Thai Journal of Mathematics
Volume 3 (2005) Number 2 : 171–184

Two-weighted Norm Inequalities for Some Anisotropic Sublinear Operators with Rough Kernel

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Abstract : In this paper, the author established several general theorems for the boundedness of some anisotropic sublinear operators with rough kernel on a weighted Lebesgue space. The conditions of these theorems are satisfied by many important operators in analysis.

Keywords : Weighted Lebesgue space, sublinear operator, anisotropic singular integral, two-weighted inequality.
2000 Mathematics Subject Classification : 42B25, 42B20.

1 Introduction

In the paper, we shall prove the boundedness of some anisotropic sublinear operators with rough kernel on a weighted L_p space. We point out that the condition (1) in the isotropic case was first introduced by Soria and Weiss in [1]. The condition (1) is satisfied by many interesting operators in harmonic analysis, such as the anisotropic Calderon-Zygmund operators, anisotropic Hardy–Littlewood maximal operators, anisotropic R. Fefferman's singular integrals, anisotropic Ricci-Stein's oscillatory singular integrals, the anisotropic Bochner–Riesz means and so on (see [1], [2]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norms

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2},$$

, let $R_0^n = \mathbb{R}^n \setminus \{0\}$, $\Sigma = \Big\{ x \in \mathbb{R}^n : |x| = 1 \Big\}$, $a = (a_1, \ldots, a_n)$, $a_i > 0$, $i = 1, \ldots, n$, $|a| = \sum_{i=1}^{n} a_i$, $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $t^a x \equiv (t^{a_1} x_1, \ldots, t^{a_n} x_n)$, $t > 0$.

Almost everywhere positive and locally integrable function $\omega : \mathbb{R}^n \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega}(\mathbb{R}^n)$ the set of all measurable functions

f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)} \equiv \|f\|_{p,\omega;\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\right)^{1/p}, \qquad 1 \leq p < \infty,$$

is finite.

For $x \in R_0^n$, let $\rho(x)$ be a positive solution to the equation $\sum_{i=1}^n x_i^2 \rho^{-2\,a_i} = 1$. Note that $\rho(x)$ is equivalent to $\sum_{i=1}^n |x_i|^{1/a_i}$, i.e.,

$$c_1 \rho(x) \leq \sum_{i=1}^n |x_i|^{1/a_i} \leq c_2 \rho(x)$$

for certain positive c_1 and c_2 (see [3]).

Definition 1.1 Function K defined on R_0^n, is said to be an anisotropic Calderon-Zygmund(ACZ) kernel in the space $\mathbb{R}^n$ if

(i) $K \in C^\infty(R_0^n)$

(ii) $K(t^a x) \equiv K\left(t^{a_1} x_1, \ldots, t^{a_n} x_n\right) = t^{-|a|} K(x),\ \ t > 0,\ x \in R_0^n$

(iii) $\int_\Sigma K(x) \sum\limits_{i=1}^n a_i x_i^2 d\sigma(x) = 0.$

2 Main Results

First, we establish the boundedness in weighted L_p spaces for a large class of anisotropic sublinear operators with rough kernel.

Theorem 2.1 *Let $p \in (1, \infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ such that, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin$* supp f

$$|Tf(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^{|a|}} |f(y)| dy, \tag{1}$$

where c_0 is independent of f and x, Ω is a-homogeneous of degree zero ($\equiv\ \Omega(t^a x) = \Omega(x)$, for all $t > 0$ and $x \in R_0^n$) and $\Omega \in L_s(\Sigma)$.

Moreover, let $s > p'$, $p' = p/(p-1)$ and $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and the following three conditions be satisfied :

(a) there exists $b > 0$ such that

$$\sup_{\rho(x)/4 < \rho(y) \leq 4\rho(x)} \omega_1(y) \leq b\,\omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

(b)

$$\mathcal{A} \equiv \sup_{r>0} \left(\int_{\rho(x)>2r} \omega_1(x) \rho(x)^{-|a|p/s'} dx\right) \left(\int_{\rho(x)<r} \omega^{1-(p/s')'}(x) dx\right)^{p/s'-1} < \infty,$$

(c)

$$\mathcal{B} \equiv \sup_{r>0} \left(\int_{\rho(x)<r} \omega_1(x)dx \right) \left(\int_{\rho(x)>2r} \omega^{1-(p/s')'}(x)\rho(x)^{-|a|p/s'} dx \right)^{p/s'-1} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(x)dx \le c \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx. \tag{2}$$

Moreover, condition (a) *can be replaced by the condition*

(a') there exist $b > 0$ *such that*

$$\omega_1(x) \left(\sup_{\rho(x)/4 \le \rho(y) \le 4\rho(x)} \frac{1}{\omega(y)} \right) \le b \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $k \in Z$, we define

$$\begin{aligned} E_k &= \Big\{ x \in \mathbb{R}^n : 2^k < \rho(x) \le 2^{k+1} \Big\}, \\ E_{k,1} &= \Big\{ x \in \mathbb{R}^n : \rho(x) \le 2^{k-1} \Big\}, \\ E_{k,2} &= \Big\{ x \in \mathbb{R}^n : 2^{k-1} < \rho(x) \le 2^{k+2} \Big\}, \\ E_{k,3} &= \Big\{ x \in \mathbb{R}^n : \rho(x) > 2^{k+2} \Big\}. \end{aligned}$$

Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^n)$, we write

$$\begin{aligned} |Tf(x)| &= \sum_{k\in Z} |Tf(x)|\chi_{E_k}(x) \\ &\le \sum_{k\in Z} |Tf_{k,1}(x)|\,\chi_{E_k}(x) + \sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(x) + \sum_{k\in Z} |Tf_{k,3}(x)|\,\chi_{E_k}(x) \\ &\equiv T_1f(x) + T_2f(x) + T_3f(x), \end{aligned} \tag{3}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$.

First we shall estimate $\|T_1f\|_{L_{p,\omega_1}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $\rho(y) \le 2^{k-1} \le \rho(x)/2$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $\rho(x - y) \ge \rho(x)/2$.

Hence by (1),

$$\begin{aligned}
T_1 f(x) &\le c_0 \sum_{k\in Z} \left(\int_{\mathbb{R}^n} \frac{|\Omega(x-y)||f_{k,1}(y)|}{\rho(x-y)^{|a|}} dy \right) \chi_{E_k} \\
&\le c_0 \int_{\rho(y)\le\rho(x)/2} \rho(x-y)^{-|a|} |\Omega(x-y)| |f(y)| dy \\
&\le 2^{|a|} c_0 \rho(x)^{-|a|} \int_{\rho(y)\le\rho(x)/2} |\Omega(x-y)||f(y)| dy \\
&\le 2^{|a|} c_0 \rho(x)^{-n} \left(\int_{\rho(y)\le\rho(x)/2} |\Omega(x-y)|^s dy \right)^{1/s} \left(\int_{\rho(y)\le\rho(x)/2} |f(y)|^{s'} dy \right)^{1/s'}
\end{aligned}$$

for any $x \in E_k$. Since

$$\left(\int_{\rho(y)\le\rho(x)/2} |\Omega(x-y)|^s dy \right)^{1/s} \le c_1 \rho(x)^{|a|/s} \|\Omega\|_s,$$

where $\|\Omega\|_s = \left(\int_\Sigma |\Omega(y')|^s d\sigma(y') \right)^{1/s}$, then we have

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x) dx \le c_2 \int_{\mathbb{R}^n} \left(\int_{\rho(y)<\rho(x)/2} |f(y)|^{s'} dy \right)^{1/s'} \rho(x)^{-|a|p/s'} \omega_1(x) dx.$$

Since $\mathcal{A} < \infty$ and $p > s'$, the Hardy inequality

$$\int_{\mathbb{R}^n} \omega_1(x) \rho(x)^{-|a|p/s'} \left(\int_{\rho(y)<\rho(x)/2} |f(y)|^{s'} dy \right)^{p/s'} dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

holds and $C \le c' \mathcal{A}$, where c' depends only on n, s and p. In fact, the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [4], [5]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_1 f(x)|^p \omega_1(x) dx \le c_3 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{4}$$

where c_3 is independent of f.

Next we estimate $\|T_3 f\|_{L_{p,\omega_1}}$. As it is easy to verify, for $x \in E_k$, $y \in E_{k,3}$ we have $\rho(y) > 2\rho(x)$ and $\rho(x-y) \ge \rho(y)/2$. Since $E_k \cap \operatorname{supp} f_{k,3} = \emptyset$, for $x \in E_k$ by

(1), we obtain

$$
\begin{aligned}
T_3 f(x) &\le c_0 \int_{\rho(y)>2\rho(x)} \frac{|\Omega(x-y)||f(y)|}{\rho(x-y)^{|a|}}\, dy \\
&\le 2^{|a|} c_0 \int_{\rho(y)>2\rho(x)} \frac{|\Omega(x-y)||f(y)|}{\rho(y)^{|a|}}\, dy \\
&= 2^{|a|} c_0 \sum_{j=1}^{\infty} \int_{2^j\rho(x)<\rho(y)\le 2^{j+1}\rho(x)} |\Omega(x-y)||f(y)|\rho(y)^{-|a|} dy \\
&\le 2^{|a|} c_0 \sum_{j=1}^{\infty} \left(\int_{2^j\rho(x)<\rho(y)\le 2^{j+1}\rho(x)} |\Omega(x-y)|^s dy \right)^{1/s} \\
&\quad \times \left(\int_{2^j\rho(x)<\rho(y)\le 2^{j+1}\rho(x)} \left(|f(y)|\rho(y)^{-|a|} \right)^{s'} dy \right)^{1/s'} \\
&\le c_4 \sum_{j=1}^{\infty} \left(2^{j+1}\rho(x) \right)^{|a|/s} \left(\int_{2^j\rho(x)<\rho(y)\le 2^{j+1}\rho(x)} \left(|f(y)|\rho(y)^{-|a|} \right)^{s'} dy \right)^{1/s'} \\
&\le 2^{|a|/s} c_4 \sum_{j=1}^{\infty} \left(\int_{2^j\rho(x)<\rho(y)\le 2^{j+1}\rho(x)} \rho(y)^{|a|s'/s} \left(|f(y)|\rho(y)^{-|a|} \right)^{s'} dy \right)^{1/s'} \\
&\le c_5 \left(\int_{\rho(y)>2\rho(x)} |f(y)|^{s'} \rho(y)^{-|a|} dy \right)^{1/s'}.
\end{aligned}
$$

Hence we have

$$
\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x) dx \le c_6 \int_{\mathbb{R}^n} \left(\int_{\rho(y)>2\rho(x)} |f(y)|^{s'} \rho(y)^{-|a|} dy \right)^{p/s'} \omega_1(x) dx.
$$

Since $\mathcal{B} < \infty$ and $p > s'$, the Hardy inequality

$$
\int_{\mathbb{R}^n} \omega_1(x) \left(\int_{\rho(y)>2\rho(x)} |f(y)|^{s'} \rho(y)^{-|a|} dy \right)^{p/s'} dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx
$$

holds and $C \le c'\mathcal{B}$, where c' depends only on n, s and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [4], [5]). Hence, we obtain

$$
\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x) dx \le c_7 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{5}
$$

where c_7 is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^n)$ boundedness of T and condition (a), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x)dx &= \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |Tf_{k,2}(x)|\, \chi_{E_k}(x)\right)^p \omega_1(x)dx \\ &= \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |Tf_{k,2}(x)|^p\, \chi_{E_k}(x)\right) \omega_1(x)dx \\ &= \sum_{k\in Z} \int_{E_k} |Tf_{k,2}(x)|^p\, \omega_1(x)dx \\ &\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n} |Tf_{k,2}(x)|^p\, dx \\ &\le \|T\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n} |f_{k,2}(x)|^p\, dx \\ &= \|T\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y) \int_{E_{k,2}} |f(x)|^p dx,\end{aligned}$$

where $\|T\| \equiv \|T\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}$. Since for $x \in E_{k,2}$, $2^{k-1} < \rho(x) \le 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} \omega_1(y) = \sup_{2^{k-1}<\rho(y)\le 2^{k+2}} \omega_1(y) \le \sup_{\rho(x)/4<\rho(y)\le 4\rho(x)} \omega_1(y) \le b\,\omega(x)$$

for almost all $x \in E_{k,2}$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x)dx &\le \|T\|^p b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p \omega(x)dx \\ &\le c_7 \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx, \qquad (6)\end{aligned}$$

where $c_7 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (3), (4), (5) and (6) imply (2), thus completing the proof. □

Let K be anisotropic Calderon–Zygmund kernel and T be the corresponding integral operator

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$

Then T satisfies the condition (1). See [1], [2] for details. Thus, we have

Corollary 2.2 *Let $p \in (1,\infty)$, K be anisotropic Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $s > p'$, $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and conditions (a), (b) and (c) be satisfied. Then inequality (2) is valid.*

Remark 2.3 Note that, Corollary 2.2 in the case $s = \infty$ was proved in [6] and for singular integral operators, defined on homogeneous groups in [7], [8] (see also [9], [10]).

Theorem 2.4 *Let $p \in (1, \infty)$, T be anisotropic sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Let also Ω is a-homogeneous of degree zero and $\Omega \in L_s(\Sigma)$. Moreover, let $s > p'$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the conditions (a) and (b).*

Then there exists a constant $c > 0$, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x)) dx \le c \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x)) dx. \tag{7}$$

Proof. Suppose that $f \in L_{p,\omega}(\mathbb{R}^n)$ and ω_1 are positive increasing functions on $(0, \infty)$ and $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfied the conditions (a) and (b).

Without loss of generality, we can suppose that ω_1 can be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda) d\lambda,$$

where $\omega_1(0+) = \lim_{t \to 0} \omega_1(t)$ and $\omega_1(t) \ge 0$ on $(0, \infty)$. In fact, there exists a sequence of increasing absolutely continuous functions ϖ_m such that $\varpi_m(t) \le \omega_1(t)$ and $\lim\limits_{m \to \infty} \varpi_m(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [9], [10], [11] and [12] for details).

We have

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x)) dx &= \omega_1(0+) \int_{\mathbb{R}^n} |Tf(x)|^p dx + \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_0^{\rho(x)} \psi(\lambda) d\lambda \right) dx \\ &= J_1 + J_2. \end{aligned}$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \ne 0$, by the boundedness of T in $L_p(\mathbb{R}^n)$, thanks to (a), we have

$$\begin{aligned} J_1 &\le \|T\|^p \omega_1(0+) \int_{\mathbb{R}^n} |f(x)|^p dx \\ &\le \|T\|^p \int_{\mathbb{R}^n} |f(x)|^p \omega_1(\rho(x)) dx \\ &\le b \, \|T\|^p \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x)) dx. \end{aligned}$$

After changing the order of integration in J_2, we have

$$\begin{aligned} J_2 &= \int_0^\infty \psi(\lambda)\left(\int_{\rho(x)>\lambda} |Tf(x)|^p dx\right) d\lambda \\ &\leq 2^{p-1}\int_0^\infty \psi(\lambda)\left(\int_{\rho(x)>\lambda} |T(f\chi_{\{\rho(x)>\lambda/2\}})(x)|^p dx\right. \\ &\quad \left. + \int_{\rho(x)>\lambda} |T(f\chi_{\{\rho(x)\leq\lambda/2\}})(x)|^p dx\right) d\lambda \\ &= J_{21} + J_{22}. \end{aligned}$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a), we have

$$\begin{aligned} J_{21} &\leq \|T\|^p \int_0^\infty \psi(t)\left(\int_{\rho(y)>\lambda/2} |f(y)|^p dy\right) dt \\ &= \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \left(\int_0^{2\rho(y)} \psi(\lambda) d\lambda\right) dy \\ &\leq \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \omega_1(2\rho(y)) dy \\ &\leq b\,\|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y)) dy. \end{aligned}$$

Let us estimate J_{22}. For $\rho(x) > \lambda$ and $\rho(y) \leq \lambda/2$, we have $\rho(x)/2 \leq \rho(x-y) \leq 3\rho(x)/2$, and so

$$\begin{aligned} J_{22} &\leq c_0 \int_0^\infty \psi(\lambda)\left(\int_{\rho(x)>\lambda}\left(\int_{\rho(y)\leq 2\lambda} \frac{|\Omega(x-y)||f(y)|}{\rho(x-y)^{|a|}} dy\right)^p dx\right) d\lambda \\ &\leq 2^{|a|} c_0 \int_0^\infty \psi(\lambda)\left(\int_{\rho(x)>\lambda}\left(\int_{\rho(y)\leq 2\lambda} |\Omega(x-y)||f(y)| dy\right)^p \rho(x)^{-|a|p} dx\right) d\lambda \\ &= c_8 \int_0^\infty \psi(\lambda)\lambda^{-|a|p+|a|}\left(\int_{\rho(y)\leq\lambda/2} |\Omega(x-y)||f(y)| dy\right)^p d\lambda \\ &\leq c_8 \int_0^\infty \psi(\lambda)\lambda^{-|a|p+|a|}\left(\int_{\rho(y)\leq\lambda/2} |\Omega(x-y)|^s dy\right)^{p/s}\left(\int_{\rho(y)\leq\lambda/2} |f(y)|^{s'} dy\right)^{p/s'} d\lambda \\ &\leq c_9 \int_0^\infty \psi(\lambda)\lambda^{-|a|p/s'+|a|}\left(\int_{\rho(y)\leq\lambda/2} |f(y)|^{s'} dy\right)^{p/s'} d\lambda. \end{aligned}$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{-|a|p/s'+|a|}\left(\int_{\rho(y)\le\lambda/2}|f(y)|^{s'}dy\right)^{p/s'}d\lambda$$

$$\le C\int_{\mathbb{R}^n}|f(y)|^p\omega(\rho(y))dy,$$

for $p > s'$ is characterized by the condition $C \le c'\mathcal{A}'$ (see [4], [5], also [13], [14]), where

$$\mathcal{A}' \equiv \sup_{\tau>0}\left(\int_{2\tau}^\infty \psi(t)t^{-|a|p/s'+|a|}d\tau\right)\left(\int_{\rho(x)<r}\omega^{1-(p/s')'}(\rho(x))dx\right)^{p/s'-1} < \infty.$$

Note that

$$\begin{aligned}\int_{2t}^\infty \psi(\tau)\tau^{-|a|p/s'+|a|}d\tau &= |a|(p/s'-1)\int_{2t}^\infty \psi(\tau)d\tau\int_\tau^\infty \lambda^{|a|-1-|a|p/s'}d\lambda\\ &= |a|(p/s'-1)\int_{2t}^\infty \lambda^{|a|-1-|a|p/s'}d\lambda\int_{2t}^\lambda \psi(\tau)d\tau\\ &\le |a|(p/s'-1)\int_{2t}^\infty \lambda^{|a|-1-|a|p/s'}\omega_1(\lambda)d\lambda\\ &= c_{10}\int_{\rho(x)>2r}\omega_1(\rho(x))\rho(x)^{-|a|p/s'}dx.\end{aligned}$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \le c_{10}\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c\int_{\mathbb{R}^n}|f(x)|^p\omega(\rho(x))dx,$$

where $c > 0$ is independent of f.

Combining the estimates of J_1 and J_2, we get (7) for

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau.$$

By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (7). The theorem is proved. □

Corollary 2.5 *Let $p \in (1,\infty)$, K be a Calderon-Zygmund kernel and T be the corresponding operator. Moreover, let $s > p'$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the conditions (a) and (b). Then inequality (7) is valid.*

Theorem 2.6 *Let $p \in (1,\infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (1). Let also Ω be a-homogeneous of degree zero and $\Omega \in L_s(\Sigma)$. Moreover, let $s > p'$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the conditions (a) and (c). Then inequality (7) is valid.*

Proof. Without loss of generality, we can suppose that ω_1 can be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$.

In fact, there exists a sequence of decreasing absolutely continuous functions ϖ_m, such that $\varpi_m(t) \leq \omega_1(t)$ and $\lim_{m\to\infty} \varpi_m(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [9], [10], [11], [12] for details).

We have

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x))dx &= \omega_1(+\infty)\int_{\mathbb{R}^n} |Tf(x)|^p dx + \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_{\rho(x)}^\infty \psi(\tau)d\tau\right) dx \\ &= I_1 + I_2.\end{aligned}$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a), we have

$$\begin{aligned}J_1 &\leq \|T\|\omega_1(+\infty)\int_{\mathbb{R}^n} |f(x)|^p dx \\ &\leq \|T\|\int_{\mathbb{R}^n} |f(x)|^p \omega_1(\rho(x))dx \\ &\leq b\,\|T\|\int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x))dx.\end{aligned}$$

After changing the order of integration in J_2 we have

$$\begin{aligned}J_2 &= \int_0^\infty \psi(\lambda)\left(\int_{\rho(x)<\lambda} |Tf(x)|^p dx\right) d\lambda \\ &\leq 2^{p-1}\int_0^\infty \psi(\lambda)\left(\int_{\rho(x)<\lambda} |T(f\chi_{\{\rho(x)<2\lambda\}})(x)|^p dx\right. \\ &\quad \left. + \int_{\rho(x)<\lambda} |T(f\chi_{\{\rho(x)\geq 2\lambda\}})(x)|^p dx\right) d\lambda \\ &= J_{21} + J_{22}.\end{aligned}$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a), we obtain

$$\begin{aligned} J_{21} &\le \|T\| \int_0^\infty \psi(t) \left(\int_{\rho(y)<2\lambda} |f(y)|^p dy \right) dt \\ &= \|T\| \int_{\mathbb{R}^n} |f(y)|^p \left(\int_{\rho(y)/2}^\infty \psi(\lambda) d\lambda \right) dy \\ &\le \|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega_1(\rho(y)/2) dy \\ &\le b\, \|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y)) dy. \end{aligned}$$

Let us estimate J_{22}. For $\rho(x) < \lambda$ and $\rho(y) \ge 2\lambda$ we have $\rho(y)/2 \le \rho(x-y) \le 3\rho(y)/2$, and so

$$\begin{aligned} J_{22} &\le c_0 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\int_{\rho(y)\ge 2\lambda} \frac{|\Omega(x-y)||f(y)|}{\rho(x-y)^{|a|}} dy \right)^p dx \right) d\lambda \\ &\le 2^{|a|} c_0 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\int_{\rho(y)\ge 2\lambda} \frac{|\Omega(x-y)||f(y)|}{\rho(y)^{|a|}} dy \right)^p dx \right) d\lambda \\ &\le 2^{|a|} c_0 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\sum_{j=1}^\infty \int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} |\Omega(x-y)||f(y)|\rho(y)^{-|a|} dy \right)^p dx \right) d\lambda \\ &\le 2^{|a|} c_0 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\sum_{j=1}^\infty \left(\int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} |\Omega(x-y)|^s dy \right)^{1/s} \right.\right. \\ &\quad \times \left.\left. \left(\int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} \left(|f(y)|\rho(y)^{-|a|} \right)^{s'} dy \right)^{1/s'} \right)^p dx \right) d\lambda \\ &\le c_{10} \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\sum_{j=1}^\infty \left(2^{j+1}\lambda\right)^{|a|/s} \right.\right. \\ &\quad \times \left.\left. \left(\int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} \left(|f(y)|\rho(y)^{-|a|} \right)^{s'} dy \right)^{1/s'} \right)^p dx \right) d\lambda \\ &\le c_{11} \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\sum_{j=1}^\infty \left(\int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} |f(y)|^{s'} \rho(y)^{-|a|} dy \right)^{1/s'} \right)^p dx \right) d\lambda \\ &\le c_{12} \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\sum_{j=1}^\infty \int_{2^j\lambda<\rho(y)\le 2^{j+1}\lambda} |f(y)|^{s'} \rho(y)^{-|a|} dy \right)^{p/s'} dx \right) d\lambda \end{aligned}$$

$$= c_{12}\int_0^\infty \psi(\lambda)\left(\int_{\rho(x)<\lambda}\left(\int_{\rho(y)>2\lambda}|f(y)|^{s'}\rho(y)^{-|a|}dy\right)^{p/s'}dx\right)d\lambda$$

$$= c_{13}\int_0^\infty \psi(\lambda)\lambda^{|a|}\left(\int_{\rho(y)>2\lambda}|f(y)|^{s'}\rho(y)^{-|a|}dy\right)^{p/s'}d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{|a|}\left(\int_{\rho(y)\geq 2\lambda}|f(y)|^{s'}\rho(y)^{-|a|}dy\right)^{p/s'}d\lambda \leq C\int_{\mathbb{R}^n}|f(y)|^p\omega(\rho(y))dy$$

for $p > s'$ is characterized by the condition $C \leq c\mathcal{B}'$ (see [4], [5], also [13], [14]), where

$$\mathcal{B}' \equiv \sup_{\tau>0}\left(\int_0^\tau \psi(t)t^{|a|}d\tau\right)\left(\int_{\rho(x)>2r}\omega^{1-(p/s')'}(x)\rho(x)^{-|a|p/s'}dx\right)^{p/s'-1} < \infty.$$

Note that

$$\begin{aligned}\int_0^\tau \psi(t)t^{|a|}dt &= |a|\int_0^\tau \psi(t)dt\int_0^t \lambda^{|a|-1}d\lambda \\ &= |a|\int_0^\tau \lambda^{|a|-1}d\lambda\int_\lambda^t \psi(\tau)d\tau \leq |a|\int_0^\tau \lambda^{|a|-1}\omega(\lambda)d\lambda \\ &= c_{14}\int_{\rho(x)<r}\omega_1(\rho(x))dx.\end{aligned}$$

Condition (c) of the theorem guarantees that $\mathcal{B}' \leq |a|\mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{14}\int_{\mathbb{R}^n}|f(x)|^p\omega(\rho(x))dx.$$

Combining the estimates of J_1 and J_2, we get (7) for $\omega_1(t) = \omega_1(+\infty)+\int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (7). The theorem is proved. □

Corollary 2.7 *Let $p \in (1,\infty)$, K be a anisotropic Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $s > p'$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the conditions (a) and (c). Then inequality (7) is valid.*

Acknowledgements

The author would like to express his thanks to Prof. Zhijian Wu and Prof. Vagif Guliyev for many helpful discussions about this subject.

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(Received 11 July 2005)

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