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# A Common Fixed Point Theorem in Fuzzy Metric Space Using Property (E. A.) and Implicit Relation

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**Abstract :** Recently, I. Altun and D. Turkoglu [Commun. Korean Math. Soc. 23 (1) (2008), 111-124] proved two common fixed point theorems for continuous compatible maps of type ($\alpha$) or ($\beta$) on a complete fuzzy metric space with an implicit relation. In this paper, our objective is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to weak compatibility and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation in fuzzy metric space. Our result generalizes the result of I. Altun and D. Turkoglu.

**Keywords :** Fuzzy metric space (FM-space); fixed point; weakly compatible maps; implicit relation; property (E. A).

**2000 Mathematics Subject Classification :** 54H25, 47H10.

## 1 Introduction

Zadeh [19] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [7] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to FM-spaces by various authors viz George and Veeramani [4], Grabiec [5], Subrahmanyam [18] and others.

In 1994, Mishra, Sharma and Singh [9] introduced the notion of compatible maps under the name of asymptotically commuting maps in FM-spaces. Singh and Jain [17] studied the notion of weak compatibility in FM-spaces (introduced by Jungck and Rhoades [6] in metric spaces). However, the study of common fixed points of noncompatible maps is also of great interest. Pant [10] initiated the study

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of common fixed points of noncompatible maps in metric spaces. In 2002, Aamri and Moutawakil [1] studied a new property for pair of maps i.e. the so-called property (E. A), which is a generalization of the concept of noncompatible maps in metric spaces. Recently, Pant and Pant [11] studied the common fixed points of a pair of noncompatible maps and the property (E. A) in FM-spaces.

Recently, implicit relations are used as a tool for finding common fixed point of contraction maps (see, [2], [8], [12], [13], [15], [16]). These implicit relations guarantee coincidence point of pair of maps that ultimately leads to the existence of common fixed points of a quadruple of maps satisfying weak compatibility criterion. In 2008, Altun and Turkoglu [3] proved two common fixed point theorems on complete FM-space with an implicit relation. In [3], common fixed point theorems have been proved for continuous compatible maps of type ($\alpha$) or ($\beta$).

Our objective is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to weak compatibility and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation in FM-space. Our result generalizes the result of Altun and Turkoglu [3].

## 2 Preliminaries

**Definition 2.1** [19] Let $X$ be any set. A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0,1]$.

**Definition 2.2** [14] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called *a continuous t-norm* if $([0,1], *)$ is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Note that among a number of possible choices for $*, a * b = \min\{a, b\}$ or simply "$* = \min$ " is the strongest possible universal $t$-norm (see [14]).

**Definition 2.3** [7] The triplet $(X, M, *)$ is a FM-space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

(FM1) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(FM2) $M(x, y, 0) = 0$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$;

(FM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

In the following example (see [4]), we know that every metric induces a fuzzy metric:

**Example 2.4** Let $(X, d)$ be a metric space. Define $a*b = ab$ (or $a*b = \min\{a, b\}$) for all $x, y \in X$ and $t > 0$,

$$M(x, y, z) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a FM-space and the fuzzy metric $M$ induced by the metric $d$ is often referred to as the standard fuzzy metric.

**Definition 2.5** [9] Let $A$ and $B$ maps from a FM-space $(X, M, *)$ into itself. The maps $A$ and $B$ are said to be *compatible* (or asymptotically commuting), if for all $t$,

$$\lim_n M(ABx_n, BAx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

From the above definition it is inferred that $A$ and $B$ are noncompatible maps from a FM-space $(X, M, *)$ into itself if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but either $\lim_n M(ABx_n, BAx_n, t) \neq 1$ or the limit does not exist.

**Definition 2.6** [17] Let $A$ and $B$ be maps from a FM-space $(X, M, *)$ into itself. The maps are said to be *weakly compatible* if they commute at their coincidence points, that is, $Az = Bz$ implies that $ABz = BAz$. Note that compatible mappings are weakly compatible but converse is not true in general.

**Definition 2.7** [11] Let $A$ and $B$ be two self-maps of a FM-space $(X, M, *)$. We say that $A$ and $B$ satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Note that weakly compatible and property (E.A) are independent to each other (see [12], Ex. 2.2).

**Lemma 2.8** [9] *If for all $x, y \in X, t > 0$ and for a number $k \in (0, 1)$ ; $M(x, y, kt) \geq M(x, y, t)$, then $x = y$.*

Throughout this paper, $(X, M, *)$ is considered to be a FM-space with condition

(FM6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

## 3 Implicit Relation

In our result, we deal with implicit relation used in [3]. In [3], Altun and Turkoglu used the following implicit relation: Let $I = [0, 1], *$ be a continuous $t$-norm and $\mathcal{F}$ be the set of all real continuous functions $F : I^6 \to \mathbb{R}$ satisfying the following conditions:

**(F-1)** $F$ is nonincreasing in the fifth and sixth variables,

**(F-2)** if, for some constant $k \in (0,1)$ we have

**(F-a)** $F\left(u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})\right) \geq 1$, or

**(F-b)** $F\left(u(kt), v(t), u(t), v(t), u(\frac{t}{2}) * v(\frac{t}{2}), 1\right) \geq 1$

for any fixed $t > 0$ and any nondecreasing functions $u, v : (0, \infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$,

**(F-3)** if, for some constant $k \in (0,1)$ we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$ for any fixed $t > 0$ and any nondecreasing function $u : (0, \infty) \to I$ then $u(kt) \geq u(t)$.

## 4 Main Results

In [3], Altun and Turkoglu proved the following:

**Theorem 4.1** *Let $(X, M, *)$ be a complete fuzzy metric space with $a*b = \min\{a, b\}$ for all $a, b \in I$ and $A, B, S$ and $T$ be maps from $X$ into itself satisfying the conditions:*

*(4.1)* $A(X) \subseteq T(X), B(X \subseteq S(X))$;

*(4.2) one of the maps $A, B, S$ and $T$ is continuous;*

*(4.3) $(A, S)$ and $(B, T)$ are compatible of type $(\alpha)$;*

*(4.4) there exist $k \in (0,1)$ and $F \in \mathcal{F}$ such that*

$$F\begin{pmatrix} M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t) \\ M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t) \end{pmatrix} \geq 1$$

*for all $x, y \in X, t > 0$.*

*Then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

Now, we prove the following

**Theorem 4.2** *Let $(X, M, *)$ be a fuzzy metric space with $a * b = \min\{a, b\}$ for all $a, b \in I$ Further, let $(A, S)$ and $(B, T)$ be weakly compatible pairs of self-maps of $X$ satisfying (4.1), (4.4) and (4.5), $(A, S)$ or $(B, T)$ satisfies the property (E.A). If the range of one of the maps $A, B, S$ or $T$ is a complete subspace of $X$ then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** If the pair $(B, T)$ satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in $X$ such that $Bx_n \to z$ and $Tx_n \to z$, for some $z \in X$ as $n \to \infty$. Sine $B(X) \subseteq S(X)$ there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = Sy_n$. Hence, $Sy_n \to z$ as $n \to \infty$. Also, since $A(X) \subseteq T(X)$, there exists a sequence $\{y'_n\}$ in $X$ such that $Ay'_n = Tx_n$. Hence, $Ay'_n \to z$ as $n \to \infty$.

Suppose that $S(X)$ is a complete subspace of $X$. Then, $z = Su$ for some $u \in X$. Subsequently, we have $Ay'_n \to Su, Bx_n \to Su, Tx_n \to Su$ and $Sy_n \to Su$ as $n \to \infty$. By (4.4), we have

$$F\begin{pmatrix} M(Au, Bx_n, kt), M(Su, Tx_n, t), M(Au, Su, t) \\ M(Bx_n, Tx_n, t), M(Au, Tx_n, t), M(Bx_n, Su, t) \end{pmatrix} \geq 1.$$

Letting $n \to \infty$, we have

$$F(M(Au, Su, kt), 1, M(Au, Su, t), 1, M(Au, Su, t), 1) \geq 1.$$

On the other hand, since

$$M(Au, Su, t) \geq M\left(Au, Su, \frac{t}{2}\right) = M\left(Au, Su, \frac{t}{2}\right) * 1$$

and $F$ is nonincreasing in the fifth variable, we have, for any $t > 0$

$$\begin{aligned} &F\left(M(Au, Su, kt), 1, M(Au, Su, t), 1, M\left(Au, Su, \frac{t}{2}\right) * 1, 1\right) \\ &\geq F(M(Au, Su, kt), 1, M(Au, Su, t), 1, M(Au, Su, t), 1) \\ &\geq 1, \end{aligned}$$

which implies, by **(F-2)** that $Au = Su$. The weak compatibility of $A$ and $S$ implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $A(X) \subseteq T(X)$, there exists a $v \in X$ such that $Au = Tv$. We now show that $Tv = Bv$. By (4.4), we have

$$F\begin{pmatrix} M(Au, Bv, kt), M(Su, Tv, t), M(Au, Su, t) \\ M(Bv, Tv, t), M(Au, Tv, t), M(Bv, Su, t) \end{pmatrix} \geq 1,$$

that is,

$$F(M(Tv, Bv, kt), 1, 1, M(Bv, Tv, t), 1, M(Bv, Tv, t)) \geq 1.$$

On the other hand, since

$$M(Bv, Tv, t) \geq M\left(Bv, Tv, \frac{t}{2}\right) = M\left(Bv, Tv, \frac{t}{2}\right) * 1$$

and $F$ is nonincreasing in the sixth variable, we have, for any $t > 0$,

$$\begin{aligned} &F\left(M(Bv, Tv, kt), 1, 1, M(Bv, Tv, t), 1, M\left(Bv, Tv, \frac{t}{2}\right) * 1\right) \\ &\geq F(M(Bv, Tv, kt), 1, 1, M(Bv, Tv, t), 1, M(Bv, Tv, t)) \\ &\geq 1, \end{aligned}$$

which implies, by **(F-2)**, that $Bv = Tv$. This implies that $Au = Su = Tv = Bv$. The weak compatibility of $B$ and $T$ implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that $Au$ is a common fixed point of $A, B, S$ and $T$. In view of (4.4), it follows

$$F\begin{pmatrix} M(AAu, Bv, kt), M(SAu, Tv, t), M(AAu, SAu, t) \\ M(Bv, Tv, t), M(AAu, Tv, t), M(Bv, SAu, t) \end{pmatrix} \geq 1,$$

that is,

$$F(M(AAu, Au, kt), M(SAu, Au, t), M(AAu, Au, t), M(Au, AAu, t)) \geq 1.$$

Thus, from **(F-3)**, we have $M(AAu, Au, kt) \geq M(AAu, Au, t)$. By the Lemma 2.8, we have, $AAu = Au$.

Therefore, $Au = AAu = SAu$ and $Au$ is a common fixed point of $A$ and $S$. Similarly, we can prove that $Bv$ is a common fixed point of $B$ and $T$. Since $Au = Bv$ we conclude that $Au$ is a common fixed point of $A, B, S$ and $T$. The proof is similar when $T(X)$ is assumed to be a complete subspace of $X$. The cases in which $A(X)$ or $B(X)$ is a complete subspace of $X$ are similar to the cases in which $T(X)$ or $S(X)$ respectively, is complete since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Sv = Tv = v$, then (4.4) gives

$$F\begin{pmatrix} M(Au, Bv, kt), M(Su, Tv, t), M(Au, Su, t), \\ M(Bv, Tv, t), M(Au, Tv, t), M(Bv, Su, t) \end{pmatrix} \geq 1,$$

that is,

$$F(M(u, v, kt), M(u, v, t), 1, 1, M(u, v, t), M(v, u, t) \geq 1.$$

Thus, from **(F-3)**, we have $M(u, v, kt) \geq M(u, v, t)$. By Lemma 2.8, we have $u = v$. Therefore, $u = v$ and the common fixed point is unique. □

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