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Analytic Solutions of a Second-Order Functional Differential Equation

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Abstract : A second-order functional differential equation

$$x''(z) = \frac{1}{c_1 x(z) + c_2 x(az + bx'(z))}$$

with the distinctive feature that the argument of the unknown function depends on the state derivative was investigated. An existence theorem was proposed for analytic solutions. The explicit analytic solutions were obtained for two different cases of b, $b = 0$ and $b \neq 0$. In the case $b \neq 0$, the Shchröder transformation was introduced to get the auxiliary equation for deriving the explicit solution.

Keywords : Functional differential equations; Local analytic solution; Shchröder transformation.
2000 Mathematics Subject Classification : 34K25.

1 Introduction

Delay differential equations have been studied rather extensively in the past forty years and are used as models to describe many physical and biological systems (see [1, 7]). Some interesting properties of these special equations were investigated and provided insights into the main theories. Since such equations are quite different from the usual differential equations, the standard existence and uniqueness theorems cannot be applied directly. It is therefore of interest to find some or all of their solutions. In [2-3, 5-6], analytic solutions of the state derivative dependent delay functional differential equations were found.

In [4], J. Si and X. Wang studied the existence of analytic solutions of the equation with state derivative dependent delay

$$x''(z) = x\left(az + bx'(z)\right), \tag{1.1}$$

where a and b are complex numbers. This equation was written in the form

$$x''(z) = f\left(x\left(z - \tau(z)\right)\right), \tag{1.2}$$

with $f(z) = z$ and $\tau(z) = (1-a)z - bx'(z)$. Here, the delay function $\tau(z)$ depends not only on the argument of unknown function, but also on the state derivative.

In [10], T. Liu and H. Li verified the existence of analytic solutions of the equation with state derivative dependent delay

$$x''(z) = \frac{1}{x\left(az + bx'(z)\right)}, \tag{1.3}$$

where a and b are complex numbers.

For more general case of (1.1), S. Pengpit, T. Kaewong and K. Kongkul investigated a second-order functional differential equation

$$x''(z) = c_1 x(z) + c_2 x\left(az + bx'(z)\right), \tag{1.4}$$

where c_1, c_2, a and b are complex numbers (see [9]). In this study, we proposed the existence theorem and obtained the explicit analytic solution of (1.4).

Consequently, we interested to find the analytic solutions of a second-order functional differential equation

$$x''(z) = \frac{1}{c_1 x(z) + c_2 x\left(az + bx'(z)\right)}, \tag{1.5}$$

where c_1, c_2, a and b are complex numbers. By the conditions of the parameter b, $b = 0$ and $b \neq 0$, we considered the explicit analytic solutions of (1.5). We derived the explicit analytic solutions of (1.5) when $b = 0$ while the Shchröder transformation was introduced in order to get the auxiliary equation and finally, we obtained the explicit solution $x(z)$ when $b \neq 0$.

2 Explicit analytic solutions

In the case $b = 0$, the functional differential equation (1.5) becomes the functional differential equation

$$x''(z) = \frac{1}{c_1 x(z) + c_2 x(az)}, \tag{2.1}$$

where c_1, c_2 and a are complex numbers. For this equation we proposed an interesting proposition as follow.

Proposition 2.1. *Suppose* $0 < |a| \leqslant 1$. *Then the functional differential equation (2.1) has an analytic solution* $x(z)$, *in a neighborhood of the origin, satisfying* $x(0)x''(0) = \frac{1}{c_1+c_2}$ *and the initial value conditions* $x(0) = \mu$, $x'(0) = \eta \in \mathbb{C}\backslash\{0\}$.

Proof. Let

$$x(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.2}$$

be the expansion of the formal solution $x(z)$ of (2.1). Substituting (2.2) into (2.1), we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (c_1 + c_2 a^k)(n+2-k)(n+1-k) a_k a_{n+2-k} \right) z^n = 1.$$

By means of the method of undetermined coefficients, we obtain

$$a_0 a_2 = \frac{1}{2(c_1 + c_2)},$$

$$\sum_{k=0}^{n} (c_1 + c_2 a^k)(n+2-k)(n+1-k) a_k a_{n+2-k} = 0, n = 1, 2, \ldots.$$

If we choose $a_0 = \mu \neq 0$, $a_1 = \eta \neq 0$ and $a_2 = \frac{1}{2(c_1+c_2)\mu}$, then the sequence $\{a_n\}_{n=3}^{\infty}$ is successively determined by

$$a_{n+2} = -\frac{1}{(c_1 + c_2)(n+2)(n+1)\mu} \sum_{k=1}^{n} (c_1 + c_2 a^k)(n+2-k)(n+1-k) a_k a_{n+2-k}, n = 1, 2, \ldots. \tag{2.3}$$

Now we show that the power series (2.2) converges in a neighborhood of the origin. First of all, note that

$$|a_{n+2}| \leqslant \frac{|c_1| + |c_2|}{|c_1 + c_2|} \frac{1}{|\mu|} \sum_{k=1}^{n} |a_k| \, |a_{n+2-k}|,$$

thus if we define recursively a sequence $\{B_n\}_{n=0}^{\infty}$ by $B_0 = |\mu|$, $B_1 = |\eta|$, $B_2 = \frac{1}{2|c_1+c_2||\mu|}$,

$$B_{n+2} = \frac{|c_1| + |c_2|}{|c_1 + c_2|} \frac{1}{|\mu|} \sum_{k=1}^{n} B_k B_{n+2-k}, n = 1, 2, \ldots, \tag{2.4}$$

then we can show that by induction

$$|a_n| \leqslant B_n, n = 1, 2, \ldots.$$

Now if we define

$$M(z) = \sum_{n=1}^{\infty} B_n z^n, \tag{2.5}$$

then

$$M^2(z) = |\eta|\, zM(z) + \frac{|c_1 + c_2|}{|c_1| + |c_2|}\, |\mu| \left(M(z) - |\eta|\, z - \frac{1}{2\, |c_1 + c_2|\, |\mu|} z^2 \right),$$

that is

$$M^2(z) - \left(\frac{|c_1 + c_2|}{|c_1| + |c_2|}\, |\mu| + |\eta|\, z \right) M(z) + \frac{|c_1 + c_2|}{|c_1| + |c_2|}\, |\mu|\, |\eta|\, z + \frac{1}{2(|c_1| + |c_2|)} z^2 = 0. \tag{2.6}$$

Let

$$R(z, w) = w^2 - \left(\frac{|c_1 + c_2|}{|c_1| + |c_2|}\, |\mu| + |\eta|\, z \right) w + \frac{|c_1 + c_2|}{|c_1| + |c_2|}\, |\mu|\, |\eta|\, z + \frac{1}{2(|c_1| + |c_2|)} z^2,$$

for (z, w) from a neighborhood of $(0, 0)$. Since $R(0, 0) = 0$, $R'_w(0, 0) = -\frac{|c_1+c_2|}{|c_1|+|c_2|}\, |\mu| \neq 0$, there exists a unique function $w(z)$, analytic in a neighborhood of zero, such that $w(0) = 0, w'(0) = |\eta|$ and $R\,(z, w(z)) = 0$. According to (2.5) and (2.6), we have $M(z) = w(z)$. It follows that the power series (2.5) converges in a neighborhood of the origin, which implies that the power series (2.2) is also converges in a neighborhood of the origin. The proof is complete. □

If we assume that $a_0 = \mu$, $a_1 = \eta$ and $a_2 = \frac{1}{2(c_1+c_2)\mu}$, we calculate the coefficients a_n by means of (2.3). Indeed the first few terms are as follows:

$$a_3 = -\frac{(c_1 + c_2 a)\eta}{6(c_1 + c_2)^2 \mu^2}, a_4 = -\frac{(c_1 + c_2 a^2) - 2(c_1 + c_2 a)^2 \eta^2}{24(c_1 + c_2)^3 \mu^3}, \ldots.$$

Thus, the explicit solution of (2.1) is

$$x(z) = \mu + \eta z + \frac{1}{2(c_1 + c_2)\mu} z^2 - \frac{(c_1 + c_2 a)\eta}{6(c_1 + c_2)^2 \mu^2} z^3 - \frac{(c_1 + c_2 a^2) - 2(c_1 + c_2 a)^2 \eta^2}{24(c_1 + c_2)^3 \mu^3} z^4 + \ldots.$$

3 Analytic solutions of the auxiliary equation

A distinctive feature of the functional differential equation (1.5) when $b \neq 0$ is that the argument of the unknown function is dependent on the state derivative $x'(z)$. We explain the existence of analytic solution of (1.5) by locally reducing the equation to another functional differential equation with proportional delays. Let

$$y(z) = az + bx'(z). \tag{3.1}$$

Then for any number z_0, we obtain

$$x(z) = x(z_0) + \frac{1}{b}\int_{z_0}^{z} (y(s) - as)\, ds, \tag{3.2}$$

and so $x(y(z)) = x(z_0) + \frac{1}{b}\int_{z_0}^{y(z)} (y(s) - as)\, ds$. From (1.5) , we can write

$$(c_1 + c_2)x(z_0) + \frac{1}{b}\left[c_1\int_{z_0}^{z} (y(s) - as)\, ds + c_2\int_{z_0}^{y(z)} (y(s) - as)\, ds\right] = \frac{b}{y'(z) - a}. \tag{3.3}$$

If z_0 is a fixed point of $y(z)$, we see that

$$x(z_0) = \frac{1}{c_1 + c_2}\frac{b}{y'(z_0) - a}. \tag{3.4}$$

Furthermore, differentiating both sides of (3.3) with respect to z, we obtain

$$-b^2 y''(z) = c_1 [y(z) - az]\, [y'(z) - a]^2 + c_2 [y(y(z)) - ay(z)]\, [y'(z) - a]^2 y'(z). \tag{3.5}$$

By the Shchröder transformation, we get the auxiliary equation

$$\begin{aligned} &\lambda b^2 [g'(\lambda z)g''(z) - \lambda g''(\lambda z)g'(z)] \\ &\quad = c_1 [g(\lambda z) - ag(z)]\, [\lambda g'(\lambda z) - ag'(z)]^2 g'(z) \\ &\qquad + c_2\lambda \left[g(\lambda^2 z) - ag(\lambda z)\right]\, [\lambda g'(\lambda z) - ag'(z)]^2 g'(\lambda z). \end{aligned} \tag{3.6}$$

The equation (3.6) satisfies the initial value conditions

$$g(0) = \mu, g'(0) = \eta \neq 0, \tag{3.7}$$

where $\lambda \neq a$, μ and η are complex numbers, and λ satisfies either

(A1) $0 < |\lambda| < 1$; or

(A2) $|\lambda| = 1$, λ is not a root of unity, and $\log|\lambda^n - 1|^{-1} \leq T \log n$, $n = 2, 3, \ldots$ for some positive constant T. Then we show that (3.5) has an analytic solution of the form

$$y(z) = g\left(\lambda g^{-1}(z)\right) \tag{3.8}$$

in a neighborhood of μ. Here $g^{-1}(z)$ denotes the inverse function of $g(z)$. We begin with the following preparatory lemma the proof of which can be followed in [8].

Lemma 3.1. *Assume that (A2) holds. Then there is a positive number δ such that $|\lambda^n - 1|^{-1} < (2n)^{\delta}$ for $n = 1, 2, \ldots$. Furthermore, the sequence $\{d_n\}_{n=1}^{\infty}$ is defined by $d_1 = 1$ and*

$$d_n = \frac{1}{|\lambda^{n-1} - 1|} \max_{\substack{n=n_1+\cdots+n_t \\ 0<n_1\leq\cdots\leq n_t, t\geq 2}} \{d_{n_1} \cdots d_{n_t}\}, n = 2, 3, \ldots$$

satisfy

$$d_n \leq (2^{5\delta+1})^{n-1} n^{-2\delta}, n = 1, 2, \ldots.$$

To find analytic solution of (3.5), we solve the auxiliary equation (3.6) satisfying the initial value conditions (3.7) to obtain an analytic solution $g(z)$.

Lemma 3.2. *Suppose (A1) holds. Then, for the initial value conditions (3.7), the auxiliary equation (3.6) has an analytic solution of the form*

$$g(z) = \mu + \eta z + \sum_{n=2}^{\infty} b_n z^n \tag{3.9}$$

in a neighborhood of the origin.

Proof. Rewrite (3.6) in the form

$$\frac{\lambda b^2}{\lambda - a} \left(\frac{g'(z) - g'(\lambda z)}{\lambda g'(\lambda z) - a g'(z)} \right) = \int_0^z c_1 \left[g(\lambda s) - a g(s) \right] g'(s) ds + \int_0^z c_2 \lambda \left[g(\lambda^2 s) - a g(\lambda s) \right] g'(\lambda s) ds.$$

Therefore, in view of $g'(0) = \eta \neq 0$ and $\lambda \neq a$, we have

$$\frac{\lambda b^2}{\lambda - a} \left(g'(z) - g'(\lambda z) \right) = \left(\lambda g'(\lambda z) - a g'(z) \right) \int_0^z c_1 \left[g(\lambda s) - a g(s) \right] g'(s) ds + \left(\lambda g'(\lambda z) - a g'(z) \right) \int_0^z c_2 \lambda \left[g(\lambda^2 s) - a g(\lambda s) \right] g'(\lambda s) ds. \tag{3.10}$$

We now solve for a solution of (3.10) in the form of a power series (3.9). By defining $b_0 = \mu$, $b_1 = \eta$ and then substituting (3.9) into (3.10), we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$b_{n+2} = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{c_1(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)}{\lambda b^2 (n-j+1)(n+2)(1 - \lambda^{n+1})} b_{j+1} b_{k+1} b_{n-j-k} + \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{c_2(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)\lambda^{n-j}}{b^2 (n-j+1)(n+2)(1 - \lambda^{n+1})} b_{j+1} b_{k+1} b_{n-j-k}, \tag{3.11}$$

$n = 0, 1, 2, \dots$ in a unique manner. We need to show that the resulting power series (3.9) converges in a neighborhood of the origin. First of all, note that

$$\left|\frac{c_1(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)}{\lambda b^2(n-j+1)(n+2)(1-\lambda^{n+1})}\right| \le \frac{|c_1|\,(1+|a|)^3}{|\lambda b^2|\,|1-\lambda^{n+1}|} \le M_1,$$

$$\left|\frac{c_2(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)\lambda^{n-j}}{b^2(n-j+1)(n+2)(1-\lambda^{n+1})}\right| \le \frac{|c_2|\,(1+|a|)^3}{|b^2|\,|1-\lambda^{n+1}|} \le M_2$$

for some positive number M_1, M_2. Let $M_3 = \max\{M_1, M_2\}$, then in view of the expression (3.11), we have

$$|b_{n+2}| \le 2M_3 \sum_{j=0}^{n} \sum_{k=0}^{n-j} |b_{j+1}||b_{k+1}||b_{n-j-k}|, n = 0, 1, 2, \dots,$$

thus if we define a sequence $\{D_n\}_{n=0}^{\infty}$ by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+2} = 2M_3 \sum_{j=0}^{n} \sum_{k=0}^{n-j} D_{j+1} D_{k+1} D_{n-j-k}, n = 0, 1, 2, \dots.$$

We can show that by induction

$$|b_n| \le D_n, n = 0, 1, 2, \dots.$$

Now if we define

$$G(z) = \sum_{n=0}^{\infty} D_n z^n, \tag{3.12}$$

then

$$G^2(z) = |\mu| \sum_{n=0}^{\infty} D_n z^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} D_{k+1} D_{n-k} z^{n+1},$$

$$G^3(z) = 2|\mu| G^2(z) + \left(\frac{1}{2M_3} - |\mu|^2\right) G(z) - \frac{1}{2M_3}\left(|\eta| z + |\mu|\right),$$

that is

$$G^3(z) - 2|\mu| G^2(z) - \left(\frac{1}{2M_3} - |\mu|^2\right) G(z) + \frac{1}{2M_3}\left(|\eta| z + |\mu|\right) = 0. \tag{3.13}$$

Let

$$R(z, w) = w^3 - 2|\mu| w^2 - \left(\frac{1}{2M_3} - |\mu|^2\right) w + \frac{1}{2M_3}\left(|\eta| z + |\mu|\right)$$

for (z, w) from a neighborhood of $(0, |\mu|)$. Since $R(0, |\mu|) = 0$ and $R'_w(0, |\mu|) = -\frac{1}{2M_3} \neq 0$, there exists a unique function $w(z)$, analytic in a neighborhood of zero, such that $w(0) = |\mu|, w'(0) = |\eta|$ and $R(z, w(z)) = 0$. By (3.12) and (3.13), we have $G(z) = w(z)$. It follows that the power series (3.12) converges in a neighborhood of the origin, and hence also (3.9), converges in a neighborhood of the origin. The proof is complete. □

Lemma 3.3. *Suppose (A2) holds. Then if $\eta \neq 0$, the auxiliary equation (3.6) has an analytic solution of the form (3.9) in a neighborhood of the origin.*

Proof. Note that

$$\left|\frac{c_1(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)}{\lambda b^2(n-j+1)(n+2)(1-\lambda^{n+1})}\right| \leq \frac{|c_1|\,(1+|a|)^3}{|b|^2}\frac{1}{|\lambda^{n+1} - 1|},$$

$$\left|\frac{c_2(\lambda - a)(j+1)(k+1)(\lambda^{j+1} - a)(\lambda^{n-j-k} - a)\lambda^{n-j}}{b^2(n-j+1)(n+2)(1-\lambda^{n+1})}\right| \leq \frac{|c_2|\,(1+|a|)^3}{|b|^2}\frac{1}{|\lambda^{n+1} - 1|}.$$

Let $M_4 = \max\{\frac{|c_1|(1+|a|)^3}{|b|^2}, \frac{|c_2|(1+|a|)^3}{|b|^2}\}$, set $b_0 = \mu$ and $b_1 = \eta$, then (3.11) again holds so that

$$b_{n+2} \leq \frac{2M_4}{|\lambda^{n+1} - 1|}\sum_{j=0}^{n}\sum_{k=0}^{n-j}|b_{j+1}||b_{k+1}||b_{n-j-k}|, n = 0, 1, 2, \ldots. \tag{3.14}$$

Let us now consider the equation

$$Q(z, w) = w^3 - 2|\mu|w^2 - \left(\frac{1}{2M_4} - |\mu|^2\right)w + \frac{1}{2M_4}\left(|\eta|z + |\mu|\right) = 0. \tag{3.15}$$

If

$$w(z) = |\mu| + |\eta|z + \sum_{n=2}^{\infty} C_n z^n, \tag{3.16}$$

where the coefficient sequence $\{C_n\}_{n=0}^{\infty}$ satisfies $C_0 = |\mu|$, $C_1 = |\eta|$,

$$C_{n+2} = 2M_4\sum_{j=0}^{n}\sum_{k=0}^{n-j}C_{j+1}C_{k+1}C_{n-j-k}, n = 0, 1, 2, \ldots,$$

then

$$w^2(z) = |\mu|\sum_{n=0}^{\infty}C_n z^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n}C_{k+1}C_{n-k}z^{n+1},$$

$$w^3(z) = 2|\mu|w^2(z) + \left(\frac{1}{2M_4} - |\mu|^2\right) w(z) - \frac{1}{2M_4}\left(|\eta|z + |\mu|\right),$$

or

$$w^3(z) - 2|\mu|w^2(z) - \left(\frac{1}{2M_4} - |\mu|^2\right) w(z) + \frac{1}{2M_4}\left(|\eta|z + |\mu|\right) = 0,$$

that is, $w(z)$ satisfies the equation (3.15) for (z, w) from a neighborhood of $(0, |\mu|)$. Since $Q(0, |\mu|) = 0$ and $Q'_w(0, |\mu|) = -\frac{1}{2M_4} \neq 0$, there exists a unique function $w(z)$, analytic in a neighborhood of zero, such that $w(0) = |\mu|, w'(0) = |\eta|$, and $Q(z, w(z)) = 0$. It follows that the power series (3.16) converges in a neighborhood of zero, and there is a positive constant T such that

$$C_n \leq T^n, n = 1, 2, \ldots. \tag{3.17}$$

By induction, we have

$$|b_n| \leq C_n d_n, n = 1, 2, \ldots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 3.1. In view of (3.17) and Lemma 3.1, we finally see that

$$|b_n| \leq T^n (2^{5\delta+1})^{n-1} n^{-2\delta}, n = 1, 2, \ldots,$$

which shows that the power series (3.9) converges for

$$|z| < \frac{1}{T 2^{5\delta+1}}.$$

The proof is complete. □

Theorem 3.1. *Suppose the conditions of Lemma 3.2 or Lemma 3.3 are satisfied. Then the equation (3.5) has an analytic solution $y(z)$ of the form (3.8) in a neighborhood of the number μ, where $g(z)$ of the form (3.9) is an analytic solution of (3.6).*

Proof. In view of Lemma 3.2 and Lemma 3.3, the function $g(z)$ of the form (3.9) is an analytic solution of (3.6) in a neighborhood of the origin. Since $g'(0) = \eta \neq 0$, the function $g^{-1}(z)$ is analytic in a neighborhood of $g(0) = \mu$. If we now define $y(z)$ by means of (3.8), then

$$y'(z) = \lambda g'\left(\lambda g^{-1}(z)\right)\left(g^{-1}(z)\right)' = \lambda \frac{g'\left(\lambda g^{-1}(z)\right)}{g'\left(g^{-1}(z)\right)},$$

$$
\begin{aligned}
&-b^2y''(z)\\
&\quad= -b^2\frac{\lambda^2 g''\left(\lambda g^{-1}(z)\right)g'\left(g^{-1}(z)\right)-\lambda g'\left(\lambda g^{-1}(z)\right)g''\left(g^{-1}(z)\right)}{\left[g'\left(g^{-1}(z)\right)\right]^3}\\
&\quad= \frac{c_1\left[g\left(\lambda g^{-1}(z)\right)-az\right]\left[\lambda g'\left(\lambda g^{-1}(z)\right)-ag'\left(g^{-1}(z)\right)\right]^2 g'\left(g^{-1}(z)\right)}{\left[g'\left(g^{-1}(z)\right)\right]^3}\\
&\qquad+\frac{c_2\lambda\left[g\left(\lambda^2 g^{-1}(z)\right)-ag\left(\lambda g^{-1}(z)\right)\right]\left[\lambda g'\left(\lambda g^{-1}(z)\right)-ag'\left(g^{-1}(z)\right)\right]^2 g'\left(\lambda g^{-1}(z)\right)}{\left[g'\left(g^{-1}(z)\right)\right]^3},
\end{aligned}
$$

and

$$
\begin{aligned}
&c_1\left[y(z)-az\right]\left[y'(z)-a\right]^2+c_2\left[y\left(y(z)\right)-ay(z)\right]\left[y'(z)-a\right]^2y'(z)\\
&\quad= \frac{c_1\left[g\left(\lambda g^{-1}(z)\right)-az\right]\left[\lambda g'\left(\lambda g^{-1}(z)\right)-ag'\left(g^{-1}(z)\right)\right]^2 g'\left(g^{-1}(z)\right)}{\left[g'\left(g^{-1}(z)\right)\right]^3}\\
&\qquad+\frac{c_2\lambda\left[g\left(\lambda^2 g^{-1}(z)\right)-ag\left(\lambda g^{-1}(z)\right)\right]\left[\lambda g'\left(\lambda g^{-1}(z)\right)-ag'\left(g^{-1}(z)\right)\right]^2 g'\left(\lambda g^{-1}(z)\right)}{\left[g'\left(g^{-1}(z)\right)\right]^3}
\end{aligned}
$$

as requirs. The proof is complete. □

We can derive the explicit form of $x(z)$, an analytic solution of (1.5), in a neighborhood of the fixed point μ of $y(z)$ by means of (3.4). Assume that $x(z)$ is of the form

$$x(z)=x(\mu)+x'(\mu)(z-\mu)+\frac{x''(\mu)}{2!}(z-\mu)^2+\cdots+\frac{x^{(n)}(\mu)}{n!}(z-\mu)^n+\cdots,$$

we need to determine the derivatives $x^{(n)}(\mu), n=0,1,2,\ldots$. First of all, in view of (3.4) and (3.1), we have

$$x(\mu)=\frac{1}{c_1+c_2}\frac{b}{\lambda-a},$$

and

$$x'(\mu)=\frac{(1-a)\mu}{b}$$

respectively. Furthermore,

$$x''(\mu)=\frac{\lambda-a}{b}.$$

Recall the formula for the higher derivatives of composition. Namely, for $n\geq 1$,

$$\left(f\left(\varphi(z)\right)\right)^{(n)}=\sum_{\substack{1\leq i\leq n,\sum_{k=1}^n i_k=i\\ \sum_{k=1}^n ki_k=n}}\frac{n!f^{(i)}}{i_1!i_2!\cdots i_n!}\left(\frac{u'}{1!}\right)^{i_1}\left(\frac{u''}{2!}\right)^{i_2}\cdots\left(\frac{u^{(n)}}{n!}\right)^{i_n},$$

where $u = \varphi(z)$, $f^{(i)} = \frac{d^i f}{du^i}$, $u^{(k)} = \frac{d^k u}{dz^k}$, we have

$$\Phi^{(n)} := \left(c_1 x(z) + c_2 x(az + bx'(z))\right)^{(n)} = c_1 \left(x(z)\right)^{(n)} + c_2 \left(x(az + bx'(z))\right)^{(n)},$$

such that

$$(x(az + bx'(z)))^{(n)} = \sum_{\substack{1 \le i \le n, \sum_{k=1}^{n} i_k = i \\ \sum_{k=1}^{n} k i_k = n}} \frac{n! x^{(i)}}{i_1! i_2! \cdots i_n!} \left(\frac{a + bx''(z)}{1!}\right)^{i_1} \left(\frac{bx'''(z)}{2!}\right)^{i_2} \cdots \left(\frac{bx^{(n+1)}(z)}{n!}\right)^{i_n}$$

for $n = 1, 2, \ldots$, and

$$\begin{aligned} x^{(n+2)}(z) &= \left(\frac{1}{c_1 x(z) + c_2 x(az + bx'(z))}\right)^{(n)} \\ &= \sum_{\substack{1 \le j \le n, \sum_{l=1}^{n} j_l = j \\ \sum_{l=1}^{n} l j_l = n}} \frac{(-1)^j n! j!}{j_1! j_2! \cdots j_n! \Phi^{j+1}} \left(\frac{\Phi'}{1!}\right)^{j_1} \left(\frac{\Phi''}{2!}\right)^{j_2} \cdots \left(\frac{\Phi^{(n)}}{n!}\right)^{j_n}. \end{aligned}$$

By means of this formula, we can obtain $x^{(n+2)}(\mu)$ for $n = 1, 2, \ldots$. It is then write out the explicit from of our solution $x(z)$:

$$x(z) = \frac{1}{c_1 + c_2} \frac{b}{\lambda - a} + \frac{(1-a)\mu}{b}(z - \mu) + \frac{\lambda - a}{2! b}(z - \mu)^2 + \sum_{n=3}^{\infty} \frac{x^{(n)}(\mu)}{n!}(z - \mu)^n.$$

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