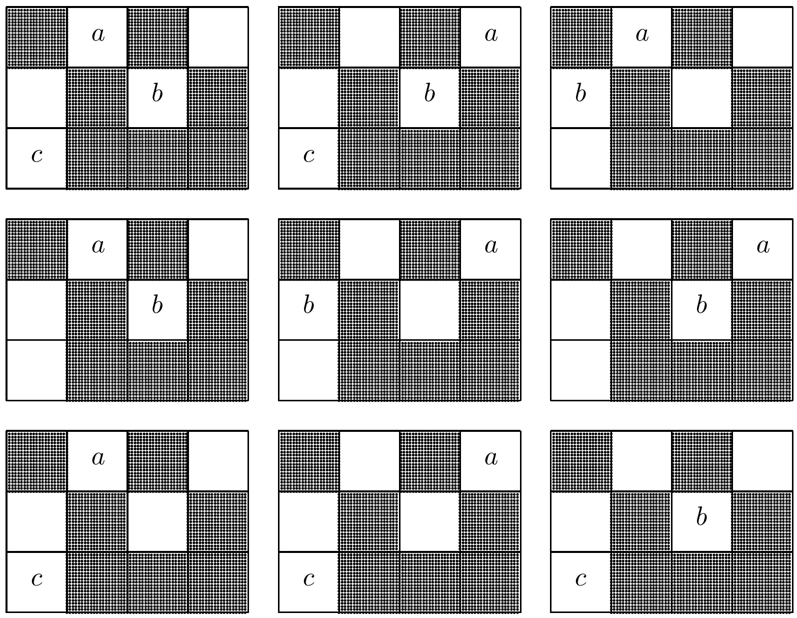


Figure 4

References

[1] V. K. Balakrishnan, Combinatorics, Mc-Graw Hill, Singapore, 1995.

[2] F. S. Roberts, Applied Combinatorics, Prentice-Hall, Englewood Cliffs, New Jersy, 1985.

[3] A. Tucker, Applied Combinatorics, 3rd Edition, John Wiley , Singapore, 1995.

(Received 15 July 2010)

Vites Longani
Department of Mathematics,
Faculty of Science,
and College of Arts, Media and Technology
Chiang Mai University,
Chiang Mai 50200, THAILAND
e-mail : `vites@chiangmai.ac.th`