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The Least Group Congruence on E-inversive Semigroups and E-inversive E-semigroups

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Abstract : In this paper, we investigated a group congruence on an E-inversive semigroup by using weakly self-conjugate subsemigroups and the least group congruence on E-inversive E-semigroups with commuting idempotents.

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1 Introduction

Let S be a semigroup and $E(S)$ denote the set of all idempotents of S. For every $a \in S$, $V(a){:=}\ \{\ x \in S \mid a = axa, x = xax\}$ is the set of all *inverses* of element a, and $W(a){:=}\ \{\ x \in S \mid x = xax\}$ is the set of all *weak inverses* of element a. An element a in a semigroup S is called *E-inversive* [7] if there exists $x \in S$ such that ax is an idempotent of S. A semigroup S is called *E-inversive* if for all $a \in S, a$ is E-inversive. A semigroup S is called an *E-semigroup* if $E(S)$ forms a subsemigroup of S. A congruence ρ on a semigroup S is called a *group congruence* if S/ρ is a group.

Basic properties and results of E-inversive E-semigroups were given by Mitsch [5], Weipoltshammer [7]. Gomes [2] described the least group congruence on a dense and unitary E-semigroup. Zheng [8] gave the least group congruence on an E-inversive semigroup which used self-conjugate subsets of a semigroup. Weipoltshammer [7] considered the least group congruence on an E-inversive E-semigroup.

In this paper, we investigated characterizations of a group congruence on an E-inversive semigroup and the least group congruence on an E-inversive semigroup which we used weakly self-conjugate subsemigroups of a semigroup and the least group congruence on an E-inversive E-semigroup with commuting idempotents.

A subset H of a semigroup S is *full* if $E(S) \subseteq H$. A subsemigroup H of a semigroup S is called *weakly self-conjugate* if for all $a \in S, x \in H, a' \in W(a), axa', a'xa \in H$. For any subset H of a semigroup S, let $H_\omega{:=}\{a \in S \mid ha \in H$ for some $h \in H\}$ which is called the *closure* of H. If H is a subsemigroup of S, then $H \subseteq H_\omega$. A subsemigroup H of a semigroup S is *closed* if $H = H_\omega$.

For a semigroup S, the following notations will be used; $\mathcal{C}$ is the class of all full and weakly self-conjugate subsemigroups of S, $\overline{\mathcal{C}}$ is the class of all closed subsemigroups of S in $\mathcal{C}$. Let $U := \bigcap_{H\in\mathcal{C}} H$. Then clearly, U is full and weakly self-conjugate. Note that U is the smallest element in $\mathcal{C}$.

For any subset H of a semigroup S, we define a *binary relation* β_H on S as follows :

$$\beta_H := \Big\{(a,b) \in S\times S \mid xa = by \text{ for some } x, y \in H\Big\}.$$

For any congruence ρ on a semigroup S, the *kernel* of ρ is the set

$$Ker\rho := \Big\{a \in S \mid a\rho \in E(S/\rho) = \{a \in S \mid (a, a^2) \in \rho\Big\}.$$

If ρ is a group congruence on a semigroup S, then $a \in Ker\rho$ if and only if $(a, e) \in \rho$ for some(all) $e \in E(S)$. For basic concepts in semigroup theory, see [1], [3] and [6].

The following results are used in this research.

Lemma 1.1 ([7]) *A semigroup S is E-inversive if and only if $W(a) \neq \emptyset$ for all $a \in S$.*

Proposition 1.2 ([7]) *For any semigroup S, the following statements are equivalent:*

(i) *S is an E-semigroup.*

(ii) *$W(ab) = W(b)W(a)$ for all $a, b \in S$.*

Proposition 1.3 [7] *Let S be an E-semigroup. Then*

(i) *for all $a \in S, a' \in W(a), e, f \in E(S), ea', a'f, fa'e \in W(a)$,*

(ii) *for all $a \in S, a' \in W(a), e \in E(S), a'ea, aea' \in E(S)$,*

(iii) *for all $e \in E(S), W(e) \subseteq E(S)$,*

(iv) *for all $e, f \in E(S), W(ef) = W(fe)$.*

2 The least group Congruence on E-inversive semigroups.

The next result, we show that β_H is a group congruence on an E-inversive semigroup.

Theorem 2.1 *If S is an E-inversive semigroup and $H \in \mathcal{C}$, then*

$$\beta_H := \Big\{(a,b) \in S\times S \mid xa = by \text{ for some } x, y \in H\Big\}$$

is a group congruence on S.

Proof. To show that β_H is a congruence on S, let $a, b, c \in S$. Let $a' \in W(a)$. Since H is full, $aa', a'a \in E(S) \subseteq H$. Note that $(aa')a = a(a'a)$, we have $a\beta_H a$.

Suppose that $a\beta_H b$. Then $xa = by$ for some $x, y \in H$. Let $b' \in W(b)$. Then $(aa'(byb'))b = a((a'xa)b'b)$. Since $aa'byb', a'xab'b \in H$, $b\beta_H a$.

Suppose that $a\beta_H b$ and $b\beta_H c$. Then $xa = by$ and $zb = cw$ for some $x, y, z, w \in H$. Thus $(zx)a = zby = c(wy)$ and $zx, wy \in H$, it follows that $a\beta_H c$.

Suppose that $a\beta_H b$ and $c \in S$. Then $xa = by$ for some $x, y \in H$. Let $b' \in W(b)$ and $c' \in W(c)$. Then $(bcc'b'x)ac = bcc'b'(xa)c = bc(c'b'byc)$. Since H is weakly self-conjugate, $bcc'b'x, c'b'byc \in H$. Hence β_H is a right compatible. Similarly, we can show that β_H is a left compatible. Hence β_H is a congruence on S.

Finally, we shall show that S/β_H is a group. Fix $x \in H$. Claim that $x\beta_H$ is the identity of S/β_H. Let $a \in S$ and $a' \in W(a)$. Then $axa', a'a \in H$ and $(axa')a = (ax)(a'a)$, so $(a, ax) \in \beta_H$. Note that $xaa', a'a \in H$ and $(xaa')a = (xa)a'a$, so $(a, xa) \in \beta_H$. Hence $x\beta_H$ is the identity of S/β_H.

Clearly, $x\beta_H = y\beta_H = e\beta_H$ for all $x, y \in H, e \in E(S)$. Then $a\beta_H a'\beta_H = (aa')\beta_H = x\beta_H = (a'a)\beta_H = a'\beta_H a\beta_H$. Therefore $a'\beta_H$ is an inverse of $a\beta_H$. Hence S/β_H is a group. □

Remark. From Theorem 2.1, we see that $H \subseteq Ker\beta_H$ for every $H \in \mathcal{C}$.

Lemma 2.2 *Let S be an E-inversive semigroup.*

(i) *If $H \in \mathcal{C}$, then $Ker\beta_H = H_\omega$.*

(ii) *If $H \in \overline{\mathcal{C}}$, then $Ker\beta_H = H = H_\omega$.*

(iii) *If ρ is a group congruence on S, then $Ker\rho \in \overline{\mathcal{C}} \subseteq \mathcal{C}$ and $\rho = \beta_{Ker\rho}$.*

Proof. (i) Suppose that $H \in \mathcal{C}$. By Theorem 2.1, β_H is a group congruence on S. Let $a \in Ker\beta_H$. Then $(a, e) \in \beta_H$ for all $e \in E(S)$. Let $e \in E(S)$. Then $xa = ey$ for some $x, y \in H$. Since $ey \in H$, we get $xa \in H$. Thus $a \in H_\omega$.

Conversely, let $a \in H_\omega$. Then there exists $h \in H$ such that $ha \in H$. For any $a' \in W(a)$, $(a(ha)a')a'a = a(haa'a'a)$ where $a(ha)a', haa'a'a \in H$, so $(a'a, a) \in \beta_H$ and $a \in Ker\beta_H$.

(ii) Clearly, by (i), $H = H_\omega = Ker\beta_H$.

(iii) Let $e \in E(S)$. Then $(e, e) \in \rho$, so $e \in Ker\rho$. Thus $Ker\rho$ is full. Let $x \in Ker\rho$. Then $(x, e) \in \rho$ for all $e \in E(S)$. Let $a \in S, a' \in W(a)$. Then $(axa', aea') \in \rho$, and $(axa')\rho = (aea')\rho = a\rho e\rho a'\rho = a\rho a'\rho = (aa')\rho$ where $e\rho$ is the identity element in S/ρ. Then $(axa', aa') \in \rho$, so $axa' \in Ker\rho$. Similarly, we can show that $a'xa \in Ker\rho$ for all $a \in S, a' \in W(a),\ x \in Ker\rho$.

Now we shall show that $Ker\rho$ is a subsemigroup of S. Let $a, b \in Ker\rho$. Then $a\rho = e\rho, b\rho = e\rho$ for all $e \in E(S)$. Thus $(ab)\rho = a\rho b\rho = e\rho e\rho = e\rho$ for all $e \in E(S)$. Hence $ab \in Ker\rho$. That is, $Ker\rho \in \mathcal{C}$. Next, we shall show that $(Ker\rho)_\omega = Ker\rho$. Note that $Ker\rho \subseteq (Ker\rho)_\omega$.

Let $x \in (Ker\rho)_\omega$. Then there exists $y \in Ker\rho$ such that $yx \in Ker\rho$. Thus $yx\rho = e\rho$ for all $e \in E(S)$ and $y\rho x\rho = (yx)\rho = e\rho$. Since $y \in Ker\rho$, $y\rho = e\rho$. Hence $x\rho = e\rho$, so $x \in Ker\rho$. Therefore $(Ker\rho)_\omega = Ker\rho$, so $Ker\rho \in \overline{\mathcal{C}}$.

Finally, we shall show that $\rho = \beta_{Ker\rho}$. Let $(a, b) \in \rho$ and $a' \in W(a)$. Then $(aa', ba') \in \rho$. We get that $ba' \in Ker\rho$, and $(ba')a = b(a'a)$. Then $(a, b) \in \beta_{Ker\rho}$ and so $\rho \subseteq \beta_{Ker\rho}$. Suppose that $(a, b) \in \beta_{Ker\rho}$. Then $xa = by$ for some $x, y \in Ker\rho$. Thus $x\rho = e\rho = y\rho$ for all $e \in E(S)$. Since ρ is a group congruence and $e\rho$ is the identity element in S/ρ, we have $a\rho = e\rho a\rho = x\rho a\rho = (xa)\rho = (by)\rho = b\rho y\rho = b\rho e\rho = b\rho$ and so $(a, b) \in \rho$. Hence $\rho = \beta_{Ker\rho}$. □

Corollary 2.3 *Let S be an E-inversive semigroup. Then ρ is a group congruence on S if and only if there exists $K \in \overline{\mathcal{C}}$ such that $\rho = \beta_K$ where $K = Ker\rho$.*

Proof. It follows from Theorem 2.1 and Lemma 2.2(iii). □

Lemma 2.4 *Let S be an E - inversive semigroup.*

(i) *If $H \subseteq K \subseteq S$, then $\beta_H \subseteq \beta_K$.*

(ii) *If $H, K \in \overline{\mathcal{C}}$ such that $\beta_H \subseteq \beta_K$ then $H \subseteq K$, (hence for $H, K \in \overline{\mathcal{C}}, H \subseteq K$ if and only if $\beta_H \subseteq \beta_K$).*

Proof. (i) Let $H \subseteq K$ and $(a, b) \in \beta_H$. Then there exist $x, y \in H \subseteq K$ such that $xa = by$. Hence $(a, b) \in \beta_K$.

(ii) Let $H, K \in \overline{\mathcal{C}}$ be such that $\beta_H \subseteq \beta_K$. By Lemma 2.2(ii), $Ker\beta_H = H$ and $Ker\beta_K = K$. Let $x \in H$. Then $x \in Ker\beta_H$. So $(x, e) \in \beta_H \subseteq \beta_K$ for all $e \in E(S)$. Therefore $x \in Ker\beta_K = K$. The proof is completed. □

By Lemma 2.2 and 2.4, we have the least group congruence on an E-inversive semigroup.

Theorem 2.5 *Let S be an E - inversive semigroup. If U is the smallest element in $\mathcal{C}$, then β_U is the least group congruence on S.*

Proof. Let ρ be an arbitrary group congruence on S. By Corollary 2.3, we obtain that $\rho = \beta_K$ where $K = Ker\rho \in \overline{\mathcal{C}}$. Since $U \subseteq K$ and by Lemma 2.4(i), $\beta_U \subseteq \beta_K = \rho$. Hence β_U is the least group congruence on S. □

We conclude this section by investigating alternative characterization of a group congruence on an E - inversive semigroup.

Proposition 2.6 *Let S be an E - inversive semigroup with $H \in \overline{\mathcal{C}}$. If $a, b \in S$, then the following statements are equivalent.*

(i) *For all $b' \in W(b)$, $ab' \in H$.*

(ii) *For all $a' \in W(a)$, $a'b \in H$.*

(iii) *For all $a' \in W(a)$, $ba' \in H$.*

(iv) *For all $b' \in W(b)$, $b'a \in H$.*

(v) *For all* $b' \in W(b)$*, there exists* $x \in H$ *such that* $axb' \in H$*.*

(vi) *For all* $a' \in W(a)$*, there exists* $x \in H$ *such that* $a'xb \in H$*.*

(vii) *For all* $a' \in W(a)$*, there exists* $x \in H$ *such that* $bxa' \in H$*.*

(viii) *For all* $b' \in W(b)$*, there exists* $x \in H$ *such that* $b'xa \in H$*.*

(ix) *There exist* $x, y \in H$ *such that* $ax = yb$*.*

(x) *There exist* $x, y \in H$ *such that* $xa = by$*.*

(xi) $HaH \cap HbH \neq \emptyset$*.*

Proof. (i)$\Rightarrow$(ii) Let $a' \in W(a)$ and $b' \in W(b)$. Then $ab' \in H$ and $a'ab'a \in H$. Now $(a'ab'a)a'b = (a'a)(b'aa'b) \in H$. Therefore $a'b \in H_\omega = H$.

(ii)$\Rightarrow$(i) It is similar to the proof of (i)$\Rightarrow$(ii).

The proof of (ii)$\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are similarly to the proof of (i) $\Leftrightarrow$ (ii).

(iv)$\Rightarrow$(v) Let $b' \in W(b)$ and $a' \in W(a)$. Then $b'a \in H$ and $b'aa'b \in H$. Now $a(b'aa'b)b' = (ab'aa')(bb') \in HE(S)E(S) \subseteq H$.

(v)$\Rightarrow$(iv) Let $a' \in W(a)$ and $b' \in W(b)$. Then there exists $x \in H$ such that $axb' \in H$. Thus $a'(xab')a \in H$ and $(a'ax)b'a = a'(xab')a \in H$. Therefore $b'a \in H_\omega = H$.

(v)$\Rightarrow$(vi) Let $a' \in W(a)$ and $b' \in W(b)$. Then there exists $x \in H$ such that $axb' \in H$. Thus $a'(axb')b = (a'a)x(b'b) \in H$. Therefore $a'yb \in H$ where $y = axb' \in H$.

(vi)$\Rightarrow$(v) It is similar to the proof of (v) $\Rightarrow$ (vi).

Again, we can show that (vi)$\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii).

(viii)$\Rightarrow$(ix) Let $a' \in W(a)$ and $b' \in W(b)$. Then there exists $x \in H$ such that $b'xa \in H$. Thus $(a'(bb'x)a)(b'b), (aa')(b(b'xa)b') \in H$ and $a(a'bb'xab'b) = (aa'bb'xab')b$. Hence $au = vb$ where $u = a'bb'xab'b,\ v = aa'bb'xab' \in H$.

(ix)$\Rightarrow$(x) Let $a' \in W(a)$ and $b' \in W(b)$. By assumption, there exist $x, y \in H$ such that $ax = yb$. Thus $(bb'axa')a = b(b'yba'a)$. Put $u = (b'b)(axa')$ and $v = (b'yb)(a'a)$. Thus $u, v \in H$.

(x)$\Rightarrow$(xi) Suppose that $xa = by$ for some $x, y \in H$. Then $x^2ay = xby^2$ which implies that $HaH \cap HbH \neq \emptyset$.

(xi)$\Rightarrow$(v) Suppose that $HaH \cap HbH \neq \emptyset$. Let $xay = x_1by_1$ for some $x, y, x_1, y_1 \in H$ and $a' \in W(a), b' \in W(b)$. Then $a'xa, by_1b' \in H$ and $(a'xa)y \in H$. Let $u = a'xay$. Then $u \in H$. Thus $a(a'xay)b' = (aa')(xay)b' = (aa')(x_1by_1)b' = (aa')x_1(by_1b') \in E(S)HH \subseteq H$. □

Remark. Let S be an E-inversive semigroup with $H \in \overline{\mathcal{C}}$ and $a, b \in S$. Then $a\beta_H b$ if and only if one of the equivalent conditions in Proposition 2.6 holds.

Theorem 2.7 *Let* S *be an* E *- inversive semigroup and* $H \in \overline{\mathcal{C}}$*. The mapping* $\phi : H \longmapsto \beta_H$ *and* $\varphi : \rho \longmapsto Ker\rho$ *are mutually - inverse inclusion - preserving mapping from* $\overline{\mathcal{C}}$ *onto the set of all group congruences on* S*.*

Proof. Let $\phi : H \longmapsto \beta_H$ and $\varphi : \rho \longmapsto Ker\rho$ be defined by $H\phi = \beta_H$ for all $H \in \overline{\mathcal{C}}$ and $\rho\varphi = Ker\rho$ for all group congruence ρ on S. We shall show that $\phi \circ \varphi = 1_{\overline{\mathcal{C}}}$ and $\varphi \circ \phi = 1_\Gamma$ where $1_{\overline{\mathcal{C}}}$ is the identity map of $\overline{\mathcal{C}}$ and 1_Γ is the identity map of Γ where Γ is the set of all group congruences on S. If $H \in \overline{\mathcal{C}}$, then $H\phi \circ \varphi = \beta_H\varphi = Ker\beta_H = H$ by Lemma 2.2(ii). Thus $\phi \circ \varphi = 1_{\overline{\mathcal{C}}}$. If $\rho \in \Gamma$, then $\rho\varphi \circ \phi = Ker\rho\phi = \beta_{Ker\rho} = \rho$ by Lemma 2.2(iii). Therefore $\varphi \circ \phi = 1_\Gamma$. If $H, K \in \overline{\mathcal{C}}$ implies $H \subseteq K$ if and only if $\beta_H \subseteq \beta_K$. It follows that ϕ and φ are inclusion - preserving. □

Finally, we have the least group congruence on an E-inversive E-semigroup with commuting idempotents.

Theorem 2.8 *If S is an E-inversive E-semigroup with commuting idempotents, then the relation*

$$\sigma^* := \Big\{(a,b) \in S \times S \mid ea = fb \ \ for\ some\ e, f \in E(S)\Big\}$$

is the least group congruence on S.

Proof. Clearly, σ^* are reflexive and symmetric. Let $a, b, c \in S$. Suppose that $a\sigma^* b$ and $b\sigma^* c$. Then $ea = fb$ and $gb = hc$ for some $e, f, g, h \in E(S)$. Thus $gea = gfb = f(gb) = fhc$. Note that $ge, fh \in E(S)$. Hence $a\sigma^* c$ and σ^* is an equivalence relation.

Suppose that $a\sigma^* b$. Then $ea = fb$ for some $e, f \in E(S)$. Thus $eac = fbc$, and so $ac\sigma^* bc$. Let $c' \in W(c)$. Since all idempotents of S commute, we have $(cec')(ca)^* = c(c'c)fb = cf(c'c)b = (cfc')(cb)$. By Proposition 1.3(ii), we have $cec', cfc' \in E(S)$. Hence $ca\sigma^* cb$. Therefore σ^* is a congruence on S. Claim that for any $e \in E(S)$, $e\sigma^*$ is the identity of S/σ^*. Let $e \in E(S), x \in S$ and $x' \in W(x)$. Since $E(S)$ is a subsemigroup of S, $(x'x)e \in E(S)$. Note that $(x'x)ex = (x'xe)x$, so $(ex, x) \in \sigma^*$. Consider $(xex')x = xe(x'x) = x(x'x)e = (xx')(xe)$. Thus $(x, xe) \in \sigma^*$. Therefore $e\sigma^*$ is the identity of S/σ^*. Clearly, $(x\sigma^*)(x'\sigma^*) = (xx')\sigma^* = e\sigma^* = (x'x)\sigma^* = x'\sigma^* x\sigma^*$. Hence σ^* is a group congruence on S. To show that σ^* is the least group congruence on S, let σ be an arbitrary group congruence on S. Let $(a, b) \in \sigma^*$. Then $ea = fb$ for some $e, f \in E(S)$. Thus $a\sigma = (e\sigma)(a\sigma) = (ea)\sigma = (fb)\sigma = (f\sigma)(b\sigma) = (b\sigma)$, so $(a, b) \in \sigma$. Hence σ^* is the least group congruence on S. □

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