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Weak and Strong Convergence Theorems for Approximating Common Fixed Points of Three Nonexpansive Mappings

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Abstract : In this paper, a new three-step iterative scheme for three nonexpansive mappings is introduced and studied. Weak and strong convergence theorems of such iterations to a common fixed point of the nonexpansive mappings are established. The results obtained in this paper extend and improve the results due to [W. Takahashi and T. Tamura, convergence theorems for a pair of nonexpansive mappings, J. Convex Anal., 5 (1995), 45–58], [K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl., 178 (1993), 301–308] and [H. F. Senter and W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., 44 (1974), 375–380] and [G. Liu, D. Lei and S. Li, Approximating fixed points of nonexpansive mappings, Internat. J. Math. & Math. Sci., 24 (2000), 173–177].

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1 Introduction

Let C be a nonempty convex subset of a real Banach space X, and let T_1, T_2 and $T_3 : C \to C$ be given mappings. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T_1 x_n + (1 - a_n)x_n, \\ y_n &= b_n T_2 z_n + c_n T_1 x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T_3 y_n + \beta_n T_2 z_n + \gamma_n T_1 x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

If $c_n = \beta_n = \gamma_n \equiv 0$ and $T_1 = T_2 = T_3$, then (1.1) reduces to the Noor iterations :

$$\begin{aligned} z_n &= a_nT_1x_n + (1-a_n)x_n, \\ y_n &= b_nT_1z_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT_1y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = \beta_n = \gamma_n \equiv 0$ and $T_1 = T_2 = T_3$, then (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= c_nT_1x_n + (1-c_n)x_n, \\ x_{n+1} &= \alpha_nT_1y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{c_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $T_1 = I$, the identity operator on C, and $\beta_n = 0$, then (1.1) reduces to the iterative scheme defined by Das and Debata [1] and Takahashi and Tomura [9]

$$\begin{aligned} y_n &= b_nT_2x_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT_3y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{b_n\}, \{\alpha_n\}$ are sequences in $[0, 1]$. Das and Debata [1] used the scheme (1.3) to approximate common fixed points of the maps when X is stricty convex. Takahashi and Tamura [9] prove weak convergence of the iterates $\{x_n\}$ defined by (1.3) in a uniformly convex Banach space X which satisfies the Opial property or whose norm is $Fre'chet$ differentiable.

If $T_1 = I$, the identity operator on C, $\beta_n = 0$ and $T := T_2 = T_3$, then (1.1) reduces to the usual Ishikawa iterative scheme:

$$\begin{aligned} y_n &= b_nTx_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nTy_n + (1-\alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

If $T_1 = T_2 = I$ the identity operator on C and $T := T_3$, then (1.1) reduces to the usual Mann iterative scheme:

$$x_{n+1} = \alpha_nTx_n + (1-\alpha_n)x_n, \quad n \geq 1.$$

If $a_n = b_n = c_n \equiv 0$, then (1.1) reduces to the iterative scheme

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= S_nx_n \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $S_n = \alpha_nT_3 + \beta_nT_2 + \gamma_nT_1 + (1-\alpha_n-\beta_n-\gamma_n)I$.

If $\alpha_n = a, \beta_n = b$ and $\gamma_n = c$ for all $n \in N$, then (1.4) reduces to the iterative scheme defined by Liu, Lei and Li [3]

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= Sx_n \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $S = aT_3 + bT_2 + cT_1 + (1 - a - b - c)I$. Liu et al. [3] showed that $\{x_n\}$ defined by (1.5) converges to a common fixed point of T_1, T_2 and T_3 in Banach space, provided that $T_i (i = 1, 2, 3)$ satisfy condition A.

The purpose of this paper is to establish weak and strong convergence of the iterative scheme (1.1) to a common fixed point of three nonexpansive mappings in a uniformly convex Banach space.

Now, we recall the well-known concepts and results.

Let X be a normed space and C a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* on C if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

A Banach space X is said to satisfy *Opial's condition* if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1. ([5], Lemma 4) *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\},\ r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\begin{aligned} \|\alpha x + \beta y + \gamma z + \lambda w\|^2 \quad &\leq \quad \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \lambda\|w\|^2 \\ &\quad -\frac{1}{3}\lambda(\alpha g(\|x - w\| + \beta g(\|y - w\| + \gamma g(\|z - q\|)), \end{aligned}$$

for all $x, y, z, w \in B_r$ and all $\alpha, \beta, \gamma, \lambda \in [0, 1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

Lemma 1.2. ([4], Lemma 1.6) *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 1.3. ([7], Lemma 2.7) *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2 Main results

In this section, we prove weak and strong convergence theorems of the iterative scheme (1.1) to a common fixed point of nonexpansive mappings T_1, T_2 and T_3. Let $F(t_i), i = 1, 2, 3$ denote the set of all fixed points of T_i, and let $F = \cap_{i=1}^{3} F(T_i)$. We first prove the following lammas.

Lemma 2.1. *Let X be a Banach space and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0, 1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0, 1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined as in (1.1). If $F \neq \emptyset$ then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. Then

$$\begin{aligned}
\|z_n - p\| &= \|a_n T_1 x_n + (1-a_n)x_n - p\| \\
&\leq a_n\|T_1 x_n - p\| + (1-a_n)\|x_n - p\| \\
&\leq a_n\|x_n - p\| + (1-a_n)\|x_n - p\| \\
&\leq \|x_n - p\| \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|b_n T_2 z_n + c_n T_1 x_n + (1-b_n-c_n)x_n - p\| \\
&\leq b_n\|T_2 z_n - p\| + c_n\|T_1 x_n - p\| + (1-b_n-c_n)\|x_n - p\| \\
&\leq b_n\|z_n - p\| + c_n\|x_n - p\| + (1-b_n-c_n)\|x_n - p\| \\
&\leq \|x_n - p\|. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n T_3 y_n + \beta_n T_2 z_n + \gamma_n T_1 x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n - p\| \\
&\leq \alpha_n\|T_3 y_n - p\| + \beta_n\|T_2 z_n - p\| + \gamma_n\|T_1 x_n - p\| \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| \\
&\leq \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| \\
&\leq \|x_n - p\|. \qquad (2.3)
\end{aligned}$$

Thus the sequence $\{\|x_n - p\|\}$ is bounded and decreasing which implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. □

The next lemma is crucial for proving the main theorems.

Lemma 2.2. *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps with $F \neq \emptyset$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined as in (1.1).*

(i) *If* $0 < \liminf_{n\to\infty}\alpha_n$, $0 < \liminf_{n\to\infty} b_n$ *and* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *then* $\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0$.

(ii) *If* $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ *and* $0 < \liminf_{n\to\infty}\alpha_n$, *then* $\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0$.

(iii) *If* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$ *and* $0 < \liminf_{n\to\infty}\beta_n$, *then* $\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0$.

(iv) *If* $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *then* $\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0$.

(v) *If* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ *and* $0 < \liminf_{n\to\infty} \alpha_n$, *then* $\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0$.

(vi) *If* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *then* $\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0$.

(vii) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *then* $\lim_{n\to\infty} \|T_3 y_n - x_n\| = 0$.

Proof. Let $p \in F$. By Lemma 2.1, $\sup_{n\geq 1} \|x_n - p\|$ exists. Choose a number $r > 0$ and $r > \sup_{n\geq 1} \|x_n - p\|$, then by (2.1), (2.2), (2.3) we have that all sequences $\{z_n - p\}, \{y_n - p\}, \{x_n - p\}, \{T_1 x_n - p\}, \{T_2 z_n - p\}, \{T_3 y_n - p\}$ belong to B_r and by Lemma 1.1 there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that

$$\begin{aligned} \|\alpha x + \beta y + \gamma z + \lambda w\|^2 &\leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \lambda\|w\|^2 - \frac{1}{3}\alpha\lambda g(\|x - w\|) \\ &\quad -\frac{1}{3}\beta\lambda g(\|y - w\|) - \frac{1}{3}\gamma\lambda g(\|z - w\|) \end{aligned} \tag{2.4}$$

for all $x, y, z, w \in B_r$ and all $\alpha, \beta, \gamma, \lambda \in [0, 1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

From (1.1) and (2.4) we have

$$\begin{aligned} \|z_n - p\|^2 &= \|a_n(T_1 x_n - p) + 0(0) + 0(0) + (1 - a_n)(x_n - p)\|^2 \\ &\leq a_n\|T_1 x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 \\ &\quad -\frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|) \\ &\leq a_n\|x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 \\ &\quad -\frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|) \\ &= \|x_n - p\|^2 - \frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \|y_n - p\|^2 &= \|b_n(T_2 z_n - p) + c_n(T_1 x_n - p) + 0(0) + (1 - b_n - c_n)(x_n - p)\|^2 \\ &\leq b_n\|T_2 z_n - p\|^2 + c_n\|T_1 x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)] \\ &\leq b_n\|z_n - p\|^2 + c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)] \\ &\leq b_n\|x_n - p\|^2 - \frac{1}{3}b_n a_n(1 - a_n)g(\|T_1 x_n - x_n\|) \\ &\quad +c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)] \end{aligned}$$

$$\begin{aligned} &= \|x_n - p\|^2 - \frac{1}{3}b_n a_n(1-a_n)g(\|T_1x_n - x_n\|) \\ &\quad -\frac{1}{3}(1-b_n-c_n)[b_n g(\|T_2z_n - x_n\|) + c_n g(\|T_1x_n - x_n\|)]. \end{aligned} \tag{2.6}$$

By (1.1), (2.4), (2.5) and (2.6), we also have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(T_3y_n - p) + \beta_n(T_2z_n - p) + \gamma_n(T_1x_n - p) + \\ &\quad (1-\alpha_n-\beta_n-\gamma_n)(x_n - p)\|^2 \\ &\leq \alpha_n\|T_3y_n - p\|^2 + \beta_n\|T_2z_n - p\|^2 + \gamma_n\|T_1x_n - p\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|(x_n - p)\|^2 \\ &\quad -\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_2z_n - x_n\|) \\ &\quad +\gamma_n g(\|T_1x_n - x_n\|)] \\ &\leq \alpha_n\|y_n - p\|^2 + \beta_n\|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|(x_n - p)\|^2 \\ &\quad -\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_2z_n - x_n\|) \\ &\quad +\gamma_n g(\|T_1x_n - x_n\|)] \\ &\leq \alpha_n\|x_n - p\|^2 - \frac{1}{3}\alpha_n b_n a_n(1-a_n)g(\|T_1x_n - x_n\|) \\ &\quad -\frac{1}{3}\alpha_n(1-b_n-c_n)[b_n g(\|T_2z_n - x_n\|) + c_n g(\|T_1x_n - x_n\|)] \\ &\quad +\beta_n\|x_n - p\|^2 - \frac{1}{3}\beta_n a_n(1-a_n)g(\|T_1x_n - x_n\|) + \gamma_n\|x_n - p\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|(x_n - p)\|^2 \\ &\quad -\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_2z_n - x_n\|) \\ &\quad +\gamma_n g(\|T_1x_n - x_n\|)] \\ &= \|x_n - p\|^2 - \frac{1}{3}\alpha_n b_n a_n(1-a_n)g(\|T_1x_n - x_n\|) \\ &\quad -\frac{1}{3}\alpha_n(1-b_n-c_n)[b_n g(\|T_2z_n - x_n\|) + c_n g(\|T_1x_n - x_n\|)] \\ &\quad -\frac{1}{3}\beta_n a_n(1-a_n)g(\|T_1x_n - x_n\|) \\ &\quad -\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_2z_n - x_n\|) \\ &\quad +\gamma_n g(\|T_1x_n - x_n\|)]. \end{aligned} \tag{2.7}$$

Thus

$$\alpha_n b_n a_n(1-a_n)g(\|T_1x_n - x_n\|) \quad \leq \quad 3[\|x_n - p\|^2 - \|x_{n+1} - p\|^2]. \tag{2.8}$$

(i) If $0 < \liminf_{n\to\infty} \alpha_n,\ 0 < \liminf_{n\to\infty} b_n$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, then there exist positive integer n_0 and reals $\eta_1, \eta_2, \eta_3, \eta_4 \in (0,1)$ such that $0 < \eta_1 \leq \alpha_n,\ 0 < \eta_2 \leq b_n,\ 0 < \eta_3 \leq a_n < \eta_4 < 1$ for all $n \geq n_0$. It follows from (2.8) that

$$\eta_1\eta_2\eta_3(1-\eta_4)g(\|T_1x_n - x_n\|) \leq \ 3[\|x_n - p\|^2 - \|x_{n+1} - p\|^2] \quad for\ all\ n \geq n_0.$$

This implies by Lemma 2.1 that $\lim_{n\to\infty} g(\|T_1x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$.

By using (2.7) and Lemma 2.1 with the same method as in (i), then (ii)-(vii) are directly obtained, respectively. □

Lemma 2.3. *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined by the iterative scheme (1.1) if*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \min\{\liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n\} \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, *or*

(iii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *or*

(iv) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \gamma_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ *or*

(v) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *and*
$0 < \liminf_{n\to\infty} b_n$, *or*

(vi) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, *or*

(vii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *or*

(viii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n, \liminf_{n\to\infty} \gamma_n\}$
$\leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,

then $\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0.$

Proof. (i) By Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = 0, \lim_{n\to\infty}\|T_2z_n - x_n\| = 0, \lim_{n\to\infty}\|T_3y_n - x_n\| = 0.$$

It follows that

$$\begin{aligned}
\|T_2x_n - x_n\| &\leq \|T_2x_n - T_2z_n\| + \|T_2z_n - x_n\| \\
&\leq \|z_n - x_n\| + \|T_2z_n - x_n\| \\
&= \|a_nT_1x_n + (1-a_n)x_n - x_n\| + \|T_2z_n - x_n\| \\
&\leq a_n\|T_1x_n - x_n\| + \|T_2z_n - x_n\| \\
&\leq \|T_1x_n - x_n\| + \|T_2z_n - x_n\| \to 0 \qquad \textit{as } n\to\infty, \textit{ and}
\end{aligned}$$

$$\begin{aligned}
\|T_3x_n - x_n\| &\leq \|T_3x_n - T_3y_n\| + \|T_3y_n - x_n\| \\
&\leq \|x_n - y_n\| + \|T_3y_n - x_n\| \\
&= \|b_nT_2z_n + c_nT_1x_n + (1-b_n-c_n)x_n - x_n\| + \|T_3y_n - x_n\| \\
&\leq b_n\|T_2z_n - x_n\| + c_n\|T_1x_n - x_n\| + \|T_3y_n - x_n\| \\
&\leq \|T_2z_n - x_n\| + \|T_1x_n - x_n\| + \|T_3y_n - x_n\| \to 0 \quad \textit{as } n\to\infty.
\end{aligned}$$

By using the same proof as in (i), (ii)- (viii) are obtained. □

Theorem 2.4. *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined by the iterative scheme (1.1) if*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \min\{\liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n\} \leq \liminf_{n\to\infty}(b_n + c_n) < 1$, *or*

(iii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *or*

(iv) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \gamma_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(b_n + c_n) < 1$ *or*

(v) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, *and*
$0 < \liminf_{n\to\infty} b_n$, *or*

(vi) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \inf_{n\to\infty} c_n \leq \liminf_{n\to\infty}(b_n + c_n) < 1$, *or*

(vii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1$, *or*

(viii) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n, \liminf_{n\to\infty} \gamma_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,

and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. (i) By lemma 2.3, we have

$$\lim_{n\to\infty} \|T_1 x_n - x_n\| = \lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0. \tag{2.9}$$

Suppose without loss of generality that T_1 is completely continuous. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1 x_{n_k}\}$ converges. Therefore from (2.9), $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of T_1 and (2.9) we have that $T_1 q = q$, so q is a fixed point of T_1. Since T_2, T_3 are continuous and $\lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0$, we obtain that $q \in F(T_2), q \in F(T_3)$, so $q \in F$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{n\to\infty} x_{n_k} = q$, so $\lim_{n\to\infty} x_n = q$. Since

$$\|y_n - x_n\| \le b_n \|T_2 z_n - x_n\| + c_n \|T_1 x_n - x_n\| \to 0$$

and

$$\|z_n - x_n\| = a_n \|T_1 x_n - x_n\| \to 0 \qquad \textit{as } n \to \infty,$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$

The proof of (ii)-(viii) is similar to that of (i). $\square$

For $c_n = \beta_n = \gamma_n = 0$ for all $n \in N$, the following result are obtained directly from Theorem 2.4.

Corollary 2.5. *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{a_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences defined by the iterative scheme (1.2). If*

$$\begin{aligned} 0 &< \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1, \\ 0 &< \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1, \\ 0 &< \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, \end{aligned}$$

and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

In the next result, we prove weak convergence for the iterative scheme (1.1) for three nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined by the iterative scheme (1.1)*

(i) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *and*
$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

(ii) *If* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

Proof. (i) If follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0.$$

Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.4, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.2, $u, v \in F$. By Lemma 2.1, $lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.3 that $u = v$. Therefor $\{x_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

(ii) The proof of (ii) is similar to that of (i). □

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