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On L_p–Inverse Theorem for a Linear Combination of Szász-Beta Operators

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Abstract : For $f \in L_p[0,\infty), 1 \leqslant p < \infty$ Gupta et al [10] introduced a sequence of linear positive operators by coupling the well-known Szász operators and beta operators called as Szász–beta operators . In this paper we obtain an inverse theorem for a linear combination of these operators.

Keywords : Linear combinations, L_p-approximation, Steklov means, modulus of continuity.
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1 Introduction

The Szász-beta operators are defined by

$$B_n(f;t) = \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \int_0^{\infty} b_{n,\nu}(u) f(u)\, du,\ \ t \in [0,\infty),$$

where

$$p_{n,\nu}(t) = \frac{e^{-nt}(nt)^{\nu}}{\nu!},\ \ b_{n,\nu}(t) = \frac{1}{B(\nu+1,n)} \frac{t^{\nu}}{(1+t)^{n+\nu+1}}$$

and $B(\nu+1,n)$ is the well known beta integral.

The operators B_n can be expressed as

$$B_n(f;t) = \int_0^{\infty} W_n(t,u) f(u)\, du,$$

where $W_n(t,u) = \sum_{\nu=0}^{\infty} p_{n,\nu}(t)\, b_{n,\nu}(u)$ is the kernel of the operators.

For some other modifications of the Szász operators we refer the reader to [7],[8] and [12].

It turns out the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators B_n, Prerna [6] considered the linear combination $B_n(f,k,x)$ of operators B_n, and obtained a direct theorem for these combinations in the L_p-norm.

The linear combination $B_n(f,k,x)$ of the operators B_n, is defined as

$$B_n(f,k,x)=\sum_{j=0}^{k}C(j,k)B_{d_jn}(f,x),$$

where

$$C(j,k)=\prod_{i=0,i\neq j}^{k}\frac{d_j}{d_j-d_i},k\neq 0 \text{ and } C(0,0)=1,$$

$d_0,d_1,...d_k$ are $(k+1)$ arbitrary but fixed distinct positive integers. Throughout this paper let $0<a<a_1<a_2<a_3<b_3<b_2<b_1<b<\infty$, $I=[a,b]$ $I_i=[a_i,b_i], i=1,2,3$ and $f\in L_p(I)$, $1\leqslant p<\infty$.

The m^{th} order integral modulus of smoothness of f is defined as

$$\omega_m\big(f,\tau,p,I\big)=\sup_{0<\delta\leqslant\tau}\|\Delta_\delta^m f(t)\|_{L_p[a,b-m\delta]},$$

where Δ_δ^m is the forward difference operator with step length δ and $0<\tau\leqslant(b-a)/m$. Further χ denotes the characteristic function of the interval I_1 and C a positive constant not necessarily the same in different cases.

In [6] it was established that for $f\in L_p[0,\infty), 1\leqslant p<\infty$ there holds

$$\|B_n(f,k,.)-f\|_{L_p(I_2)}\leqslant C\left(\omega_{2k+2}\big(f,n^{-1/2},p,I_1\big)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right),$$

where C depends on k and p, but is independent of f and n.

In the present paper, we obtain a corresponding inverse result, i.e.,the characterization of the class of functions for which $\|B_n(f,k,.)-f\|_{L_p(I_1)}=O\big(n^{-\alpha/2}\big)$ as $n\to\infty$, where $0<\alpha<2k+2$. Thus we prove the following theorem (*inverse theorem*):

Theorem 1.1. *Let $f\in L_p[0,\infty), p\geqslant 1,\ 0<\alpha<2k+2$ and $\|B_n(f,k,.)-f\|_{L_p(I_1)}=O(n^{-\alpha/2})$ as $n\to\infty$. Then, $\omega_{2k+2}(f,\tau,p,I_2)=O(\tau^\alpha)$ as $\tau\to 0$.*

2 Preliminaries

In this section we give some results which are useful in establishing our main theorem.

Lemma 2.1. *[6] For the function $\lambda_{n,m}(x)$ defined by*

$$\lambda_{n,m}(x)=\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\left(\frac{\nu}{n}-x\right)^m,$$

we have $\lambda_{n,0}(x)=1,\ \lambda_{n,1}(x)=0,$ *and there holds the recurrence relation:*

$$n\lambda_{n,m+1}(x)=x\big(m\lambda_{n,m-1}(x)+\lambda'_{n,m}(x)\big),\ m\geqslant 1.$$

Consequently, we have

(i) $\lambda_{n,m}(x)$ *is a polynomial in* x *of degree* $[m/2]$;

(ii) $\lambda_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right),\ x\in[0,\infty),$

where $[\beta]$ *is the integer part of* β.

Lemma 2.2. *[6] For* $m\in N^0$ *(the set of non-negative integers), the* m*th order moment for the operators* B_n *be defined as*

$$V_{n,m}(t)=B_n\left((u-t)^m;t\right).$$

Then $V_{n,0}(t)=1$, $V_{n,1}(t)=\frac{t+1}{n-1}$, $V_{n,2}(t)=\frac{(n+2)t^2+2(n+2)t+2}{(n-1)(n-2)}$, *and there holds the recurrence relation,*

$(n-m-1)V_{n,m+1}(t)=t\,V^{(1)}_{n,m}(t)+\{(m+1)(2t+1)-t\}\,V_{n,m}(t)+mt(2+t)V_{n,m-1}(t),\ m\geqslant 1.$

Consequently, for each $t\geqslant 0$,

$$V_{n,m}(t)=O\big(n^{-[(m+1)/2]}\big).$$

For sufficiently small $\eta>0$ the Steklov mean $f_{\eta,m}$ of m th order corresponding to f is defined as follows:

$$f_{\eta,m}(t)=\eta^{-m}\int_{-\eta/2}^{\eta/2}...\int_{-\eta/2}^{\eta/2}\Big(f(t)+(-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)\Big)\prod_{i=1}^m dt_i, t\in I_1.$$

Lemma 2.3. *For the function* $f_{\eta,m}$*, we have*

(a) $f_{\eta,m}$ *has derivatives up to order* m *over* I_1*;*

(b) $\|f^{(r)}_{\eta,m}\|_{L_p(I_1)}\leqslant C_r\,\eta^{-r}\omega_r(f,\eta,I), r=1,2,...,m;$

(c) $\|f-f_{\eta,m}\|_{L_p(I_1)}\leqslant C_{m+1}\ \omega_m(f,\eta,I);$

(d) $\|f_{\eta,m}\|_{L_p(I_1)}\leqslant C_{m+2}\ \eta^{-m}\|f\|_{L_p(I)};$

(e) $\|f^{(r)}_{\eta,m}\|_{L_p(I_1)}\leqslant C_{m+3}\ \|f\|_{L_p(I)}$

where C_i'*s are certain constants that depend on* i *but are independent of* f *and* η.

Following [Theorem 18.17, [2]], or [pp.163-165,[1]], the proof of the above lemma easily follows hence the details are omitted.

We establish the following Lorentz [3] type lemma:

Lemma 2.4. *[6] There exist the polynomials $Q_{i,j,r}(t)$ independent of n and ν such that*

$$t^r\frac{d^r p_{n,\nu}(t)}{dt^r}=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i(\nu-nt)^j Q_{i,j,r}(t)p_{n,\nu}(t).$$

Lemma 2.5. *Let $h\in L_p[0,\infty), p\geqslant 1$ has a compact support, $i,j\in N^0$ and $m>0$ be fixed. Then, for a constant C independent of n and h there holds*

$$\left\|\int_0^\infty\sum_{\nu=0}^\infty p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^i\int_t^u b_{n,\nu}(u)(u-w)^j h(w)\,dw\,du\right\|_{L_p(I_2)}$$
$$\leqslant C\left\{n^{-(i+j+1)/2}\|h\|_{L_p(I_1)}+n^{-m}\|h\|_{L_p[0,\infty)}\right\}.$$

Proof. Defining $s=jp+p-1$ and using Jensen's inequality repeatedly, we get

$$\left|\int_0^\infty\sum_{\nu=0}^\infty p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^i\int_t^u b_{n,\nu}(u)(u-w)^j h(w)\,dw\,du\right|^p$$
$$\leqslant\sum_{\nu=0}^\infty p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{ip}\int_0^\infty b_{n,\nu}(u)(u-t)^s\left|\int_t^u|h(w)|^p\,dw\right|du$$
$$\leqslant\sum_{\nu=0}^\infty p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{ip}\int_0^\infty \psi(u)b_{n,\nu}(u)|u-t|^s\left|\int_t^u|h(w)|^p\,dw\right|du$$
$$+\sum_{\nu=0}^\infty p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{ip}\int_0^\infty (1-\psi(u))b_{n,\nu}(u)|u-t|^s\left|\int_t^u|h(w)|^p\,dw\right|du(2.1)$$

In the first integral we divide integration in $'u'$ over $[t+\frac{l}{\sqrt{n}},t+\frac{(l+1)}{\sqrt{n}}]$, $l=0,\pm1,...,\pm r$; where $r=r(n)\in N$ satisfies $rn^{-1/2}\leqslant\max(b_1-a_2,b_2-a_1)\leqslant(r+1)\frac{1}{\sqrt{n}}$. A typical element of the 1st term of (2.1) is now L_p- bounded by

$$\frac{n_2}{l^4}\int_{a_2}^{b_2}\left[\sum_{\nu=0}^\infty p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{ip}\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{(l+1)}{\sqrt{n}}} b_{n,\nu}(u)|u-t|^{s+4}\,du\left(\int_t^{t+\frac{(l+1)}{\sqrt{n}}}\psi(w)|h(w)|^p\,dw\right)\right]dt.$$

We now use Hölder's inequality for infinite sum coupled with moment estimates and finally Fubini's theorem to obtain estimate. The presence of factor $(1-\psi(u))$ in second term in (2.1) implies $|u-t|/\delta>1$. This gives arbitrary order $O\big(n^{-m}\big)$. This completes the proof.

Lemma 2.6. *Let $h \in L_p[0,\infty), p \geqslant 1$ and $\operatorname{supp} h \subset I_2$. Then*

$$\left\| B_n^{(2k+2)}(h,.) \right\|_{L_p(I_2)} \leqslant C\, n^{k+1} \|h\|_{L_p(I_2)}. \tag{2.2}$$

Moreover, if $h^{(2k+1)} \in A.C.(I_2)$ and $h^{(2k+2)} \in L_p(I_2)$, then

$$\left\| B_n^{(2k+2)}(h,.) \right\|_{L_p(I_2)} \leqslant C' \left\| h^{(2k+2)} \right\|_{L_p(I_2)}, \tag{2.3}$$

the constants C and C' are independent of n and h.

Proof. Since $Q_{i,j,2k+2}$ and $t^{-(2k+2)}$ are bounded on I_2, it follows from Lemmas 2.1 and 2.4 that for $h \in L_1[0,\infty)$

$$\left\| B_n^{(2k+2)}(h,.) \right\|_{L_1(I_2)} \leqslant C\, n^{k+1} \, \|h\|_{L_1(I_2)} .$$

If $h \in L_\infty[0,\infty)$, then by Lemma 2.4 and moment estimates we get

$$\left\| B_n^{(2k+2)}(h,.) \right\|_{L_\infty(I_2)} \leqslant C\, n^{k+1} \, \|h\|_{L_\infty(I_2)} .$$

Now, using Riesz-Thorin interpolation theorem [4], we obtain(2.2) To obtain (2.3), the differentiability properties of h imply that

$$h(u) = \sum_{r=0}^{2k+1} \frac{(u-t)^r}{r!} h^{(r)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)\, dw.$$

Using Lemma 2.4 we have

$$\begin{aligned} B_n^{(2k+2)}(h,t) &= \frac{1}{(2k+1)!t^r} \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \sum_{i,j} n^i (\nu - nt)^j Q_{i,j,2k+2}(t) \times \\ &\qquad \times \int_0^\infty \int_t^u b_{n,\nu}(u)(u-w)^{2k+1} h^{(2k+2)}(w)\, dw\, du. \end{aligned}$$

Now, applying Lemma 2.5 in above we obtain

$$\begin{aligned} B_n^{(2k+2)}(h,t) &\leqslant C \sum_{i,j} n^{i+j} \left\| \int_0^\infty \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \Big(\frac{\nu}{n} - t\Big)^j Q_{i,j,2k+2}(t) \times \right. \\ &\qquad \left. \times \int_t^u b_{n,\nu}(u)(u-w)^{2k+1} h^{(2k+2)}(w)\, dw\, du \right\|_{L_p(I_2)} \\ &\leqslant C \left\| h^{(2k+2)} \right\|_{L_p(I_2)} . \end{aligned}$$

Thus, we get (2.3).

3 Proof of Main Theorem

Proof. We choose points $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$ and a function $g \in C_0^{2k+2}$ such that $\operatorname{supp} g \subset (x_2, y_2)$, $g(t) = 1$ on $[x_3, y_3]$ and $[x_i, y_i] \subset [x_{i-1}, y_{i-1}]$, $i = 2, 3.$ with $[x_i, y_i] \subset I_1$. Writing $fg = \mathcal{F}$, for all values of $r \leqslant \gamma$ we have

$$\begin{aligned}\|\Delta_r^{2k+2}\mathcal{F}\|_{L_p[x_2,y_2]} &\leqslant \|\Delta_r^{2k+2}(\mathcal{F} - B_n(\mathcal{F}, k, .))\|_{L_p[x_2,y_2]} \\ &+ \|\Delta_r^{2k+2}B_n(\mathcal{F}, k, .)\|_{L_p[x_2,y_2]}.\end{aligned}$$

On a repeated application of Jensen's inequality and then Fubini's theorem we obtain

$$\begin{aligned}\|\Delta_r^{2k+2}\mathcal{F}\|_{L_p[x_2,y_2]} &\leqslant \|\Delta_r^{2k+2}(\mathcal{F} - B_n(\mathcal{F}, k, .))\|_{L_p[x_2,y_2]} \\ &\quad + r^{2k+2}\|B_n^{(2k+2)}(\mathcal{F}, k, .)\|_{L_p[x_2,y_2+(2k+2)r]}. \qquad (3.1)\end{aligned}$$

In second term of (3.1) we write $\mathcal{F} = \big(\mathcal{F} - \mathcal{F}_{\eta,2k+2}\big) + \mathcal{F}_{\eta,2k+2}$, where $\mathcal{F}_{\eta,2k+2}$ is the $(2k+2)$th order Steklov mean of $\mathcal{F}$ and then use Lemma 2.6. It follows from the properties of the Steklov mean that for sufficiently small $\eta > 0$,

$$\begin{aligned}\|\Delta_r^{2k+2}\mathcal{F}\|_{L_p[x_2,y_2]} &\leqslant \|\Delta_r^{2k+2}(\mathcal{F} - B_n(\mathcal{F}, k, .))\|_{L_p[x_2,y_2]} \\ &\quad + C\, r^{2k+2}\Big(n^{k+1} + \eta^{-(2k+2)}\Big)\,\omega_{2k+2}\big(\mathcal{F}, \eta, p, [x_2, y_2]\big).\end{aligned}$$

Now, following the lemma of Berens and Lorentz [5] we can complete the proof once it is established that

$$\|\Delta_r^{2k+2}(\mathcal{F} - B_n(\mathcal{F}, k, .))\|_{L_p[x_2,y_2]} = O\big(n^{-\alpha/2}\big), \quad n \to \infty. \qquad (3.2)$$

Thus,

$$\omega_{2k+2}\big(\mathcal{F}, \tau, p, [x_2, y_2]\big) = O\big(\tau^{\alpha}\big), \quad \tau \to 0.$$

Therefore, as $\mathcal{F} = f$ for $t \in [x_3, y_3]$, $\omega_{2k+2}\big(f, \tau, p, [x_2, y_2]\big) = O\big(\tau^{\alpha}\big)$, $\tau \to 0$ as required.

We prove this by induction on α. Consider the case $\alpha \leqslant 1$.

$$\begin{aligned}\|B_n(fg, k, t) - (fg)(t)\|_{L_p[x_2,y_2]} &\leqslant \|g(t)B_n(f(u) - f(t)), k, t))\|_{L_p[x_2,y_2]} \\ &\quad + \|B_n(f(u)(g(u) - g(t)), k, t)\|_{L_p[x_2,y_2]}.\end{aligned}$$

Now, $g(u) - g(t) = (u - t)g'(\theta)$ for some θ lying between u and t. Using Jensen's inequality, Fubini's theorem, moment estimates and the compactness of f to estimate the second term and statement of the theorem we get

$$\|B_n(fg, k, t) - (fg)(t)\|_{L_p[x_2,y_2]} = O\big(n^{-\alpha/2}\big) + O\big(n^{-1/2}\big) = O\big(n^{-\alpha/2}\big).$$

This proves (3.2) when $\alpha \leqslant 1$.

Now, we assume (3.2) to hold true for all values of α satisfying $r-1<\alpha<r$ and prove that the same holds true for $r<\alpha<r+1$. Thus we have

$$\omega_{2k+2}\big(\mathcal{F},\tau,p,[c,d]\big)=O\big(\tau^{r-1+\beta}\big),\quad \tau\to 0,\ 0<\beta<1,$$

for any $[c,d]\subset(a_1,b_1)$.

Hence, following [1] it follows that f coincides a.e. on $[x_2,y_2]\subset(c,d)$ with a function F possessing an absolutely continuous derivative $F^{(r-2)}$ and the $(r-1)$th derivative $F^{(r-1)}\in L_p[x_2,y_2]$. Let χ denote the characteristic function of the interval $[x_1,y_1]$.

Therefore, we have

$$\begin{aligned}
&\|B_n(fg,k,t)-(fg)(t)\|_{L_p[x_2,y_2]}\\
&\leqslant\sum_{i=0}^{r-2}\frac{1}{i!}\|f^{(i)}(t)B_n(u-t)^i(g(u)-g(t)),k,t)\|_{L_p[x_2,y_2]}\\
&+\frac{1}{(r-2)!}\Bigg\|B_n\Bigg(\chi(u)(g(u)-g(t))\Bigg(\int_t^u(u-w)^{r-2}\times\\
&\qquad\times\Big(f^{(r-1)}(w)-f^{(r-1)}(t)\Big)\,dw\Bigg),k,t\Bigg)\Bigg\|_{L_p[x_2,y_2]}\\
&+\|B_n(F(u,t)(1-\chi(u)))(g(u)-g(t),k,t)\|_{L_p[x_2,y_2]}\\
&=J_1+J_2+J_3,\qquad\text{say,}
\end{aligned}$$

where $F(u,t)=f(u)-\sum_{i=0}^{r-2}\frac{(u-t)^i}{i!}f^{(i)}(t);\ u\in[0,\infty),\ t\in[x_2,y_2]$. Using mean value theorem on g the direct theorem 3.1 [6] and the moment estimates we get $J_1,J_3=O\big(n^{-(k+1)}\big)$, $n\to\infty$. By repeated application of Jensen's inequality, mean value theorem on g and breaking $[t,u]$ as in Lemma 2.5, we have

$$\begin{aligned}
\|J_2\|^p&=\int_{x_2}^{y_2}\Bigg|B_n\Bigg(\chi(u)(g(u)-g(t))\int_t^u(u-w)^{r-2}\times\\
&\qquad\times\Big(f^{(r-1)}(w)-f^{(r-1)}(t)\Big)\,dw,k,t\Bigg)\Bigg|^p dt\\
&\leqslant C\int_{x_2}^{y_2}\int_{x_1}^{y_1}W_n(t,u)|u-t|^{rp-1}\int_t^u\chi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(t)\right|^p dw\,du\,dt\\
&\leqslant C\sum_{l=1}^{r}\int_{x_2}^{y_2}\Bigg[\Bigg(\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{(l+1)}{\sqrt{n}}}\int_t^{t+\frac{(l+1)}{\sqrt{n}}}+\int_{t-\frac{(l+1)}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}}\int_{t-\frac{(l+1)}{\sqrt{n}}}^{t}\Bigg)\left(\frac{n^2}{l^4}\right)^p W_n(t,u)\times\\
&\qquad\times|u-t|^{rp+4p-1}\chi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(t)\right|^p dw\,du\Bigg]\,dt
\end{aligned}$$

$$+ \int_{x_2}^{y_2} \Big[\int_{x_2-\frac{1}{\sqrt{n}}}^{y_2+\frac{1}{\sqrt{n}}} \int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}} W_n(t,u)|u-t|^{rp-1}\chi(w) \times$$

$$\times \left| f^{(r-1)}(w) - f^{(r-1)}(t) \right|^p dw\, du \Big] dt$$

$$\leqslant C \sum_{l=1}^{r} \left(\frac{n^2}{l^4} \right)^p \int_0^{t+\frac{l}{\sqrt{n}}} \omega\big(f^{(r-1)}, w, p, [x_1, y_1]\big)^p dw$$

$$+ n^{-(rp-1)/2} \int_0^{\frac{1}{\sqrt{n}}} \omega\big(f^{(r-1)}, w, p, [x_1, y_1]\big)^p dw,$$

on using moment estimates and then interchanging order of integration in t and w. Lastly, utilizing $\omega\big(f^{(r-1)}, w, p, [x_1, y_1]\big) = O(w^p)$ we find

$$J_2 = O\big(n^{-(r+\beta)/2}\big), \quad n \to \infty.$$

Combining the estimates of J_1, J_2 and J_3, we obtain (3.2). The proof of (3.2) shows that

$$\omega_{2k+2}(f, \tau, p, [x_2, y_2]) = O(\tau^{\alpha}), \quad \alpha < 2k+2, \ \alpha \neq 2, 3, .., 2k+1. \tag{3.3}$$

This very statement implies that it is also true for integer values 2,3,...$2k+1$. To prove this, let $\alpha = r$ where r takes value from 2,3,...,$2k+1$.

Then, since (3.3) is true for $(r, r+1)$, it follows that

$$\begin{aligned} \omega_{2k+2}(f, \tau, p, [x_2, y_2]) &= O(\tau^{r+\beta}), \quad 0 < \beta < 1 \\ &= O(\tau^r). \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.1. *Similar results can be obtained for the operators $L_n(f, x)$ and the operators $M_{n,\alpha,\beta}(f, x)$ for $\alpha = \beta = 1$ and $I_n = \{0\}$ defined as follows:*

$$L_n(f; x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)\, dt,$$

where $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}$, $b_{n,k}(t) = \frac{t^k}{B(k+1,n)(1+t)^{n+k+1}}$ *and*

$$M_{n,\alpha,\beta}(f, x) = (n-\alpha+1) \sum_{k=\beta}^{n-\alpha+\beta} p_{n,k}(x) \int_0^1 p_{n-\alpha,k-\beta}(t) f(t)\, dt + \sum_{k \in I_n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

in [9] and [11] respectively.

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