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A Fixed Point Approach to the Generalized Hyers-Ulam Stability of Reciprocal Difference and Adjoint Functional Equations

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Abstract : In this paper, we investigate the generalized Hyers-Ulam stability of Reciprocal Difference and Adjoint Functional equations by applying fixed point method.

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1 Introduction

The stability problem of functional equations originates from the fundamental question: When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?

In connection with the above question, in 1940, S. M. Ulam [51] raised a question concerning the stability of homomorphisms. Let G be a group and let G' be a metric group with $d(.,.)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there is a homomorphism $H : G \to G'$ with $d(f(x), H(x))$ for all $x \in G$?

The first partial solution to Ulam's question was given by D.H. Hyers [18]. He considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\|$$

for all $x, y \in E$, it was shown that the limit

$$a(x) =_{n\to\infty}^{lim} \frac{f(2^n x)}{2^n}$$

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exists for all $x \in E$ and that $a : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon.$$

Moreover, it was proved that if $f(tx)$ is continuous in t for each fixed $x \in E$, then a is linear. In this case, the Cauchy additive functional equation $f(x+y) = f(x) + f(y)$ is said to satisfy the Hyers-Ulam stability.

In 1978, Th. M. Rassias [45] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. He proved the following theorem.

Theorem 1.1. [**Th. M. Rassias**] *If a function $f : E \to E'$ between Banach spaces satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(|x\|^p + \|y\|^p\right) \tag{1.1}$$

for some $\theta \geq 0$, $0 \leq p < 1$ and for all $x, y \in E$, then there exists a unique additive function $a : E \to E'$ such that

$$\|f(x) - a(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continous in t for each fixed $x \in E$, then a is linear.

A particular case of Th. M. Rassias' theorem regarding the Hyers-Ulam stability of the additive mappings was proved by T. Aoki [3]. The theorem of Rassias was later extended to all $p \neq 1$ and generalized by many mathematicians (see [9-12],[14],[19-20]). The phenomenon that was introduced and proved by Th. M. Rassias is called the Hyers-Ulam-Rassias stability. The Hyers-Ulam-Rassias stability for various functional equations have been extensively investigated by numerous authors; one can refer to (see [9], [13-14], [16], [18-19], [21], [48]). In 1994, a generalization of the Th.M. Rassias' theorem was obtained by P.Gâvruta [14], who replaced the bound $\theta\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\phi(x, y)$.

In 1982-1994, a generalization of the Hyers result was established by J. M. Rassias with a weaker condition controlled by a product of different powers of norms. However, there was a singular case. Then for this singularity, a counter example was given by P. Gâvruta [15]. This stability is called Hyers-Ulam-Rassias stability involving a product of different powers of norms.

In 1996, Isac and Th. M. Rassias [21] were the first to provide applications of stability theorem of functional equations for the proof of new fixed point theorems with applications.

Usually, the stability problem for functional equations is solved by direct method in which the exact solution of the functional equation is explicitly constructed as a limit of a (Hyers) sequence, starting from the given approximate solution f ([2], [11], [19], [24-25]). In 2003, Radu [37] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. This method was recently been used by many authors (see [26-29], [32], [34-35]).

Cădariu and Radu ([5], [6]) applied a fixed point method to investigate the Jensen's and Cauchy additive functional equations.

Very recently, S. M. Jung [28] applied a fixed point method for proving the Hyers-Ulam stability for the reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}. \tag{1.3}$$

In 2008, K. Ravi and B. V. Senthil Kumar [43] investigated some results on Ulam-Gavruta-Rassias stability of the functional equation (1.3). It was proved that the reciprocal function $r(x) = \frac{c}{x}$ is a solution of the functional equation (1.3).

Later, J. M. Rassias and et.al., [44] introduced the Reciprocal Difference Functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.4}$$

and the Reciprocal Adjoint Functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{1.5}$$

and investigated the Hyers-Ulam stability of the equations (1.4) and (1.5). Further in the same paper, it was proved that the functional equations (1.3), (1.4) and (1.5) are equivalent.

In this paper, we apply fixed point method and prove the generalized Hyers-Ulam stability of the functional equation (1.4) and (1.5).

2 Preliminaries

In this section, we present an important definition and result like generalized metric and fundamental result of fixed point theory.

Let A be a set. A function $d : A \times A \to [0, \infty]$ is called a generalized metric on A if d satisifes the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in A$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in A$.

We note that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.

The following theorem is very useful for proving our main result which is due to Margolis and Diaz [31].

Theorem 2.1. *[31] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{2.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1. *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set* $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
4. *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, let X and Y be sets of non-zero real numbers. For convenience, let us define

$$R_1 f(x, y) = f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}$$

and

$$R_2 f(x, y) = f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}$$

for all $x, y \in X$.

3 GENERALIZED HYERS-ULAM STABILITY OF (1.4) BY FIXED POINT METHOD

In the following theorem, we will set $\frac{0^2}{0} = 0$.

Theorem 3.1. *Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$|R_1 f(x, y)| \leq \varphi(x, y) \tag{3.1}$$

for all $x, y \in X$, where $\varphi : X^2 \to Y$ is a given function.

If there exists $L < 1$ such that the mapping

$$x \to \psi(x) = \varphi\Big(\frac{x}{2}, \frac{x}{2}\Big)$$

has the property

$$\psi\Big(\frac{x}{2}\Big) \leq 2L\psi(x), \text{for all} \quad x \in X$$

and the mapping φ has the property

$$\lim_{n\to\infty} 2^{-n}\varphi(2^{-n}x, 2^{-n}y) = 0, \text{for all} \quad x, y \in X, \tag{3.2}$$

then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$|r(x) - f(x)| \leq \frac{1}{1-L}\psi(x) \tag{3.3}$$

for all $x \in X$.

Proof. Define a set S by

$$S = \{h : X \to Y | h \text{ is a function}\}$$

and introduce the generalized metric d on S as follows:

$$d(g,h) = d_\psi(g,h) = \inf\{C \in \mathbb{R}_+ : |g(x) - h(x)| \leq C\psi(x), \text{for all } x \in X\}. \tag{3.4}$$

Now, we show that (S,d) is complete. Using the idea from [27], we prove the completeness of (S,d). Let $\{h_n\}$ be a Cauchy sequence in (S,d). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $d(h_m, h_n) \leq \epsilon$ for all $m, n \geq N_\epsilon$. From (3.4), we arrive

$$\forall \epsilon > 0, \exists\ N_\epsilon \in \mathbb{N}, \forall m, n \geq N_\epsilon, \forall x \in X : |h_m(x) - h_n(x)| \leq \epsilon\psi(x). \tag{3.5}$$

If x is a fixed number, (3.5) implies that $\{h_n(x)\}$ is a Cauchy sequence in $(Y, |.|)$. Since $(Y, |.|)$ is complete, $\{h_n(x)\}$ converges for all $x \in X$. Therefore, we can define a function $h : X \to Y$ by

$$h(x) =_{n\to\infty}^{lim} h_n(x)$$

and hence $h \in S$. Letting $m \to \infty$ in (3.5), we have

$$\forall \epsilon > 0, \exists\ N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon, \forall x \in X : |h(x) - h_n(x)| \leq \epsilon\psi(x).$$

By considering (3.4), we arrive

$$\forall \epsilon > 0, \exists\ N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon : d(h, h_n) \leq \epsilon,$$

which implies that the Cauchy sequence $\{h_n\}$ converges to h in (S,d). Hence (S,d) is complete.

Define a mapping $\sigma : S \to S$ by

$$\sigma h(x) = \frac{1}{2}h\left(\frac{x}{2}\right) \qquad (x \in X) \tag{3.6}$$

for all $h \in S$. We claim that σ is strictly contractive on S. For any given $g, h \in S$, let $C_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq C_{gh}$. Hence

$$\begin{aligned} d(g,h) < C_{gh} &\Longrightarrow |g(x) - h(x)| \leq C_{gh}\psi(x), \forall x \in X \\ &\Longrightarrow \left|\frac{1}{2}g\Big(\frac{x}{2}\Big) - \frac{1}{2}h\Big(\frac{x}{2}\Big)\right| \leq \frac{1}{2}C_{gh}\psi\Big(\frac{x}{2}\Big), \forall x \in X \\ &\Longrightarrow \left|\frac{1}{2}g\Big(\frac{x}{2}\Big) - \frac{1}{2}h\Big(\frac{x}{2}\Big)\right| \leq LC_{gh}\psi(x), \forall x \in X \\ &\Longrightarrow d(\sigma g, \sigma h) \leq LC_{gh}. \end{aligned}$$

Therefore, we see that

$$d(\sigma g, \sigma h) \leq L d(g,h), \quad \text{for all} \quad g,h \in S$$

that is, σ is strictly contractive mapping of S, with the Lipschitz constant L.

Now, replacing (x,y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (3.1), we get

$$\left|\frac{1}{2} f\left(\frac{x}{2}\right) - f(x)\right| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \psi(x)$$

for all $x \in X$. Hence (3.4) implies that $d(\sigma f, f) \leq 1$. Hence by applying the fixed point alternative Theorem 2.1, there exists a function $r : X \to Y$ satisfying the following:

(1) r is a fixed point of σ, that is

$$r\left(\frac{x}{2}\right) = 2r(x) \tag{3.7}$$

for all $x \in X$. The mapping r is the unique fixed point of σ in the set

$$\mu = \{g \in S : d(f,g) < \infty\}.$$

This implies that r is the unique mapping satisfying (3.7) such that there exists $C \in (0, \infty)$ satisfying

$$|r(x) - f(x)| \leq C\psi(x), \forall x \in X.$$

(2) $d(\sigma^n f, r) \to 0$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} 2^{-n} f(2^{-n}x) = r(x) \tag{3.8}$$

for all $x \in X$.

(3) $d(r,f) \leq \frac{1}{1-L} d(r, \sigma f)$, which implies

$$d(r,f) \leq \frac{1}{1-L}.$$

Thus the inequality (3.3) holds. Hence from (3.1), (3.2) and (3.8), we have

$$\begin{aligned} |R_1 r(x,y)| &= \lim_{n\to\infty} 2^{-n} |R_1 f(2^{-n}x, 2^{-n}y)| \\ &\leq \lim_{n\to\infty} 2^{-n} \varphi(2^{-n}x, 2^{-n}y) \\ &= 0 \end{aligned}$$

for all $x, y \in X$. Hence r is a solution of the functional equation (1.4). By Theorem 2.1 [44], $r : X \to Y$ is a reciprocal mapping.

Next, we show that r is the unique reciprocal mapping satisfying (1.4) and (3.3). Suppose, let $R : X \to Y$ be another reciprocal function satisfying (1.4) and (3.3). Then from (1.4), we have that R is a fixed point of σ. Since $d(f, R) < \infty$, we have

$$R \in S^* = \{g \in S | d(f, g) < \infty\}.$$

From Theorem 2.1(3) and since both r and R are fixed points of σ, we have $r = R$. Therefore r is unique. Hence, there exists a unique reciprocal mapping $r : X \to Y$ satisfying (1.4) and (3.3), which completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $f : X \to Y$ be a mapping for which there exists a constant ϵ (independent of x,y) ≥ 0 such that the functional inequality*

$$|R_1 f(x, y)| \leq \epsilon \tag{3.9}$$

holds for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.4) and

$$|r(x) - f(x)| \leq 2\epsilon \tag{3.10}$$

for all $x \in X$.

Proof. Taking $\varphi(x, y) = \epsilon$, for all $x, y \in X$ and choosing $L = \frac{1}{2}$ in Theorem 3.1, the proof of Corollary 3.2 follows immediately. □

Corollary 3.3. *Let $f : X \to Y$ be a mapping and let there exist real numbers $p > -1$ and $\theta \geq 0$ such that*

$$|R_1 f(x, y)| \leq \theta \left(|x|^p + |y|^p\right) \tag{3.11}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.4) and

$$|r(x) - f(x)| \leq \frac{4\theta}{2^{p+1} - 1} |x|^p \tag{3.12}$$

for all $x \in X$.

Proof. The proof is similar to that of Theorem 3.1 by taking $\varphi(x, y) = \theta \left(|x|^p + |y|^p\right)$, for all $x, y \in X$ and $L = 2^{-p-1}$. □

Corollary 3.4. *Let $f : X \to Y$ be a mapping and there exist real numbers a, b : $\rho = a + b > -1$. If there exists $c_1 \geq 0$ such that*

$$|R_1 f(x, y)| \leq c_1 |x|^a |y|^b \tag{3.13}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.4) and

$$|r(x) - f(x)| \leq \frac{2c_1}{2^{\rho+1} - 1} |x|^\rho \tag{3.14}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x,y) = c_1|x|^a|y|^b$, for all $x, y \in X$. Then we can choose $L = 2^{-\rho-1}$ to get the desired result. □

Corollary 3.5. *Let $c_2 > 0$ and $\alpha > -\frac{1}{2}$ be real numbers, and $f : X \to Y$ be a mapping satisfying the functional inequality*

$$|R_1 f(x,y)| \leq c_2 \left[|x|^\alpha|y|^\alpha + \left(|x|^{2\alpha} + |y|^{2\alpha}\right)\right] \tag{3.15}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.4) and

$$|r(x) - f(x)| \leq \frac{6c_2}{2^{2\alpha+1} - 1}|x|^\alpha \tag{3.16}$$

for all $x \in X$.

Proof. The proof goes through the same way as in Theorem 3.1 by considering $\varphi(x,y) = c_2\left[|x|^\alpha|y|^\alpha + \left(|x|^{2\alpha} + |y|^{2\alpha}\right)\right]$, for all $x, y \in X$ and selecting $L = 2^{-2\alpha-1}$. □

4 GENERALIZED HYERS-ULAM STABILITY OF (1.5) BY FIXED POINT METHOD

Theorem 4.1. *Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$|R_2 f(x,y)| \leq \varphi(x,y) \tag{4.1}$$

for all $x, y \in X$, where $\varphi : X^2 \to Y$ is a given function.

If there exists $L < 1$ such that the mapping

$$x \to \psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

has the property

$$\psi\left(\frac{x}{2}\right) \leq 2L\psi(x), \text{for all} \quad x \in X$$

and the mapping φ has the property

$$\lim_{n\to\infty} 2^{-n}\varphi(2^{-n}x, 2^{-n}y) = 0, \text{for all} \quad x, y \in X, \tag{4.2}$$

then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$|f(x) - r(x)| \leq \frac{1}{1-L}\psi(x) \tag{4.3}$$

for all $x \in X$.

Proof. The proof of Theorem 4.1 is similar to that of Theorem 3.1. □

Corollary 4.2. *Let $f : X \to Y$ be a mapping for which there exists a constant ϵ (independent of x,y) ≥ 0 such that the functional inequality*

$$|R_2 f(x, y)| \leq \epsilon \tag{4.4}$$

holds for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.5) and

$$|f(x) - r(x)| \leq 2\epsilon \tag{4.5}$$

for all $x \in X$.

Proof. Corollary 4.2 can be proved by similar arguments as in Corollary 3.2. □

Corollary 4.3. *Let $f : X \to Y$ be a mapping and let there exist real numbers $p > -1$ and $\theta \geq 0$ such that*

$$|R_2 f(x, y)| \leq \theta \left(|x|^p + |y|^p\right) \tag{4.6}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.5) and

$$|f(x) - r(x)| \leq \frac{4\theta}{2^{p+1} - 1} |x|^p \tag{4.7}$$

for all $x \in X$.

Proof. The proof of Corollary 4.3 can be done by similar arguments as in Corollary 3.3. □

Corollary 4.4. *Let $f : X \to Y$ be a mapping and there exist real numbers a, b : $\rho = a + b > -1$. If there exists $c_1 \geq 0$ such that*

$$|R_2 f(x, y)| \leq c_1 |x|^a |y|^b \tag{4.8}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.5) and

$$|f(x) - r(x)| \leq \frac{2c_1}{2^{\rho+1} - 1} |x|^\rho \tag{4.9}$$

for all $x \in X$.

Proof. The proof of Corollary 4.4 goes through the same way as in Corollary 3.4. □

Corollary 4.5. *Let $c_2 > 0$ and $\alpha > -\frac{1}{2}$ be real numbers, and $f : X \to Y$ be a mapping satisfying the functional inequality*

$$|R_2 f(x, y)| \leq c_2 \left[|x|^\alpha |y|^\alpha + \left(|x|^{2\alpha} + |y|^{2\alpha}\right)\right] \tag{4.10}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying the functional equation (1.5) and

$$|f(x) - r(x)| \leq \frac{6c_2}{2^{2\alpha+1} - 1}|x|^\alpha \tag{4.11}$$

for all $x \in X$.

Proof. The proof of Corollary 4.5 is similar to that of Corollary 3.5. □

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