Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 459–467

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A Common Fixed Point Theorem for Single-Valued and Set-Valued Mappings Satisfying A Generalized Contractive Condition

A. Aliouche and A. Djoudi

Abstract : In this paper, we prove a common fixed point theorem for two pairs of single-valued and set-valued mappings satisfying a generalized contractive condition and a property using the concept of weak compatibility in metric spaces which generalizes Theorem 1 of [1] and Corollary 3 of [2].

Keywords : Weakly compatible mappings; Common fixed point; Set-valued mappings; Property (E.A).
2000 Mathematics Subject Classification : 54H25; 47H10.

1 Introduction

Let S and T be self-mappings of a metric space (X, d). S and T are commuting if $STx = TSx$ for all $x \in X$. Sessa [13] defined S and T to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leq d(Tx, Sx). \tag{1.1}$$

Jungck [9] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [9] and [13].

Jungck [10] defined S and T to be weakly compatible if they commute at their coincidence points.

It was proved that if S and T are compatible, then they are weakly compatible and the converse is not true in general.

Let $B(X)$ be the set of all nonempty bounded subsets of X.

As in [7] and [8], let $\delta(A,B)$ and $D(A,B)$ be the functions defined by

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}, \text{ for all } A,B \in B(X).$$

If A consists of a single point a, we write $\delta(A,B) = \delta(a,B)$. If B consists also of a single point b, we write $\delta(A,B) = d(a,b)$.

It follows immediately from the definition of δ that for all $A,B,C \in B(X)$.

$$\delta(A,B) = \delta(B,A) \geq 0,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(C,B),$$
$$\delta(A,B) = 0 \text{ iff } A = B = \{a\},$$
$$\delta(A,A) = diamA.$$

Sessa [14] extended (1.1) to single-valued and set-valued mappings as follows:

The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be weakly commuting on X if $IFx \in B(X)$ and

$$\delta(FIx, IFx) \leq \max\{\delta(Ix,Fx), diam(IFx)\} \text{ for all } x \in X. \tag{1.3}$$

Note that if F is a single-valued mapping, then the set IFx consists of a single point. Therefore,

$diam(IFx) = 0$ for all $x \in X$ and condition (1.3) reduces to the condition (1.1)

Two commuting mappings F and I clearly weakly commute but two weakly commuting F and I do not necessarily commute as it was shown in [14].

Jungck and Rhoades [11] defined the concepts of $\delta-$compatible and weakly compatible mappings which extend the concept of compatible mappings and weakly compatible in the single-valued setting to set-valued mappings as follows:

The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be $\delta-$compatible if

$$\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ as $n \to \infty$ for some t in X.

The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at their coincidence points, i.e., for each point x in X such that $Fu = \{Iu\}$, we have $FIu = IFu$.

Note that the equation $Fx = \{Ix\}$ implies that Fx is a singleton.

It can be seen that any $\delta-$compatible pair (F,I) is weakly compatible. Examples of weakly compatible pairs which are not $\delta-$compatible were given in [11].

Definition 1.1. *[7] A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if*

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, ...,$ and $\{a_n\}$ converges to a.

(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subset A_\epsilon$ for $n > N$ where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

Lemma 1.2. *[7] If $\{A_n\}$and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$converges to $\delta(A, B)$.*

Lemma 1.3. *[8] Let $\{A_n\}$be a sequence in $B(X)$ and y a point in X such that $\delta(A_n, y) \to 0$. Then, the sequence $\{A_n\}$converges to the set $\{y\}$ in $B(X)$.*

Lemma 1.4. *[8] A set-valued mapping F of X into $B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X.*

F is said to be continuous on X if it is continuous at every point in X.

Lemma 1.5. *[8] Let $\{A_n\}$ be a sequence of nonempty subsets of X and z in X such that $\lim_{n\to\infty} a_n = z$, z independent of the particular choice of each $a_n \in A_n$. If a self-map I of X is continuous, then $\{Iz\}$ is the limit of the sequence $\{IA_n\}$.*

Definition 1.6. *[1] Two self-mappings S and T of a metric space (X, d) satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \textit{ for some } z \in X.$$

It is clear from the definition of compatibility that S and T are noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, but, $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or does not exist. Therefore, two noncompatible self-mappings of a metric space (X, d) satisfy the property (E.A).

2 Preliminaries

Definition 2.1. *Two mappings $f : X \to X$ and $T : X \to B(X)$ satisfy the property $(E.A)$ if if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} fx_n = z \quad \textit{and} \quad \lim_{n\to\infty} Tx_n = \{z\} \textit{ for some } z \in X.$$

Example 2.2. *Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$ and $T : X \to B(X)$ by $f(x) = x+1$ and $Tx = [2, x+1]$. Consider the sequence $\{x_n\}$ such that $x_n = 1 + \dfrac{1}{n}$, $n = 1, 2, ...$ Clearly, $\lim_{n\to\infty} fx_n = 2$ and $\lim_{n\to\infty} Tx_n = \{2\}$. Therefore, f and T satisfy the property $(E.A)$.*

Example 2.3. *Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$ and $T : X \to B(X)$ by $f(x) = x + 1$ and $Tx = \{x + 2\}$. Suppose that f and T satisfy property (E.A). Then, there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} fx_n = z$ and $\lim_{n\to\infty} Tx_n = \{z\}$. Therefore, $\lim_{n\to\infty} x_n = z - 1 = z + 2$ for some $z \in X$ which is a contradiction. Hence, f and T do not satisfy the property (E.A).*

It is clear from the definition of $\delta-$compatibility that f and T are noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = z$ and $\lim_{n\to\infty} Tx_n = \{z\}$ for some $z \in X$ but, $\lim_{n\to\infty} \delta(fTx_n, Tfx_n)$ is either non-zero or does not exist. Therefore, two mappings $f : X \to X$ and $T : X \to B(X)$ of a metric space (X, d) which are not $\delta-$compatible satisfy the property (E.A).

Several authors proved fixed point theorems and common fixed point theorems for mappings satisfying contractive conditions of integral type, see [2, 3, 4, 5, 6, 12, 15]. Recently, Zhang [16] and Aliouche [3] proved common fixed point theorems using generalized contractive conditions in metric spaces. These theorems extend well-known results in [4], [5], [12] and [15].

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,

(ii) F is increasing on R_A^+,

(iii) F is continuous.

Define $\mathcal{F}[0, A) = \{F : F \text{ satisfies (i)–(iii)}\}$.

The following examples were given by [16].

1) Let $F(t) = t$, then $F \in \mathcal{F}[0, A)$ for each $A \in (0, +\infty]$.

2) Suppose that φ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, A).$$

Let $F(t) = \int_0^t \varphi(s)ds$, then $F \in [0, A)$.

3) Suppose that ψ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \psi(t)dt > 0 \text{ for each } \epsilon \in (0, A)$$

and φ is nonnegative, Lebesgue integrable on $[0, \int_0^A \psi(s)ds)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, \int_0^A \psi(s)ds).$$

Let $F(t) = \int_0^{\int_0^t \psi(s)ds} \varphi(u)du$, then $F \in \mathcal{F}[0, A)$.

4) If $G \in [0, A)$ and $F \in \digamma[0, G(A-0))$, then a composition mapping $F \circ G \in \digamma[0, A)$. For instance, let $H(t) = \int_0^{F(t)} \varphi(s)ds$, then $H \in \digamma[0, A)$ whenever $F \in \digamma[0, A)$ and φ is nonnegative, Lebesgue integrable on $\digamma[0, F(A-0))$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, F(A-0)).$$

Lemma 2.4. *[16] Let $A \in (0, +\infty]$ and $F \in \digamma[0, A)$. If $\lim_{n\to\infty} F(\epsilon_n) = 0$ for $\epsilon_n \in R_A^+$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

Let $A \in (0, +\infty]$, $\psi : R_A^+ \to \mathbb{R}_+$ satisfying
(i) $\psi(t) < t$ for each $t \in (0, A)$,
(ii) ψ is nondecreasing and upper semi-continuous.
Define $\Psi[0, A) = \{\psi : \psi$ satisfies (i) and (ii) above$\}$.

Lemma 2.5. *[16] If $\psi \in \Psi[0, A)$, then $\psi(0) = 0$.*

The following Theorem was proved by [1].

Theorem 2.6. *Let A, B, S and T be self-mappings of a metric space (X, d) such that*

$$d(Ax, By) \leq \phi(\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\})$$

for all $x, y \in X$. Suppose that $A(X) \subset T(X)$, $B(X) \subset S(X)$ and the (A, S) or (B, T) satisfies the property (E.A). If the range of one of the mappings A, B, S and T is a complete subspace of X, then A, B, S and T have a unique common fixed point in X.

It is our purpose in this paper to extend Theorem 2.6 for two pairs of single-valued and set-valued mappings and prove a common fixed point theorem using a generalized contractive condition and a property (E.A).

3 Main Results

Let $D = \sup\{d(x, y) : x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$.

Theorem 3.1. *Let f and g be self-mappings of a metric space (X, d) and S and T be mappings from X into $B(X)$ satisfying*

$$\cup S(X) \subset g(X) \quad and \quad \cup T(X) \subset f(X) \tag{3.1}$$

$$F(\delta(Sx, Ty) \leq \psi(F(\max\{d(fx, gy), \delta(fx, Sx), \delta(gy, Ty), d(fx, Ty), d(Sx, gy)\})) \tag{3.2}$$

for all $x, y \in X$, where $F \in [0, A)$ and $\psi \in \Psi[0, \digamma(A-0))$. Suppose that the pair (f, S) or (g, T) satisfies the property (E.A), (f, S) and (g, T) are weakly compatible and $f(X)$ or $g(X)$ or $S(X)$ or $T(X)$ is a closed subset of X. Then, f, g, S and T have a unique common fixed point in X.

Proof. Suppose that the pair (g, T) satisfies the property (E.A). Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} gx_n = z$ and $\lim_{n\to\infty} Tx_n = \{z\}$ for some $z \in X$. Therefore, we have $\lim_{n\to\infty} \delta(gx_n, Tx_n) = 0$. Since $T(X) \subset f(X)$, there exists in X a sequence $\{y_n\}$ such that $fy_n \in Tx_n$. Assume that $\limsup_{n\to\infty} \delta(Sy_n, Tx_n) = l > 0$. Using (3.2) we have

$$\begin{aligned} F(\delta(Sy_n, Tx_n)) &\leq \psi(F(\max\{d(fy_n, gx_n), \delta(fy_n, Sy_n), \\ &\quad \delta(gx_n, Tx_n), d(fy_n, Tx_n), d(Sy_n, gx_n)\})) \\ &\leq \psi(F(\max\{\delta(gx_n, Tx_n), \delta(Sy_n, Tx_n), d(gx_n, Tx_n) \\ &\quad \delta(Sy_n Tx_n) + \delta(gx_n, Tx_n\})). \end{aligned}$$

Letting $n \to \infty$ we get $F(l) \leq \psi(F(l)) < F(l)$.

Which is a contradiction. Hence, $\lim_{n\to\infty} F(\delta(Sy_n, Tx_n) = 0$ and Lemma 2.4 implies that $\lim_{n\to\infty} \delta(Sy_n, Tx_n) = 0$; i.e., $\lim_{n\to\infty} Sy_n = \{z\}$.

Suppose that $f(X)$ is a complete subspace of X. Then, $z = fu$ for some $u \in X$.

If $Su \neq \{z\}$, applying (3.2) we get

$$\begin{aligned} F(\delta(Su, Tx_n)) &\leq \psi(F(\max\{d(fu, gx_n), \delta(fu, Su), \delta(gx_n, Tx_n), \\ &\quad d(fu, Tx_n), d(Su, gx_n)\})). \end{aligned}$$

Letting $n \to \infty$ we obtain

$$\begin{aligned} F(\delta(Su, z)) &\leq \psi(F(\delta(Su, z)) \\ &< F(\delta(Su, z)) \end{aligned}$$

and so $Su = \{fu\} = \{z\}$. Since $S(X) \subset g(X)$, there exists $v \in X$ such that $Su = \{gv\} = \{z\}$.

If $Tv \neq \{z\}$, using (3.2) we have

$$\begin{aligned} F(\delta(z, Tv)) &= F(\delta(Su, Tv)) \\ &\leq \psi(F(\delta(z, Tv))) \\ &< F(\delta(z, Tv)) \end{aligned}$$

which implies that $Tv = \{gv\} = \{z\}$. As the pairs (f, S) and (B, T) are weakly compatible, we get $Sz = \{fz\}$ and $Tz = \{gz\}$. If $Sz \neq \{z\}$, using (3.2) we obtain

$$\begin{aligned} F(\delta(Sz, z)) &= F(\delta(Sz, Tv)) \\ &\leq \psi(F(\delta(Sz, z))) \\ &< F(\delta(Sz, z)) \end{aligned}$$

and so $Sz = \{fz\} = \{z\}$. Similarly, we can prove that $Tz = \{gz\} = \{z\}$.

The proof is similar when $g(X)$ is assumed to be a closed subset of X. By (3.1), the cases in which $S(X)$ or $T(X)$ is a closed subset of X are similar to the cases in which $f(X)$ or $g(X)$ is a closed subset of X. The uniqueness of z follows from (3.2). $\square$

If S and T are single-valued mappings in Theorem 3.1 and $F(t) = \int_0^t \varphi(s)ds$, where φ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon \in (0, A)$, we get a generalization of Corollary 3 of [2].

If S and T are single-valued mappings and $F(t) = t$ in Theorem 3.1, we get a generalization of Theorem 2.5.

If $S = T$ and $g = f$ in Theorem 3.1, we get the following Corollary.

Corollary 3.2. *Let f be a self-mapping of a metric space (X, d) and T be a mapping from X into $B(X)$ such that*

$$\cup T(X) \subset f(X)$$

$$F(\delta(Tx, Ty)) \leq \psi(F(\max\{d(fx, fy), \delta(fx, Tx), \delta(fy, Ty), d(fx, Ty), d(Tx, fy)\}))$$

for all $x, y \in X$, where $F \in [0, A)$ and $\psi \in \Psi[0, \digamma(A - 0))$. Suppose that the pair (f, T) satisfies the property (E.A), (f, T) is weakly compatible and $f(X)$ or $T(X)$ is a closed subset of X. Then, f and T have a unique common fixed point in X.

If $F(t) = t$ in Theorem 3.1, we get the following Corollary.

Corollary 3.3. *Let f and g be self-mappings of a metric space (X, d) and S, T be mappings from X into $B(X)$ satisfying* (3.1) *and*

$$\delta(Sx, Ty) \leq \psi(\max\{d(fx, fy), \delta(fx, Tx), \delta(fy, Ty), d(fx, Ty), d(Tx, fy)\})$$

for all $x, y \in X$. Suppose that the pair (f, S) or (g, T) satisfies the property (E.A), (f, S) and (g, T) are weakly compatible and one of $f(X)$ or $g(X)$ or $S(X)$ or $T(X)$ is a closed subset of X. Then, f, g, S and T have a unique common fixed point in X.

Example 3.4. *Let $X = [0, 1]$ endowed with the Euclidean metric d. Define $S, T : X \to B(X)$ and $f, g : X \to X$ by*

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x+1}{4} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}, \quad gx = \begin{cases} 1 - x & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$

$$Sx = \left\{\frac{1}{2}\right\} \text{ for all } x \in [0, 1] \text{ and } Tx = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0, \frac{1}{2}] \\ (\frac{3}{8}, \frac{1}{2}] & \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$

let $F(s) = s^{\frac{1}{s}}$ and $\psi(t) = \dfrac{1}{8^{\frac{16}{3}} \cdot 3^{\frac{8}{3}}} t$. Then, $F \in \digamma[0, A)$ and $\psi \in \Psi[0, e^{\frac{1}{e}}]$, where $A = e > D$.

We have

$$\cup S(X) = \left\{\frac{1}{2}\right\} \subset g(X) = [\tfrac{1}{2}, 1] \cup \{0\} \quad \textit{and}$$
$$\cup T(X) = (\tfrac{3}{8}, \tfrac{1}{2}] = f(X).$$

On the other hand, if $x \in X$ and $y \in [0, \frac{1}{2}]$, then

$$\begin{aligned} F(\delta(Sx,Ty)) &= 0 \\ &\leq \psi(F(\max\{d(fx,gy),\delta(fx,Sx),\delta(gy,Ty),d(fx,Ty),d(Sx,gy)\})). \end{aligned}$$

If $x \in X$ and $y \in (\frac{1}{2}, 1]$, then

$$\delta(Sx,Ty) \leq \tfrac{1}{8} \quad \text{and} \quad d(fx,gy) \geq \tfrac{3}{8}.$$

Hence

$$F(\delta(Sx,Ty)) \leq (\tfrac{1}{8})^8 \quad \text{and} \quad F(d(fx,gy)) \geq (\tfrac{3}{8})^{\frac{8}{3}}$$

and so

$$\begin{aligned} F(\delta(Sx,Ty)) &\leq \frac{1}{8^{\frac{16}{3}} \cdot 3^{\frac{8}{3}}} F(d(fx,gy)) \\ &= \psi(F(d(fx,gy))) \\ &\leq \psi(F(\max\{d(fx,gy),\delta(fx,Sx),\delta(gy,Ty),d(fx,Ty),d(Sx,gy)\})). \end{aligned}$$

$g(X)$ is a closed subset of X and the pairs (f,S) and (g,T) are weakly compatible. Taking $x_n = \frac{1}{2} - \frac{1}{2n}$, the pair (f,S) satisfies the property $(E.A)$ with $z = \frac{1}{2}$ Consequently, by Theorem 3.1, $\frac{1}{2}$ is the unique common fixed point of f, g, S and T.

References

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002), 181–188.

[2] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl., 322 (2) (2006), 796–802.

[3] A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions, J. Math Anal. Appl., 341 (1) (2008), 707–719.

[4] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29 (9) (2002), 531–536.

[5] A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math Anal. Appl., 329 (1) (2007), 31–45.

[6] A. Djoudi and F. Merghadi, Common fixed point theorems for maps under a contractive condition of integral type, J. Math. Anal. Appl., 341 (2) (2008), 953–960.

[7] B. Fisher, Common fixed points of mappings and set-valued mappings, Rostock. Math. Kolloq. 18 (1981), 69–77.

[8] B. Fisher and S. Sessa, Two common fixed point theorems for weakly commuting mappings, Period. Math. Hungar. 20 (1989), 207–218.

[9] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771–779.

[10] G. Jungck, Common fixed points for non-continuous non-self maps on non metric spaces, Far East J. Math. Sci., 4 (2) (1996), 199–215.

[11] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity. Indian J. Pure. Appl. Math. 29 (3) (1998), 227–238.

[12] B. E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 63 (2003), 4007–4013.

[13] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd), 32 (46) (1982), 149–153.

[14] S. Sessa, M. S. Khan and M. Imdad, Common fixed point theorem with a weak commutativity condition, Glas. Mat., 21 (41) (1986), 225–235.

[15] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 15 (2005), 2359–2364.

[16] X. Zhang, common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl., 333 (2) (2007), 780–786.

(Received 23 October 2008)

Abdelkrim Aliouche
Department of Mathematics,
University of Larbi Ben M'Hidi,
Oum-El-Bouaghi, 04000, Algeria.
e-mail : alioumath@yahoo.fr

Ahcene Djoudi
Université de Annaba,
Faculté des sciences,
Département de mathématiques,
B. P. 12, 23000, Annaba, Algérie.
e-mail : adjoudi@yahoo.com