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Weak and Strong Convergence of an Implicit Iteration Process for Two Families of Asymptotically Nonexpansive Mappings

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Abstract : In this paper, we consider an implicit iterative process for two finite families of asymptotically nonexpansive mappings in the framework of a real uniformly convex Banach space. The results presented in this paper mainly improve and extend the recent ones announced in [S. S. Chang, K. K. Tan, H. W. J. Lee and C. K. Chan, On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings, J. Math. Anal. Appl., 313 (2006), 273–283] and [S. H. Khan, I. Yildirim and M. Ozdemir, Convergence of an implicit algorithm for two families of nonexpansive mappings, Comput. Math. Appl., 59 (2010), 3084–3091].

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1 Introduction and Preliminaries

Throughout this paper, we always assume that E is a real Banach space. Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

Let C be a nonempty subset of E and $T : C \to C$ be a mapping. Denote by $F(T)$ the set of fixed points of the mapping T. Recall that the mapping T is said to be

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nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recall that the mapping T is said to be *asymptotically nonexpansive* if there exists a positive sequence $\{t_n\} \subset [1, \infty)$ with $t_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq t_n \|x - y\|, \quad \forall x, y \in C, \, n \geq 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings.

Recently, convergence problems of implicit iterative processes have been investigated by many authors.

In 2001, Xu and Ori [13] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_1, T_2, \cdots, T_N\}$ with a real sequence $\{\alpha_n\}$ in $(0, 1)$ and an initial point $x_0 \in C$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\ &\cdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\ &\cdots, \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $T_n = T_{n(modN)}$ (here the mod N takes values in $\{1, 2, \cdots, N\}$).

They proved the following weak theorem based on the iterative process (1.1).

Theorem XO. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $T_i : C \to C$ be a nonexpansive self-mapping on C such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ for each $i \in \{1, 2, \cdots, N\}$. Let $\{x_n\}$ be defined by* (1.1). *If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^N$.*

Recently, Khan, Yildirim and Ozdemir [9] considered convergence problems of an implicit iterative algorithm for two families of nonexpansive mappings. Their implicit algorithm is expressed as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_1 + \gamma_1 T_1 x_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_2 + \gamma_2 T_2 x_2, \\ &\cdots \\ x_N &= \alpha_N x_{N-1} + \beta_N S_N x_N + \gamma_N T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_{N+1} x_{N+1} + \gamma_{N+1} T_{N+1} x_{N+1}, \\ &\cdots, \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n S_n x_n + \gamma_n T_n x_n, \quad \forall n \geq 1, \tag{1.2}$$

where $S_n = S_{n(modN)}$ and $T_n = T_{n(modN)}$ (here the mod N takes values in $\{1, 2, \cdots, N\}$).

Recall that a space X is said to satisfy *Opial's condition* [10] if, for each sequence $\{x_n\}$ in X, the convergence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X \ (y \neq x).$$

It is well known [10] that each l^p $(1 \leq p < \infty)$ satisfies Opial's condition. It is also known [4] that any separable Banach space can be equivalently renormed to that it satisfies Opial's condition.

Khan, Yildirim and Ozdemir [9] obtained the following weak convergence theorem with the help of Opial's condition.

Theorem KYO. *Let E be a real uniformly convex Banach space which satisfies The Opial's condition and C be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings of C with a nonempty fixed point set $F := \left(\bigcap_{j=1}^N F(T_j)\right) \bigcap \left(\bigcap_{j=1}^N F(S_j)\right)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n+\beta_n+\gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Then the sequence $\{x_n\}$ defined by* (1.2) *converges weakly to $q \in F$.*

We remark that, from the view of computation, the implicit iterative processes (1.1) and (1.2) is often impractical since, in many cases, to solve the operation equation exactly is difficult. For each step, we must solve a nonlinear operator equation. Therefore, one of the interesting and important problems in the theory of implicit iterative processes is to consider the iterative processes with errors. That is an efficient iterative process to compute approximately fixed point of nonlinear mappings.

In this paper, motivated by the above results, we introduce the following iterative process for two families of asymptotically nonexpansive mappings:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_1 + \gamma_1 T_1 x_1 + \delta_1 u_1,\\
x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_2 + \gamma_2 T_2 x_2 + \delta_2 u_2,\\
&\cdots\\
x_N &= \alpha_N x_{N-1} + \beta_N S_N x_N + \gamma_N T_N x_N + \delta_N u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_1^2 x_{N+1} + \gamma_{N+1} T_1^2 x_{N+1} + \delta_{N+1} u_{N+1},\\
&\cdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} S_N^2 x_{2N} + \gamma_{2N} T_N^2 x_{2N} + \delta_{2N} u_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} S_1^3 x_{2N+1} + \gamma_{2N+1} T_1^3 x_{2N+1} + \delta_{2N+1} u_{2N+1},\\
&\cdots,
\end{aligned}$$

where x_0 is the initial value, $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_n + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1. \tag{1.3}$$

The purpose of this paper is to study the weak and strong convergence of implicit iterative sequences generated in the implicit iterative process (1.3). Weak and strong theorems are established in the framework of a uniformly convex Banach space. The results of this paper improve and extend the corresponding results of Chang et al. [1], Chidume and Shahzad [3], Guo and Cho [7], Kahn, Yildirim and Ozdemir [9], Thianwan and Suantai [12], Xu and Ori [13] and Zhou and Chang [14].

Next, we state the following useful lemmas.

Lemma 1.1. ([2,6]) *Let E be a uniformly convex Banach Space, C be a nonempty closed convex subset of E and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., for each sequence $\{x_n\} \in C$, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I-T)x_n\}$ converges strongly to* 0*, then* $(I-T)q = 0$.

Lemma 1.2. ([11]) *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ *exists;*

(b) *if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$.*

Lemma 1.3. ([8]) *Let E be a uniformly convex Banach space, $s > 0$ be a positive number and $B_s(0)$ be a closed ball of E. There exits a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|ax + by + cz + dw\|^2 \le a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|)$$

for all $x, y, z, w \in B_s(0) = \{x \in E : \|x\| \le s\}$ and $a, b, c, d \in [0,1]$ such that $a + b + c + d = 1$.

2 Main Results

Now, we are ready to give our main results.

Theorem 2.1. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \ge 1$ be a positive integer and $I = \{1, 2, 3, \cdots, N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(s_{n,i} - 1) < \infty$ and let T_i be an asymptotically nonexpansive mapping with the sequence $\{t_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(t_{n,i} - 1) < \infty$ for each $i \in I$. Let $k_n = \max\{s_n, t_n\}$, where $s_n = \max\{s_{n,i} : i \in I\}$ and $t_n = \max\{t_{n,i} : i \in I\}$. Assume that $F = \left(\bigcap_{i=1}^{N} F(S_i)\right) \bigcap \left(\bigcap_{i=1}^{N} F(T_i)\right) \ne \emptyset$. Let $\{x_n\}$ be the sequence generated in (1.3). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences satisfying $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \ge 1$ and $\{u_n\}$ be a bounded sequence in C. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $[0,1]$ satisfy the following restrictions*

(a) $\sum_{n=1}^{\infty} \delta_n < \infty$;

(b) *there exist constants $a, b, c \in (0,1)$ such that $a \le \alpha_n, b \le \beta_n$ and $c \le \gamma_n$, $\forall n \ge 1$;*

(c) $(\beta_n + \gamma_n)L < 1$, *where $L = \sup_{n\ge1}\{k_n\}$, $\forall n \ge 1$.*

Then the sequence $\{x_n\}$ converges weakly to a point in F.

Proof. First, we show that the sequence generated in the implicit iterative process (1.3) is well defined. For each $n \ge 1$, define a mapping $R_n : C \to C$ by

$$R_n(x) = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x + \gamma_n T_{i(n)}^{h(n)} x + \delta_n u_n, \quad \forall x \in C.$$

Notice that

$$\begin{aligned}
&\|R_n(x) - R_n(y)\| \\
&= \big\|\big(\alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x + \gamma_n T_{i(n)}^{h(n)} x + \delta_n u_n\big) \\
&\qquad - \big(\alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} y + \gamma_n T_{i(n)}^{h(n)} y + \delta_n u_n\big)\big\| \\
&\le \beta_n \big\|S_{i(n)}^{h(n)} x - S_{i(n)}^{h(n)} y\big\| + \gamma_n \big\|T_{i(n)}^{h(n)} x - T_{i(n)}^{h(n)} y\big\| \\
&\le (\beta_n + \gamma_n)L\|x - y\|, \quad \forall x, y \in C.
\end{aligned}$$

From the restriction (c), we see that R_n is a contraction for each $n \geq 1$. By Banach contraction principle, we see that there exists a unique fixed point $x_n \in C$ such that

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_n + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1.$$

This shows that the sequence generated in the implicit iterative process (1.3) is well defined.

Next, we show the sequence $\{x_n\}$ is bounded. For any $p \in F$, we have

$$\begin{aligned}
\|x_n - p\| &= \big\|\big(\alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_n + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n\big) - p\big\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \big\|S_{i(n)}^{h(n)} x_n - p\big\| + \gamma_n \big\|T_{i(n)}^{h(n)} x_n - p\big\| + \delta_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (\beta_n + \gamma_n) k_{h(n)} \|x_n - p\| + \delta_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) k_{h(n)} \|x_n - p\| + \delta_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (k_{h(n)} - \alpha_n) \|x_n - p\| + \delta_n \|u_n - p\|,
\end{aligned}$$

which yields from the restriction (b) that

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{(k_{h(n)} - 1)}{a} \|x_n - p\| + \frac{\delta_n}{a} \|u_n - p\|.$$

Note that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. It follows that there exists some positive integer n_0 such that for any $h(n) \geq n_o$, $k_{h(n)} - 1 < \frac{a}{2}$. Therefore we get that

$$\begin{aligned}
\|x_n - p\| &\leq \frac{a}{a + 1 - k_{h(n)}} \|x_{n-1} - p\| + \frac{\delta_n}{a + 1 - k_{h(n)}} \|u_n - p\| \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{a + 1 - k_{h(n)}}\right) \|x_{n-1} - p\| + \frac{\delta_n}{a + 1 - k_{h(n)}} \|u_n - p\| \qquad (2.1) \\
&\leq \left(1 + \frac{2(k_{h(n)} - 1)}{a}\right) \|x_{n-1} - p\| + \frac{2\delta_n}{a} \|u_n - p\|
\end{aligned}$$

for all such n, where n is such that $h(n) \geq n_0$. In view of Lemma 1.2 this means by the restriction (a) that the limit of $\lim_{n\to\infty} \|x_n - p\|$ exists. It follows that the sequence $\{x_n\}$ is bounded.

On the other hand, we obtain from Lemma 1.3 that

$$\begin{aligned}
\|x_n - p\|^2 &= \big\|\big(\alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_n + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n\big) - p\big\|^2 \\
&\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n \big\|S_{i(n)}^{h(n)} x_n - p\big\|^2 + \gamma_n \big\|T_{i(n)}^{h(n)} x_n - p\big\|^2 \\
&\quad + \delta_n \|u_n - p\|^2 - \alpha_n \beta_n g_1\big(\big\|x_{n-1} - S_{i(n)}^{h(n)} x_n\big\|\big) \\
&\leq \alpha_n \|x_{n-1} - p\|^2 + (\beta_n + \gamma_n) k_{h(n)}^2 \|x_n - p\|^2 + \delta_n \|u_n - p\|^2 \\
&\quad - \alpha_n \beta_n g_1\big(\big\|x_{n-1} - S_{i(n)}^{h(n)} x_n\big\|\big),
\end{aligned}$$

which in turn yields that

$$\begin{aligned}&\alpha_n\beta_n g_1\big(\|x_{n-1} - S_{i(n)}^{h(n)} x_n\|\big)\\ &\le \alpha_n\|x_{n-1} - p\|^2 + \big((\beta_n + \gamma_n)k_{h(n)}^2 - 1\big)\|x_n - p\|^2 + \delta_n\|u_n - p\|^2\\ &\le \alpha_n\big(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\big) + (\beta_n + \gamma_n)\big(k_{h(n)}^2 - 1\big)\|x_n - p\|^2\\ &\quad + \delta_n\|u_n - p\|^2.\end{aligned}$$

From the restrictions (a) and (b) and since $\lim_{n\to\infty}\|x_n - p\|$ exists, we obtain that

$$\lim_{n\to\infty}\|x_{n-1} - S_{i(n)}^{h(n)} x_n\| = 0. \tag{2.2}$$

Reconsidering Lemma 1.3, we also have

$$\begin{aligned}\|x_n - p\|^2 \le \alpha_n\|x_{n-1} - p\|^2 &+ (\beta_n + \gamma_n)k_{h(n)}^2\|x_n - p\| + \delta_n\|u_n - p\|^2\\ &- \alpha_n\gamma_n g_2\big(\|x_{n-1} - T_{i(n)}^{h(n)} x_n\|\big),\end{aligned}$$

which in turn yields that

$$\begin{aligned}&\alpha_n\gamma_n g_2\big(\|x_{n-1} - T_{i(n)}^{h(n)} x_n\|\big)\\ &\le \alpha_n\big(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\big) + (\beta_n + \gamma_n)\big(k_{h(n)}^2 - 1\big)\|x_n - p\|^2\\ &\quad + \delta_n\|u_n - p\|^2.\end{aligned}$$

In view of the restrictions (a) and (b), we see that

$$\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}^{h(n)} x_n\| = 0. \tag{2.3}$$

Notice that

$$\|x_n - x_{n-1}\| \le \beta_n\|S_{i(n)}^{h(n)} x_n - x_{n-1}\| + \gamma_n\|T_{i(n)}^{h(n)} x_n - x_{n-1}\| + \delta_n\|u_n - x_{n-1}\|.$$

It follows from (2.2) and (2.3) that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.4}$$

For each $j \in I$, we have

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0, \quad \forall j \in I. \tag{2.5}$$

Next, we show that $x_n - S_n x_n \to 0$ as $n \to \infty$. Since for any positive integer $n > N$, it can be written as $n = (h(n) - 1)N + i(n)$, where $i(n) \in \{1, 2, \cdots, N\}$.

$$\begin{aligned}\|x_{n-1} - S_n x_n\| &\le \|x_{n-1} - S_{i(n)}^{h(n)} x_n\| + \|S_{i(n)}^{h(n)} x_n - S_n x_n\|\\ &= \|x_{n-1} - S_{i(n)}^{h(n)} x_n\| + \|S_{i(n)}^{h(n)} x_n - S_{i(n)} x_n\|\\ &\le \|x_{n-1} - S_{i(n)}^{h(n)} x_n\| + L\|S_{i(n)}^{h(n)-1} x_n - x_n\|\\ &\le \|x_{n-1} - S_{i(n)}^{h(n)} x_n\| + L\|S_{i(n)}^{h(n)-1} x_n - S_{i(n-N)}^{h(n)-1} x_{n-N}\|\\ &\quad + L\|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{n-N-1}\| + L\|x_{n-N-1} - x_n\|.\end{aligned} \tag{2.6}$$

Since for each $n > N$, $n = (n-N)(\text{mod } N)$, on the other hand, we obtain from $n = (h(n)-1)N+i(n)$ that $n-N = \big((h(n)-1)-1\big)N+i(n) = (h(n-N)-1)N+i(n-N)$. That is,

$$h(n-N) = h(n) - 1 \quad \text{and} \quad i(n-N) = i(n).$$

Therefore we see

$$\begin{aligned}\big\|S_{i(n)}^{h(n)-1}x_n - S_{i(n-N)}^{h(n)-1}x_{n-N}\big\| &= \big\|S_{i(n)}^{h(n)-1}x_n - S_{i(n)}^{h(n)-1}x_{n-N}\big\| \\ &\le L\|x_n - x_{n-N}\|\end{aligned} \tag{2.7}$$

and

$$\big\|S_{i(n-N)}^{h(n)-1}x_{n-N} - x_{n-N-1}\big\| = \big\|S_{i(n-N)}^{h(n-N)}x_{n-N} - x_{n-N-1}\big\|. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), we get that

$$\begin{aligned}\|x_{n-1} - S_n x_n\| \le{}& \big\|x_{n-1} - S_{i(n)}^{h(n)}x_n\big\| + L^2\|x_n - x_{n-N}\| \\ &+ L\big\|S_{i(n-N)}^{h(n-N)}x_{n-N} - x_{n-N-1}\big\| + L\|x_{n-N-1} - x_n\|.\end{aligned}$$

It follows from (2.2), (2.4) and (2.5) that

$$\lim_{n\to\infty}\|x_{n-1} - S_n x_n\| = 0. \tag{2.9}$$

Notice that

$$\|x_n - S_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_n\|.$$

This implies from (2.4) and (2.9) that

$$\lim_{n\to\infty}\|x_n - S_n x_n\| = 0. \tag{2.10}$$

For any $j \in I$, we see that

$$\begin{aligned}\|x_n - S_{n+j}x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j}x_{n+j}\| + \|S_{n+j}x_{n+j} - S_{n+j}x_n\| \\ &\le (1+L)\|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j}x_{n+j}\|.\end{aligned}$$

In view of (2.5) and (2.10), we obtain that

$$\lim_{n\to\infty}\|x_n - S_{n+j}x_n\| = 0, \quad \forall j \in I. \tag{2.11}$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n\to\infty}\|x_n - S_l x_n\| = 0, \quad \forall l \in I. \tag{2.12}$$

In a similar way, we can obtain that

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0, \quad \forall l \in I. \tag{2.13}$$

Noting that E is uniformly convex and $\{x_n\}$ is bounded, we have that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $\omega \in C$. In view of Lemma 1.1, we obtain from (2.12) and (2.13) that $\omega \in F$.

Next, we show $\{x_n\}$ converges weakly to ω. Suppose the contrary. Then there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\bar{\omega} \in C$, where $\omega \neq \bar{\omega}$. In the same way, we can show that $\bar{\omega} \in F$. Notice that we have proved that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} \|x_n - \omega\| = Q$, where Q is a nonnegative number. By virtue of Opial's condition of E, we have

$$\begin{aligned} Q &= \liminf_{n_i\to\infty} \|x_{n_i} - \omega\| < \liminf_{n_i\to\infty} \|x_{n_i} - \bar{\omega}\| \\ &= \liminf_{n_j\to\infty} \|x_{n_j} - \bar{\omega}\| < \liminf_{n_j\to\infty} \|x_{n_j} - \omega\| = Q. \end{aligned}$$

This is a contradiction. Hence $\omega = \bar{\omega}$. This completes the proof. □

Remark 2.2. Comparing with Theorem KYO in section 1, we have the following:

(a) From point of view of mappings, the class of nonexpansive mappings is extended to the class of asymptotically nonexpansive mappings.

(b) From point of view of computation, the implicit iterative process (1.3) with errors is more efficient than the implicit iterative process (1.2).

If $S_i = T_i$ for each $i \in I$ in Theorem 2.1, we can get the following results easily.

Corollary 2.3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, \cdots, N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(s_{n,i} - 1) < \infty$ for each $i \in I$. Let $s_n = \max\{s_{n,i} : i \in I\}$. Assume that $F = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C. Let $\{x_n\}$ be the sequence generated in the following iterative process:*

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_n + \gamma_n u_n, \quad \forall n \geq 1. \tag{2.14}$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ satisfy the following restrictions

(a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(b) *there exist constants $a, b \in (0, 1)$ such that $a \leq \alpha_n$ and $b \leq \beta_n$, $\forall n \geq 1$;*

(c) $\beta_n L < 1$, *where $L = \sup_{n\geq 1}\{s_n\}$, $\forall n \geq 1$.*

Then $\{x_n\}$ converges weakly to a point in F.

In 2005, Chidume and Shahzad [3] introduced the following conception. Recall that a family $\{T_i\}_{i=1}^{N} : C \to C$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy *Condition* (B) on C if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1\leq i\leq N} \{\|x - T_i x\|\} \geq f(d(x, F)).$$

Based on Condition (B), we introduced the following conception for two finite families of asymptotically nonexpansive mappings. Recall that two families $\{S_i\}_{i=1}^N : C \to C$ and $\{T_i\}_{i=1}^N : C \to C$ with $F = \big(\bigcap_{i=1}^N F(S_i)\big) \bigcap \big(\bigcap_{i=1}^N F(T_i)\big) \neq \emptyset$ is said to satisfy *Condition* (B') on C if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in C$

$$\max_{1\le i\le N}\{\|x - S_i x\|\} + \max_{1\le i\le N}\{\|x - T_i x\|\} \ge f(d(x,F)).$$

Next, we give strong convergence theorems with the help of Condition (B').

Theorem 2.4. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \ge 1$ be a positive integer and $I = \{1,2,3,\cdots,N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^\infty (s_{n,i} - 1) < \infty$ and T_i be an asymptotically nonexpansive mapping with the sequence $\{t_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^\infty (t_{n,i} - 1) < \infty$ for each $i \in I$. Let $k_n = \max\{s_n, t_n\}$, where $s_n = \max\{s_{n,i} : i \in I\}$ and $t_n = \max\{t_{n,i} : i \in I\}$. Assume that $F = \big(\bigcap_{i=1}^N F(S_i)\big) \bigcap \big(\bigcap_{i=1}^N F(T_i)\big) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in* (1.3). *Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences satisfying $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ for each $n \ge 1$ and $\{u_n\}$ be a bounded sequence in C. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $[0,1]$ satisfy the following restrictions*

(a) $\sum_{n=1}^\infty \delta_n < \infty$;

(b) *there exist constants $a, b, c \in (0,1)$ such that $a \le \alpha_n$, $b \le \beta_n$ and $c \le \gamma_n$,* $\forall n \ge 1$;

(c) $(\beta_n + \gamma_n)L < 1$, *where* $L = \sup_{n\ge1}\{k_n\}$, $\forall n \ge 1$.

If $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ satisfy Condition (B'), then the sequence $\{x_n\}$ converges strongly to a point in F.

Proof. In view of Condition (B'), we obtain from (2.12) and (2.13) that $f(d(x_n,F)) \to 0$, which implies $d(x_n,F) \to 0$. Next, we show that the sequence $\{x_n\}$ is Cauchy. Notice that (2.1) can be rewritten as

$$\begin{aligned}\|x_n - p\| &\le \left(1 + \frac{2(k_{h(n)} - 1)}{a}\right)\|x_{n-1} - p\| + \frac{2\delta_n}{a}\|u_n - p\| \\ &\le e^{\lambda_n}\|x_{n-1} - p\| + \eta_n,\end{aligned}$$

where $\lambda_n = \frac{2(k_{h(n)}-1)}{a}$, $\eta_n = \frac{2\delta_n}{a}\|u_n - p\|$ and n is such that $h(n) \ge n_0$,. For any n, where n is such that $h(n) \ge n_0$, we have

$$\begin{aligned}\|x_{m+n} - x_n\| &\le \|x_{m+n} - p\| + \|x_n - p\| \\ &\le \big(1 + e^{\sum_{k=n+1}^{m+n}\lambda_k}\big)\|x_n - p\| + \sum_{k=n+1}^{m+n} \eta_k e^{\sum_{j=k+1}^{m+n-2}\lambda_j} + \eta_{m+n}.\end{aligned}$$

Since F is closed and convex, we see that $x^* \in F$. This completes the proof. □

If $S_i = T_i$ for each $i \in I$ in Theorem 2.4, we can get the desired results easily.

Corollary 2.5. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, \cdots, N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(s_{n,i} - 1) < \infty$. Let $s_n = \max\{s_{n,i} : i \in I\}$. Assume that $F = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in* (2.14)*. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ satisfy the following restrictions*

(a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(b) *there exist constants $a, b \in (0, 1)$ such that $a \leq \alpha_n$ and $b \leq \beta_n$, $\forall n \geq 1$;*

(c) *$\beta_n L < 1$, where $L = \sup_{n\geq1}\{s_n\}$, $\forall n \geq 1$.*

If $\{S_i\}_{i=1}^{N}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to a point in F.

Recall that a mapping $T : C \to C$ is said to be *semicompact* if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x \in C$.

Next, we give strong convergence theorems with the help of the semicompactness.

Theorem 2.6. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, \cdots, N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(s_{n,i} - 1) < \infty$ and T_i be an asymptotically nonexpansive mapping with the sequence $\{t_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(t_{n,i} - 1) < \infty$ for each $i \in I$. Let $k_n = \max\{s_n, t_n\}$, where $s_n = \max\{s_{n,i} : i \in I\}$ and $t_n = \max\{t_{n,i} : i \in I\}$. Assume that $F = \left(\bigcap_{i=1}^{N} F(S_i)\right) \bigcap \left(\bigcap_{i=1}^{N} F(T_i)\right) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in* (1.3)*. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences satisfying $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $[0, 1]$ satisfy the following restrictions*

(a) $\sum_{n=1}^{\infty} \delta_n < \infty$;

(b) *there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n, b \leq \beta_n$ and $c \leq \gamma_n$, $\forall n \geq 1$;*

(c) *$(\beta_n + \gamma_n)L < 1$, where $L = \sup_{n\geq1}\{k_n\}$, $\forall n \geq 1$.*

If one of $\{S_i\}_{i=1}^{N}$ or one of $\{T_i\}_{i=1}^{N}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to a point in F.

Proof. Without loss of generality, we may assume that S_1 is semicompact. From (2.12), we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to $x \in C$. For each $l \in I$, we get that $\|x - S_l x\| = \lim_{n_i\to\infty} \|x_{n_i} - S_l x_{n_i}\| = 0$. This implies $x \in \bigcap_{l=1}^{N} F(S_l)$. In a similar way, we can get $x \in \bigcap_{l=1}^{N} F(T_l)$. This means that $x \in F$. In view of Theorem 2.1, we obtain that $\lim_{n\to\infty} \|x_n - x\|$ exists. Therefore, we can obtain the desired conclusion immediately. □

If $S_i = T_i$ for each $i \in I$ in Theorem 2.6, we can get the following results easily.

Corollary 2.7. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed convex subset of E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, \cdots, N\}$. Let S_i be an asymptotically nonexpansive mapping with the sequence $\{s_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(s_{n,i} - 1) < \infty$ for each $i \in I$. Let $s_n = \max\{s_{n,i} : i \in I\}$. Assume that $F = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in (2.14). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ satisfy the following restrictions*

(a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(b) *there exist constants $a, b \in (0, 1)$ such that $a \leq \alpha_n$ and $b \leq \beta_n$, $\forall n \geq 1$;*

(c) $\beta_n L < 1$, *where $L = \sup_{n\geq 1}\{s_n\}$, $\forall n \geq 1$.*

If one of $\{S_i\}_{i=1}^{N}$ is semicompact, then $\{x_n\}$ converges strongly to a point in F.

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