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Annihilator Conditions Relative to a Class of Modules

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Abstract : Annihilator conditions relative to a class of modules are studied and used to characterize the relative extending modules. In particular, dual rings relative to the class of all small right ideals, called right small-dual rings, are investigated and some known results on the dual rings are generalized to the case of small-dual rings.

Keywords : Annihilator condition, Extending module, Small-dual ring.
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1 Introduction

A ring R is called a right dual ring, if every right ideal I of R is a right annihilator, that is, $r_R l_R(I) = I$. Analogously a left dual ring is defined, a left and right dual ring is called a dual ring in [8]. For a given one-sided ideal of a ring R, it may or may not be easy to check if it is an annihilator. Moreover, for any ring R, in general there will be some one-sided ideals which do have the property that they are annihilators. Thus the annihilator conditions are limited to a special class of one-sided ideals of a ring. For instance, if every maximal right ideal of R is a right annihilator, then R is called a right Kasch ring[2]; if every essential right ideal of R is a right annihilator, then R is called a right quasi-dual ring[11].

Using similar thought, Doğruöz and Smith in [5] introduce extending modules with respect to modules classes. Let $\mathscr{L}$ be a class of right R-modules, according to [5], an $\mathscr{L}$-submodule N of M means that N is a submodule of M with $N \in \mathscr{L}$; a right R-module M is type 2 $\mathscr{L}$-extending[5] if for every $\mathscr{L}$-submodule N of M, every closure of N in M is a direct summand of M; a right R-module M is called weak type 2 $\mathscr{L}$-extending if every $\mathscr{L}$-submodule of M is essential in a direct summand of M. For a special class of right R-modules, we recently investigate the weak type 2 $\mathscr{L}$-extending modules in [14], which is extending relative to the class $\mathscr{L}$ of finitely generated submodules of M.

Motivated by these, in this paper we investigate annihilator conditions of a module M with respect to a general class of right R-modules, and obtain that these annihilator conditions are closely connected with the relative extending modules in [5]. In Section 2, to built the consistency of each relative dual module and ring, we define the relative annihilator conditions of a module, that is, an $\mathscr{L}$-dual module, and obtain some characterizations of weak type 2 $\mathscr{L}$-extending modules [5] by $\mathscr{L}$-dual modules. As applications, in Section 3, a dual ring relative to the class of all small right ideals, that is, a small-dual ring, is studied, and some known results on the dual rings are generalized to the case of small-dual rings.

In the sequel sections, the notion $A \subseteq^e B$ (resp. $A \subseteq^{\oplus} B$) means that A is an essential submodule (resp. a direct summand) of B. Assume that M is a right R-module and $S = \text{End}(M_R)$, let $l_S(N) = \{f \in S \mid f(n) = 0, \forall n \in N\}$ be the left annihilator of N in S. Similarly, $r_M(I) = \{m \in M \mid f(m) = 0, \forall f \in I\}$ be the right annihilator of I in M. By a class $\mathscr{L}$ of right R-modules we mean a collection of right R-modules which contains the zero module and which is closed under isomorphisms. For other terminology we refer to [1] and [6].

2 Relative dual modules

Definition 2.1. *Let $\mathscr{L}$ be a class of right R-modules, a right R-module M is called an $\mathscr{L}$-dual module if $r_M l_S(N) = N$ for each $\mathscr{L}$-submodule N of M. In particular, if R_R is an $\mathscr{L}$-dual module, then R is called a right $\mathscr{L}$-dual ring. Similarly, $\mathscr{L}$-dual left R-modules and left $\mathscr{L}$-dual rings are defined.*

Example 2.2. *Let R be a ring.*

(1) *If $\mathscr{L}$ is the class of all right ideals of R, then the right $\mathscr{L}$-dual ring R is called a right dual ring[8]. If R is a left and right dual ring, then R is called a dual ring;*

(2) *If $\mathscr{L}$ is the class of all maximal right ideals of R, then the right $\mathscr{L}$-dual ring R is called a right Kasch ring[2];*

(3) *If $\mathscr{L}$ is the class of all essential right ideals of R, then the right $\mathscr{L}$-dual ring R is called a right quasi-dual ring[11].*

Let $\mathscr{L}$ be a class of right R-modules, according to [5] $\mathscr{L}^e$ denotes the class of right R-modules which contain an essential $\mathscr{L}$-submodule, and so $\mathscr{L} \subseteq \mathscr{L}^e$. It is proved that M is type 2 $\mathscr{L}$-extending if and only if M is weak type 2 $\mathscr{L}^e$-extending[5, Theorem 2.7]. In the following, let $\mathscr{L}^c$ denote the class of right R-modules, which contain an essential $\mathscr{L}$-submodule and have no essential extensions. Note that $\mathscr{L}^c \subseteq \mathscr{L}^e$ and every $\mathscr{L}^c$-submodule of a right R-module M is closed in M, thus

Proposition 2.3. *Let $\mathscr{L}$ be a class of right R-modules. If M is a type 2 $\mathscr{L}$-extending module, then M is an $\mathscr{L}^c$-dual module.*

Proof. Note that for each $\mathscr{L}^c$-submodule N of M, there is a submodule $N_0 \in \mathscr{L}$ such that N_0 is essential in N. Since N is a closed submodule of M, thus it is a direct summand of M by hypothesis, it follows that $N = r_M l_S(N)$, so M is an $\mathscr{L}^c$-dual module. □

We firstly give some general characterizations of $\mathscr{L}$-dual modules. For each $L \in \mathscr{L}$, M/L is called an *$\mathscr{L}$-dense factor module* of M.

Proposition 2.4. *The following are equivalent:*

(1) *M is an $\mathscr{L}$-dual module;*
(2) *For each $\mathscr{L}$-submodule N of M and $m \in M$, if $l_S(N) \subseteq l_S(m)$, then $m \in N$;*
(3) *every $\mathscr{L}$-dense factor module of M is cogenerated by M (i.e., can be embedded in M^I, where I is an index set).*

Proof. (1) ⇒ (2). For each $\mathscr{L}$-submodule N of M and $m \in M$, if $l_S(N) \subseteq l_S(m)$, then $m \in r_M l_S(m) \subseteq r_M l_S(N)$. From (1) we have $N = r_M l_S(N)$, hence $m \in N$.

(2) ⇒ (1). Let N be an $\mathscr{L}$-submodule of M. For each $m \in r_M l_S(N)$, we have

$$l_S(N) = l_S r_M l_S(N) \subseteq l_S(m).$$

From (2) $m \in N$, so that $r_M l_S(N) \subseteq N$. Clearly, $N \subseteq r_M l_S(N)$. So $N = r_M l_S(N)$, that is, M is an $\mathscr{L}$-dual module.

(1) ⇔ (3). By [1, Lemma 24.4 and P109] for each submodule N of M we have

$$r_M l_S(N)/N = Rej_{M/N}(M) = \cap\{\ker h \mid h \in \mathrm{Hom}(M/N, M)\ \}.$$

Thus M is an $\mathscr{L}$-dual module if and only if $\mathrm{Rej}_{M/N}(M) = 0$ for each $\mathscr{L}$-submodule N of M, if and only if M/N is cogenerated by M for each $\mathscr{L}$-submodule N of M, that is, every $\mathscr{L}$-dense factor module of M is cogenerated by M. □

A class $\mathscr{L}$ of right R-modules is said to be *closed under endomorphisms* of M, if for each $f \in \mathrm{End}(M_R)$ and $\mathscr{L}$-submodule N of M we have $f(N) \in \mathscr{L}$.

Proposition 2.5. *Let $\mathscr{L}$ be closed under endomorphisms of M_R. Then M is an $\mathscr{L}$-dual module if and only if $r_M(Sb \cap l_S(N)) = r_M(b) + N$ for each $\mathscr{L}$-submodule N of M and $b \in S$.*

Proof. The sufficiency is clear. Conversely, it is obvious that $r_M(b)+N \subseteq r_M(Sb \cap l_S(N))$ for each $\mathscr{L}$-submodule N of M. Suppose that $x \in r_M(Sb \cap l_S(N))$ and $y \in l_S(bN)$, then $ybN = 0$, hence $yb \in Sb \cap l_S(N)$, so $ybx = 0$, that is, $y \in l_S(bx)$. Therefore $l_S(bx) \supseteq l_S(bN)$. Since $\mathscr{L}$ is closed under endomorphisms of M, we have $bN \in \mathscr{L}$, hence $bx \in r_M l_S(bx) \subseteq r_M l_S(bN) = bN$ for M is an $\mathscr{L}$-dual module. Thus there is an $n \in N$ such that $bx = bn$, i.e., $x - n \in r_M(b)$. So $x \in N + r_M(b)$, as required. □

We now provide some characterizations of non-singular $\mathscr{L}^c$-dual module, which will be necessary in the last theorem.

Lemma 2.6. *Let M be non-singular. Then $r_M l_S(N)$ is a closed submodule of M for each submodule N.*

Proof. Suppose that $r_M l_S(N) \subseteq^e B$ and $r_M l_S(N) \neq B$, then there is a $0 \neq b \in B \backslash r_M l_S(N)$ and $L \subseteq^e R_R$ such that $0 \neq b \cdot L \subseteq r_M l_S(N)$. Hence $l_S(N) b \cdot L = 0$. Since M_R is non-singular, we have $l_S(N) \cdot b = 0$, i.e., $b \in r_M l_S(N)$. This is a contradiction. So $r_M l_S(N)$ is a closed submodule of M. □

Theorem 2.7. *Suppose that M_R is non-singular. The following are equivalent:*

(1) *M_R is an $\mathscr{L}^c$-dual module;*
(2) *$N \subseteq^e r_M l_S(N)$ for each $\mathscr{L}$-submodule N of M;*
(3) *$l_S(N) \neq 0$ for each non-essential $\mathscr{L}$-submodule N of M;*
(4) *for each $\mathscr{L}$-submodule N of M, N is an essential submodule of M if and only if $l_S(N) = 0$.*

Proof. (1) $\Rightarrow$ (2). For each $\mathscr{L}$-submodule N of M, let N_0 be a closure of N in M, that is, $N \subseteq^e N_0$ and $N_0 \in \mathscr{L}^c$. From (1) we have $N \subseteq r_M l_S(N) \subseteq r_M l_S(N_0) = N_0$, hence $N \subseteq^e r_M l_S(N)$.

(2) $\Rightarrow$ (1). Let N be an $\mathscr{L}^c$-submodule of M, by Lemma 2.6 $r_M l_S(N)$ is a closed submodule. Thus $r_M l_S(N)$ is a closure of N in M, whence $N = r_M l_S(N)$, that is, M is an $\mathscr{L}^c$-dual module.

(1) $\Rightarrow$ (3). Suppose that N is a non-essential $\mathscr{L}$-submodule of M, there is an $N_0 \neq M$ such that N_0 is a closure of N. From (1) $N_0 = r_M l_S(N_0)$, hence $l_S(N_0) \neq 0$. Since $l_S(N_0) \subseteq l_S(N)$ we have $l_S(N) \neq 0$.

(3) $\Rightarrow$ (4). The sufficiency is clear. Conversely, if $N \in \mathscr{L}$ and $N \subseteq^e M$, then $N \subseteq^e r_M l_S(N) \subseteq^e M$. By Lemma 2.6 we have $r_M l_S(N) = M$, hence $l_S(N) = 0$.

(4) $\Rightarrow$ (3). Clearly.

(3) $\Rightarrow$ (1). Let N be an $\mathscr{L}$-submodule of M and N_0 a closure of N in M. It is to show that $N_0 = r_M l_S(N_0)$. Obviously $N_0 \subseteq r_M l_S(N_0)$. Suppose that $N_0 \neq r_M l_S(N_0)$, then N is non-essential in $r_M l_S(N_0)$, thus there is a $0 \neq A_R \subseteq r_M l_S(N_0)$ such that $N_0 \cap A = 0$. By [7, 1.10] there exists B such that $N_0 \subseteq B$ and $A \oplus B \subseteq^e M_R$. Clearly, B is non-essential in M_R. From (3) we have $l_S(B) \neq 0$. Let $0 \neq x \in l_S(B)$. Then $xB = 0$ implies $xN_0 = 0$. Hence $x \in l_S(N_0) = l_S r_M l_S(N_0)$, so $xA = 0$, that is, $x(A \oplus B) = 0$. On the other hand, since $A \oplus B \subseteq^e M_R$, for each $0 \neq m \in M_R$, there is an $L \subseteq^e R_R$ such that $mL \subseteq A \oplus B$, hence $xmL = 0$. But since M is non-singular, we have $xm = 0$. That is, $xM = 0$, also $x = 0$. A contradiction. Therefore $N_0 = r_M l_S(N_0)$. □

Lemma 2.8. *Let M_R be a non-singular module. Then M is a weak type 2 $\mathscr{L}$-extending module if and only if each $\mathscr{L}^c$-submodule N_0 of M is a direct summand of M.*

Proof. The sufficiency is obvious. Conversely, let N_0 be an $\mathscr{L}^c$-submodule of M, then there is an $N \in \mathscr{L}$ such that N_0 is a closure of N. Since M is weak type 2 $\mathscr{L}$-extending, there exists a direct summand M_0 such that $N \subseteq^e M_0$. For M is non-singular, we have $N_0 = M_0$. □

Now we characterize the relative extending modules by the annihilator conditions as follows.

Theorem 2.9. *Suppose that M is a non-singular right R-module and $S = \mathrm{End} M_R$ is the endomorphism ring of M. Then M is a weak type 2 $\mathscr{L}$-extending module if and only if M is an $\mathscr{L}^c$-dual module and $l_S(N)$ is a direct summand of ${}_SS$ for each $\mathscr{L}$-submodule N of M.*

Proof. ($\Rightarrow$). Let N_0 be an $\mathscr{L}^c$-submodule of M, by Lemma 2.8, there is an idempotent $e \in S$ such that $N_0 = eM$. Thus $l_S(N_0) = S(1-e)$, so $r_M l_S(N_0) = r_M(S(1-e)) = eM = N_0$, that is, M is an $\mathscr{L}^c$-dual module. Moreover, since M is non-singular, for each $\mathscr{L}$-submodule N if N_0 is a closure of N in M, then $l_S(N) = l_S(N_0)$. Thus $l_S(N_0) \subseteq^{\oplus} {}_SS$ by Lemma 2.8, so $l_S(N)$ is a direct summand of ${}_SS$.

($\Leftarrow$). Let N be an $\mathscr{L}$-submodule of M, by Theorem 2.7 $N \subseteq^e r_M l_S(N)$. Since $l_S(N) \subseteq^{\oplus} {}_SS$, we have $r_M l_S(N) \subseteq^{\oplus} M_R$, that is, M is a weak type 2 $\mathscr{L}$-extending module. □

3 Small-dual rings

Let I be a right ideal of R, if for each right ideal K such that $K + I = R_R$ we have $K = R_R$, then I is called a small (or superfluous) right ideal of R. As we know, the small one-side ideals are very important in the study of rings, especially the largest small ideal, i.e., the Jacobson radical $J = J(R)$. In this section, we mainly investigate the right $\mathscr{L}$-dual ring for the class $\mathscr{L}$ of all small right ideals of R.

Definition 3.1. *A ring R is called a right small-dual ring, if $r_R l_R(I) = I$ for each small right ideal I.*

Obviously, every semiprimitive ring (i.e., $J(R) = 0$) is a small-dual ring (e.g., $\mathbb{Z}$).

Proposition 3.2. *Let R be a right small-dual ring.*

(1) *If I is a small right ideal of R, then $l_R(I) \subseteq^e {}_RR$;*
(2) $\mathrm{Soc}({}_RR) \subseteq l_R(J(R)) \subseteq^e {}_RR$;
(3) $J(R) \subseteq Z({}_RR)$.

Proof. (1) For $b \in R$, if $l_R(I) \cap Rb = 0$, then since bI is a small right ideal, by Proposition 2.5, we have $I + r_R(b) = R$. Thus $r_R(b) = R$, so $b = 0$, that is, $l_R(I) \subseteq^e {}_RR$.

(2),(3) Note that $J(R)$ is the largest small ideal of R. □

Corollary 3.3. *Suppose that R is a right small-dual ring, and satisfies ACC on left annihilators, then $J(R) = Z({}_RR)$ is nilpotent.*

Proof. By the well-known result, if R satisfies ACC on left annihilators, then $Z({}_RR)$ is nilpotent. Thus $Z({}_RR) \subseteq J(R)$, so $J(R) = Z({}_RR)$ is nilpotent by Proposition 3.2(3). □

As we note that, the Jacobson radical $J(R)$ need not be nilpotent in a small-dual ring R. In fact, a self-injevtive dual ring R is given in [8, 6.2Example] such that $0 \neq J(R) = J(R)^2$.

Theorem 3.4. *If R is a right small-dual ring, and satisfies ACC on essential left ideals, then $J(R)$ is nilpotent.*

Proof. If R satisfies ACC on essential left ideals, then R/Soc_RR is left noetherian by [3, Proposition 4]. Let $J = J(R)$ and consider the descending chain of ideals

$$J \supseteq J^2 \supseteq J^3 \supseteq \cdots ,$$

then from Proposition 3.2 we have an ascending chain of essential left ideals

$$\mathrm{Soc}_RR \subseteq l_R(J) \subseteq l_R(J^2) \subseteq \cdots .$$

Thus there is an $m \in \mathbb{N}$ such that $l_R(J^m) = l_R(J^{m+1})$. Since R is right small-dual, we have $J^m = J^{m+1}$. Also since R/Soc_RR is a Noetherian left R-module, then $(J^m + \mathrm{Soc}_RR)/\mathrm{Soc}_RR$ is finitely generated, and note that

$$(J^m + \mathrm{Soc}_RR)/\mathrm{Soc}_RR = J \cdot ((J^m + \mathrm{Soc}_RR)/\mathrm{Soc}_RR).$$

Thus by Nakayama Lemma we have $(J^m + \mathrm{Soc}_RR)/\mathrm{Soc}_RR = 0$, that is, $J^m \subseteq \mathrm{Soc}_RR$. So $J^{m+1} \subseteq J\mathrm{Soc}_RR = 0$. □

Corollary 3.5. *If R is a right small-dual semilocal left noetherian ring, then R is left aritian.*

Proof. Since R is right small-dual left noetherian ring, by Theorem 3.4 $J(R)$ is nilpotent. So R is semiprimary, thus R is left aritian by Hopkins' Theorem. □

According to [12], a ring R is called *a left C_2-ring*, if every left ideal, which is isomorphic to a direct summand of R, is a direct summand of R. If R is a left extending and left C_2-ring, then R is called a left continuous ring. A semiperfect left continuous ring R satisfying $\mathrm{Soc}_RR = \mathrm{Soc}R_R \subseteq^e_R R$ is studied in detail in [12], and it was proved that a right Kasch ring R is a left C_2 ring. A ring R is said to be *semiregular*, if $R/J(R)$ is regular and idempotents modulo $J(R)$ can be lifted. Obviously every semiperfect ring is semiregular.

Proposition 3.6. *Suppose that R is a semiregular right small-dual ring. Then*

(1) *R is a left C_2-ring;*

(2) $J(R) = Z({}_RR)$.

Proof. (1) Let I be a left ideal of R, and $I \cong Re$, where e is an idempotent of R. Since R is semiregular, there is a direct sum decomposition $R_R = C \oplus D$ such that C$\subseteq I$ and $I \cap D$ is a small submodule of ${}_RR$. Thus $I = C \oplus (I \cap D)$ and $I \cap D \subseteq J(R)$. Also since R is right small-dual, we have $I \cap D \subseteq Z({}_RR)$ by Proposition 3.2, that is, $I \cap D$ is a finitely generated projective singular module, this is impossible. So $I \cap D = 0$. Therefore $I =$ C is a direct summand of ${}_RR$, that is, R is a left C$_2$-ring.

(2) Since R is a right small-dual ring, $J(R) \subseteq Z({}_RR)$ by Proposition 3.2. For each $a \in Z({}_RR)$ since $l_R(a) \cap l_R(1-a) = 0$, we have $l_R(1-a) = 0$. Thus $R(1-a) \cong R$, whence $R(1-a)$ is a direct summand of R by (1), which implies $(1-a)R$ is also a direct summand of R. Let $(1-a)R = eR, e = e^2$, then $1 - e \in l_R(1-a) = 0$, hence $(1-a)R = R$. So $Z({}_RR) \subseteq J(R)$. □

Theorem 3.7. *Suppose that R is a semilocal right small-dual ring. Then* $J(R) = r_R(\mathrm{Soc}(R_R))$ *and* $\mathrm{Soc}({}_RR) \subseteq \mathrm{Soc}(R_R) = l_R r_R(\mathrm{Soc}(R_R)) \subseteq^e {}_RR$.

Proof. Since R is semilocal, we have that $\mathrm{Soc}(R_R) = l_R(J(R))$, thus

$$J(R) = r_R l_R(J(R)) = r_R(\mathrm{Soc}(R_R))$$

and $l_R r_R(\mathrm{Soc}(R_R)) = l_R(J(R)) = \mathrm{Soc}(R_R) \subseteq^e {}_RR$. □

Corollary 3.8. *Suppose that R is a semilocal small-dual ring. Then*

$$\mathrm{Soc}({}_RR) = r_R(J(R)) = l_R(J(R)) = \mathrm{Soc}(R_R)$$

is an essential ideal of R, and

$$r_R(\mathrm{Soc}(R_R)) = J(R) = l_R(\mathrm{Soc}({}_RR)).$$

A ring R is called a left PP ring, if every cyclic left ideal is projective, equivalently, the left annihilator of each element of is a direct summand of ${}_RR$.

Proposition 3.9. *Suppose that R is a left PP ring. Then R is a right small-dual ring if and only if R is a semiprimitive ring.*

Proof. The sufficiency is clear. Conversely, for each $x \in J(R)$, xR is a small right ideal of R. Thus $l_R(x) \subseteq^e {}_RR$ by Proposition 3.2. Since R is left PP, we have $l_R(x) \subseteq^{\oplus} {}_RR$. Whence $l_R(x) = {}_RR$, so $xR = r_R l_R(x) = 0$, that is, $J(R) = 0$. □

It is showed in [14] that every right semihereditary dual ring is semisimple. We generalize it as follows.

Corollary 3.10. *Every left* (*or right*) *PP dual ring is semisimple.*

Proof. From [8, Theorem 3.9] every dual ring R is semiperfect. It follows from Proposition 3.9 that R is semisimple. □

It is proved in [1] that an artinian ring R is QF if and only if R is a dual ring, and in [8] that every left perfect dual ring is QF. Note that every dual ring is small-dual ring, we have

Theorem 3.11. *If R is left (or right) perfect, then R is small-dual if and only if R is dual, hence every left (or right) perfect small-dual ring is QF.*

Proof. Suppose that R is small-dual ring. Let I be a right ideal of R. Since R is right perfect, hence semiperfect, there is a projective cover P of R/I. By [1, 17.17] there is a direct sum decomposition $R_R = P_1 \oplus P_2$ such that $P_1 \cong P$. Thus R is a projective cover of $R/I \oplus P_2$. By Proposition 2.4, $R/I \oplus P_2$ is cogenerated by R_R, hence R/I is cogenerated by R_R. So by Proposition 2.4, R is a right dual ring. Similarly, R is a left dual ring, so R is a dual ring.

By [8, Theorem 5.3] every cyclic right R-module is finite Goldie dimensional. Since R is right perfect, every cyclic right R-module has an essential socle. Thus every cyclic right R-module has a finitely generated essential socle, hence R is right artinian. So by [1, Ex24.11,13], R is QF. □

Hopkins' theorem asserts that R is a right artinian ring if and only if R is a right noetherian semilocal ring and $J(R)$ is nilpotent. In presence of a small-dual ring we have

Corollary 3.12. *If R is a semilocal small-dual ring satisfying ACC on essential left (or right) ideals, then R is QF.*

Proof. By Theorem 3.4 $J(R)$ is nilpotent, so that R is right (or left) perfect. It follows from Theorem 3.11 that R is QF. □

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