Thai Journal of Mathematics
Volume 8 (2010) Number 2 : 405–417

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Existence and Iterative Approximation of a Unique Solution of a System of General Quasi-Variational Inequality Problems

K.R. Kazmi[1], F.A. Khan[2] and Mohd. Shahzad

Abstract : This paper deals with the existence and uniqueness of solution of a system of general quasi-variational inequality problems in a uniformly smooth Banach space. Further, a Mann-type partial implicit iterative algorithm is proposed for the system of general quasi-variational inequality problems and discussed its convergence analysis. The iterative algorithms and results presented here improve the iterative algorithms and results discussed in [3,7,8].

Keywords : System of general quasi-variational inequality problems, Mann-type partial implicit iterative algorithms, strongly accretive mapping, Lipschitz continuous mapping, relaxed cocoercive mapping.
2000 Mathematics Subject Classification : 49J40, 47J20, 65K10.

1 Introduction

In recent years, the theory of variational inequalities has become an effective and powerful tool for studying a wide range of problems arising in many diverse fields of pure and applied sciences. One of the most interesting and important problems in the theory of variational inequalities is the development of an efficient iterative algorithm to compute approximate solutions of variational inequality problems.

One of the efficient numerical techniques for solving variational inequality

[1]Corresponding author; E-mail: krkazmi@gmail.com; This work has been done under a Major Research Project No. F.36-7/2008 (SR) sanctioned by the University Grants Commission, Government of India, New Delhi

[2] Supported by NBHM, Department of Atomic Energy, Government of India under Grants-in-aid for Post-doctoral fellowship (Reference no. 40/2/2007-R&D II/3910)

problems in Hilbert spaces is the projection method and its variant forms. Since the standard projection method strictly defined on the inner product property of Hilbert spaces, it can no longer be applied for variational inequality problems in Banach spaces. This fact motivates us to develop alternative method to study iterative algorithms for approximating the solutions of variational inequality problems in Banach spaces.

In 2004, Verma [8] studied the convergence analysis of a iterative algorithm for approximating the solution of system of variational inequality problems involving relaxed cocoercive mappings in Hilbert spaces. Since then, many authors developed and studied different iterative methods for various classes of variational inequality problems and systems of variational inequality problems involving relaxed cocoercive mappings in Hilbert spaces, see for example [3-7].

We remark that the conditions used in the main results of [3-8] reduced the relaxed cocoercive mappings into strongly monotone mappings and thus the results are actually for variational inequality problems for strongly monotone mappings, see Lemma 2.5.

In this paper, we consider a system of general quasi-variational inequality problems (in short, SGQVIP) in a uniformly smooth Banach space. Further, using retraction method, we prove the existence of unique solution for SGQVIP. Furthermore, a Mann-type partial implicit iterative algorithm is given for SGQVIP and discussed its convergence analysis.

2 Preliminaries

We assume that E is a real Banach space equipped with norm $\|\cdot\|$; E^* is the topological dual space of E; 2^E is the power set of E; $\langle\cdot,\cdot\rangle$ is the dual pair between E and E^* and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) \ = \ \{f \in E^* : \ \langle x, f\rangle = \|x\|^2, \ \ \|x\| = \|f\|\}, \ \ x \in E.$$

First, we recall and define the following concepts and results which are needed in the sequel.

Definition 2.1[1]. A Banach space E is called *smooth* if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) \ = \ \sup\Big\{\frac{(\|x+y\| + \|x-y\|)}{2} - 1 : \ x, y \in E, \ \ \|x\| = 1, \ \ \|y\| = \tau\Big\}.$$

Definition 2.2[10]. The Banach space E is said to be *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Throughout the rest of the paper, we assume that E is a real uniformly smooth Banach space. We observe that if E is smooth then J is single-valued and if $E = H$, a Hilbert space then J is the identity map on H.

Lemma 2.1[2]. Let $J : E \to E^*$ be the normalized duality mapping. Then for all $x, y \in E$, we have

(a) $\|x+y\|^2 \;\leq\; \|x\|^2 + 2\langle y,\; J(x+y)\rangle$;

(b) $\langle x-y,\; J(x)-J(y)\rangle \;\leq\; 2d^2\rho_E\Big(\dfrac{4\|x-y\|}{d}\Big)$, where $d = \sqrt{(\|x\|^2+\|y\|^2)/2}$.

Definition 2.3[2]. Let K be a nonempty, closed and convex subset of E. A mapping $P_K : E \to K$ is said to be

(i) *retraction* if $P_K^2 = P_K$;

(ii) *nonexpansive retraction* if $\|P_K(x) - P_K(y)\| \;\leq\; \|x-y\|, \;\; \forall x, y \in E$;

(iii) *sunny retraction* if $P_K(P_K(x) - t(x - P_K(x)) \;=\; P_K(x), \;\; \forall x \in E,\, t \in \mathbb{R}_+$.

Lemma 2.2[2]. A retraction P_K is sunny and nonexpansive if and only if

$$\langle x - P_K(x),\; J(P_K(x) - y)\rangle \;\geq 0, \;\; \forall x, y \in E.$$

Definition 2.3. A mapping $T : E \to E$ is said to be:

(i) *β-Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \;\leq\; \beta\|x-y\|, \quad \forall x, y \in E;$$

(ii) *r-strongly accretive* if there exists a constant $r > 0$ such that

$$\langle Tx - Ty,\; J(x-y)\rangle \;\geq\; r\|x-y\|^2, \quad \forall x, y \in E;$$

(iii) *r-relaxed accretive* if there exists a constant $r > 0$ such that

$$\langle Tx - Ty,\; J(x-y)\rangle \;\geq\; -r\|x-y\|^2, \quad \forall x, y \in E;$$

(iv) *α-cocoercive* (or *α-inverse strongly accretive*) if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty,\; J(x-y)\rangle \;\geq\; \alpha\|Tx-Ty\|^2, \quad \forall x, y \in E;$$

(v) *relaxed (γ, r)-cocoercive* if there exist constants $\gamma, r > 0$ such that

$$\langle Tx - Ty,\; J(x-y)\rangle \;\geq\; -\gamma\|Tx-Ty\|^2 + r\|x-y\|^2, \quad \forall x, y \in E;$$

(vi) *λ-expansive* if there exists a constant $\lambda > 0$ such that

$$\|Tx - Ty\| \;\geq\; \lambda\|x-y\|, \quad \forall x, y \in E.$$

Remark 2.1. We observe that a r-strongly accretive mapping must be a relaxed (γ, r)-cocoercive mapping, or a γ-cocoercive mapping must be a relaxed (γ, r)-cocoercive mapping whenever $r = 0$, but the converse is not true, see [8].

Lemma 2.3[9]. Suppose $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality

$$\delta_{n+1} \ \leq \ (1 - a_n)\delta_n + \sigma_n, \qquad \forall\ n \geq 0,$$

with $a_n \in [0, 1]$, $\sum\limits_{n=0}^{\infty} a_n = \infty$, and $\sigma_n = o(a_n)$. Then $\lim\limits_{n\to\infty} \delta_n = 0$.

Next, we prove the following results.

Lemma 2.4. If the mapping $T : E \to E$ is relaxed (γ, r)-cocoercive then T is $\left(\dfrac{2r-1}{1+2\gamma}\right)^{1/2}$-expansive for $r > \dfrac{1}{2}$.

Proof. Since T is relaxed (γ, r)-cocoercive mapping then there exist constants $r, \gamma > 0$ such that

$$\begin{aligned} -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2 \leq&\ \langle Tx - Ty,\ J(x - y)\rangle \\ \leq&\ \|Tx - Ty\|\,\|x - y\| \\ \leq&\ \frac{1}{2}\Big(\|Tx - Ty\|^2 + \|x - y\|^2\Big). \end{aligned}$$

Hence, we have

$$(1 + 2\gamma)\|Tx - Ty\|^2 \ \geq \ (2r - 1)\|x - y\|^2,$$

or

$$\|Tx - Ty\| \ \geq \ \left(\frac{2r-1}{1+2\gamma}\right)^{1/2}\|x - y\|, \ \text{ for } \ r > \frac{1}{2}.$$

This completes the proof.

Lemma 2.5. Let the mapping $T : E \to E$ be relaxed (γ, r)-cocoercive and β-Lipschitz continuous.

(i) If $\gamma\beta^2 < r$ then T is $(r - \gamma\beta^2)$-strongly accretive.

(ii) If $r < \gamma\beta^2$ then T is $(r - \gamma\beta^2)$-relaxed accretive.

Proof. Since T is relaxed (γ, r)-cocoercive and β-Lipschitz continuous, we have

$$\begin{aligned} \langle Tx - Ty,\ J(x - y)\rangle \geq&\ -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2 \\ \geq&\ -\gamma\beta^2\|x - y\|^2 + r\|x - y\|^2 \\ =&\ (r - \gamma\beta^2)\|x - y\|^2. \end{aligned}$$

This completes the proof.

Remark 2.2.(i) In the spirit of Lemma 2.5(i), Theorem 3.1 [5], Theorem 3.1 [6,7], Theorem 2.1 [8], Theorem 2.1 [4] and Theorem 3.1 [3] are actually for variational inequality problems for strongly monotone mappings in a Hilbert space.
(ii) It is easily observed from Lemma 2.5 that $\left|\beta - \frac{1}{\gamma}\right| \geq \frac{\sqrt{4r\gamma - 1}}{2\gamma}$ and $\gamma r > \frac{1}{4}$.

3 System of general quasi-variational inequality problems

Let $K : E \times E \to 2^E$ be a set-valued mapping such that for each $(x, y) \in E \times E$, $K(x, y)$ be a nonempty, closed and convex set in E. Let $g : E \to E$ be a single-valued mapping and $F, G : E \times E \to E$ be nonlinear mappings. We consider the problem of finding $x^*, y^* \in E$ with $g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*) \neq \emptyset$ such that

$$\begin{cases} \langle \rho_1 F(y^*, x^*) + g(x^*) - g(y^*),\ J(z_1 - g(x^*)) \rangle \geq 0, & \forall\, z_1 \in K(y^*, x^*), \\ \langle \rho_2 G(x^*, y^*) + g(y^*) - g(x^*),\ J(z_2 - g(y^*)) \rangle \geq 0, & \forall\, z_2 \in K(x^*, y^*), \end{cases} \tag{3.1}$$

which we call a system of general quasi-variational inequality problems (SGQVIP).

Special cases:

(1) If $g \equiv I$, identity mapping and $E = H$, Hilbert space, then SGQVIP (3.1) reduces to a system of quasi-variational inequality problems of finding $x^*, y^* \in K(x^*, y^*) \cap K(y^*, x^*) (\neq \emptyset)$ such that

$$\begin{aligned} \langle \rho_1 F(y^*, x^*) + x^* - y^*,\ z_1 - x^* \rangle &\geq 0, \quad \forall z_1 \in K(y^*, x^*), \\ \langle \rho_2 G(x^*, y^*) + y^* - x^*,\ z_2 - y^* \rangle &\geq 0, \quad \forall z_2 \in K(x^*, y^*), \end{aligned}$$

which is the correct form of the system (1)-(2) studied by Noor and Huang [7]. We remark that to find the solution (x^*, y^*) of system (1)-(2) [7] is not equivalent to find (x^*, y^*) such that

$$\begin{aligned} x^* &= P_{K(y^*, x^*)}[y^* - \rho_1 F(y^*, x^*)] \\ y^* &= P_{K(x^*, y^*)}[x^* - \rho_2 G(x^*, y^*)], \end{aligned}$$

unless $x^*, y^* \in K(y^*, x^*) \cap K(x^*, y^*) (\neq \emptyset)$.

(2) If $F = G = T$, SGQVIP (3.1) is equivalent to finding $x^*, y^* \in E$ with $g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*) (\neq \emptyset)$ such that

$$\begin{aligned} \langle \rho_1 T(y^*, x^*) + g(x^*) - g(y^*),\ J(z_1 - g(x^*)) \rangle &\geq 0, \quad \forall z_1 \in K(y^*, x^*), \\ \langle \rho_1 T(x^*, y^*) + g(y^*) - g(x^*),\ J(z_2 - g(y^*)) \rangle &\geq 0, \quad \forall z_2 \in K(x^*, y^*), \end{aligned}$$

which appears to be new one.

(3) If $K(x^*, y^*) = K(y^*, x^*) = K$, a nonempty, closed and convex set in E, then SGQVIP (3.1) reduces to the following system of variational inequality problems of finding $x^*, y^* \in K$ such that

$$\begin{aligned}\langle \rho_1 F(y^*, x^*) + g(x^*) - g(y^*),\ J(z - g(x^*))\rangle &\geq 0, \quad \forall z \in K,\\ \langle \rho_2 G(x^*, y^*) + g(y^*) - g(x^*),\ J(z - g(y^*))\rangle &\geq 0, \quad \forall z \in K.\end{aligned}$$

For appropriate and suitable choice of operators F, G and the set-valued mapping K, one can obtain a number of new and previously known problems from the SGQVIP (3.1) as special cases.

Next, we have the following Lemma.

Lemma 3.1. $(x^*, y^*) \in E \times E$ with $g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*)$, is a solution of SGQVIP (3.1) if and only if (x^*, y^*) satisfies

$$\begin{aligned}g(x^*) &= P_{K(y^*,x^*)}[g(y^*) - \rho_1 F(y^*, x^*)],\\ g(y^*) &= P_{K(x^*,y^*)}[g(x^*) - \rho_2 G(x^*, y^*)],\end{aligned}$$

where $\rho_1, \rho_2 > 0$ are constants.

4 Existence and uniqueness of solution for SGQVIP (3.1)

First, we define the following concepts.

Definition 4.1. Let $g : E \to E$ be a nonlinear mapping. A mapping $F : E \times E \to E$ is said to be:

(i) *α-strongly accretive with respect to g in the first argument* if there exists a constant $\alpha > 0$ such that

$$\langle F(x_1, y_1) - F(x_2, y_2),\ J(g(x_1) - g(x_2))\rangle \geq \alpha \|x_1 - x_2\|^2, \quad \forall\, x_1, x_2, y_1, y_2 \in E.$$

(ii) *β-Lipschitz continuous in the first argument* if there exists a constant $\beta > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\| \leq \beta \|x_1 - x_2\|, \quad \forall x_1, x_2, y_1, y_2 \in E.$$

We remark that if g is μ-Lipschitz continuous, then $\alpha \leq \mu\beta$.

Now, we prove the existence and uniqueness of solution for SGQVIP(3.1).

Theorem 4.1. Let E be a uniformly smooth Banach space with $\rho_E(t) \leq ct^2$ for some $c > 0$; K be a nonempty, closed and convex set in E and let the mapping

$g : E \to E$ be σ-strongly accretive and μ-Lipschitz continuous. Let $F, G : K \times K \to E$ be two mappings such that F be α_1-strongly accretive with respect to g and β_1-Lipschitz continuous in the first argument and G be α_2-strongly accretive with respect to g and β_2-Lipschitz continuous in the first argument. Suppose that there is a constant $\lambda > 0$ such that

$$\|P_{K(x_1,y_1)}(x) - P_{K(x_2,y_2)}(x)\| \leq \lambda\|x_1 - x_2\|, \tag{4.1}$$

$\forall (x_1, y_1), (x_2, y_2) \in E \times E, x \in E$ and $\rho_1, \rho_2 > 0$ satisfy the following conditions:

$$\theta + \theta_2 < 1 \quad \text{and} \quad \theta + \sqrt{\theta_1} < 1, \tag{4.2}$$

where $\theta := \sqrt{(1 - 2\sigma + 64c\mu^2)}$; $\theta_1 := \sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)} + \lambda$;

$\theta_2 := \sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2)} + \lambda.$

Then SGQVIP (3.1) has a unique solution.

Proof. For $(x, y) \in E \times E$, define a mapping $Q : E \times E \to E \times E$ by

$$Q(x, y) = (T(x, y), S(x, y)), \quad \forall (x, y) \in E \times E, \tag{4.3}$$

where $T, S : E \times E \to E$ are defined by

$$T(x, y) = x - g(x) + P_{K(y,x)}[g(y) - \rho_1 F(y, x)], \tag{4.4}$$

and

$$S(x, y) = y - g(y) + P_{K(x,y)}[g(x) - \rho_2 G(x, y)], \tag{4.5}$$

where $\rho_1, \rho_2 > 0$ are some constants.

For any $(x_1, y_1), (x_2, y_2) \in E \times E$, it follows that from (4.1) and (4.4) that

$$\begin{aligned} & \|T(x_1, y_1) - T(x_2, y_2)\| \\ \leq\; & \|x_1 - x_2 - (g(x_1) - g(x_2))\| \\ & + \|P_{K(y_1,x_1)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2,x_2)}[g(y_2) - \rho_1 F(y_2, x_2)]\| \\ \leq\; & \|x_1 - x_2 - (g(x_1) - g(x_2))\| \\ & + \|P_{K(y_1,x_1)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2,x_2)}[g(y_1) - \rho_1 F(y_1, x_1)]\| \\ & + \|P_{K(y_2,x_2)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2,x_2)}[g(y_2) - \rho_1 F(y_2, x_2)]\| \\ \leq\; & \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \lambda\|y_1 - y_2\| \\ & + \|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\|. \end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned} & \|S(x_1, y_1) - S(x_2, y_2)\| \\ \leq\; & \|y_1 - y_2 - (g(y_1) - g(y_2))\| + \lambda\|x_1 - x_2\| \\ & + \|g(x_1) - g(x_2) - \rho_2(G(x_1, y_1) - G(x_2, y_2))\|. \end{aligned} \tag{4.7}$$

Since g is σ-strongly accretive and μ-Lipschitz continuous, by using Lemma 2.1, we have

$$\begin{aligned}
& \|x_1 - x_2 - (g(x_1) - g(x_2))\|^2 \\
\leq\ & \|x_1 - x_2\|^2 - 2\langle g(x_1) - g(x_2),\ J(x_1 - x_2 - g(x_1) - g(x_2)))\rangle \\
=\ & \|x_1 - x_2\|^2 - 2\langle g(x_1) - g(x_2),\ J(x_1 - x_2)\rangle \\
& -2\langle g(x_1) - g(x_2),\ J(x_1 - x_2 - (g(x_1) - g(x_2))) - J(x_1 - x_2)\rangle \\
\leq\ & \|x_1 - x_2\|^2 - 2\sigma\|x_1 - x_2\|^2 + 64c\mu^2\|x_1 - x_2\|^2 \\
=\ & (1 - 2\sigma + 64c\mu^2)\|x_1 - x_2\|^2. \qquad (4.8)
\end{aligned}$$

Similarly, we have

$$\|y_1 - y_2 - (g(y_1) - g(y_2))\|^2 \ \leq\ (1 - 2\sigma + 64c\mu^2)\|y_1 - y_2\|^2. \qquad (4.9)$$

Since F is α_1-strongly accretive with respect to g in the first argument and β_1-Lipschitz continuous in the first argument, and G is α_2-strongly accretive with respect to g in the first argument and β_2-Lipschitz continuous in the first argument, we have

$$\begin{aligned}
& \|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\|^2 \\
\leq\ & \|g(y_1) - g(y_2)\|^2 - 2\rho_1\langle F(y_1, x_1) - F(y_2, x_2),\ J(g(y_1) - g(y_2)\rangle \\
& -2\rho_1\langle F(y_1, x_1) - F(y_2, x_2),\ J(g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))) \\
& -J(g(y_1) - g(y_2))\rangle \\
\leq\ & \mu^2\|y_1 - y_2\|^2 - 2\rho_1\alpha_1\|y_1 - y_2\|^2 + 64c\rho_1^2\beta_1^2\|y_1 - y_2\|^2 \\
=\ & (\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)\|y_1 - y_2\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
& \|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\| \\
\leq\ & \sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)}\|y_1 - y_2\|. \qquad (4.10)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \|g(x_1) - g(x_2) - \rho_2(G(x_1, y_1) - G(x_2, y_2))\| \\
\leq\ & \sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2)}\|x_1 - x_2\|. \qquad (4.11)
\end{aligned}$$

From (4.6), (4.8) and (4.10), we have

$$\begin{aligned}
& \|T(x_1, y_1) - T(x_2, y_2)\| \\
\leq\ & \sqrt{(1 - 2\sigma + 64c\mu^2)}\|x_1 - x_2\| + \left(\sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)} + \lambda\right)\|y_1 - y_2\|. \qquad (4.12)
\end{aligned}$$

Also, from (4.7), (4.9) and (4.11), we have

$$\|S(x_1, y_1) - S(x_2, y_2)\|$$

$$\leq \sqrt{(1-2\sigma+64c\mu^2)}\|y_1-y_2\|+\Big(\sqrt{(\mu^2-2\rho_2\alpha_2+64c\rho_2^2\beta_2^2)}+\lambda\Big)\|x_1-x_2\|. \quad (4.13)$$

From (4.12) and (4.13), we have

$$\begin{aligned}\|T(x_1,y_1)-T(x_2,y_2)\|+\|S(x_1,y_1)-S(x_2,y_2)\| & \\ \leq & \ k_1\|x_1-x_2\|+k_2\|y_1-y_2\| \\ \leq & \ \max\{k_1,k_2\}\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big), \quad (4.14)\end{aligned}$$

where

$$k_1 := \theta+\theta_2 \ \text{ and } \ k_2 := \theta+\theta_1. \quad (4.15)$$

Now, define the norm $\|\cdot\|_*$ on $E\times E$ by

$$\|(x,y)\|_* = \|x\|+\|y\|, \ \forall (x,y)\in E\times E. \quad (4.16)$$

We can easily observe that $(E\times E, \|\cdot\|_*)$ is a Banach space. Hence, it follows from (4.3), (4.14) and (4.16) that

$$\|Q(x_1,y_1)-Q(x_2,y_2)\|_* \leq \max\{k_1,k_2\}\|(x_1,y_1)-(x_2,y_2)\|_*. \quad (4.17)$$

Since $k_1<1$, $k_2<1$ by condition (4.2), it follows from (4.17) that Q is a contraction mapping. Hence, by the Banach contraction principle, there exists a unique $(x,y)\in E\times E$ such that $Q(x,y)=(x,y)$, which implies that

$$\begin{aligned} g(x) &= P_{K(y,x)}[g(y)-\rho_1 F(y,x)], \\ g(y) &= P_{K(x,y)}[g(x)-\rho_2 G(x,y)]. \end{aligned}$$

It follows from Lemma 3.1, that (x,y) is the unique solution of SGQVIP (3.1). This completes the proof.

5 Iterative algorithms and convergence analysis

In this section, we suggest that the fixed-point formulation for SGQVIP(3.1), see Lemma 3.1, and Theorem 4.1 are very important from the numerical approximation point of view and help to suggest the following iterative algorithm for SGQVIP(3.1).

Mann-type partially implicit iterative algorithm (in short, MTPIIA) 5.1. For a given point $(x_0,y_0)\in E\times E$, compute an approximate solution (x_n,y_n) given by iterative schemes:

$$x_{n+1} = (1-a_n)x_n+a_n[x_n-g(x_n)+P_{K(y_n,x_n)}(g(y_n)-\rho_1 F(y_n,x_n))], \quad (5.1)$$

$$y_{n+1} = (1-b_n)x_{n+1}+b_n[y_{n+1}-g(y_{n+1})+P_{K(x_{n+1},y_n)}(g(x_{n+1})-\rho_2 G(x_{n+1},y_n))], \quad (5.2)$$

where $\rho_1,\rho_2>0$ are constants and a_n,$b_n\in(0,1]$, $n\geq 0$ with $\sum_{n=0}^{\infty} a_n=\infty$ and $\lim_{n\to\infty} b_n=1$.

Special cases:

(1) If $E \equiv H$ and $g \equiv I$, MTPIIA 5.1 reduces to the following iterative algorithm. For a given point $(x_0, y_0) \in H \times H$ compute an approximate solution (x_n, y_n) given by

$$x_{n+1} = (1 - a_n)x_n + a_n P_{K(y_n,x_n)}(y_n - \rho_1 F(y_n, x_n)),$$

$$y_{n+1} = (1 - b_n)x_{n+1} + b_n P_{K(x_{n+1},y_n)}(x_{n+1} - \rho_2 G(x_{n+1}, y_n)),$$

which is mainly due to Noor and Huang [7].

(2) If $K(x, y) \equiv K(y, x) \equiv K$, the nonempty, closed and convex set in E and $g \equiv I$, MTPIIA 5.1 reduces to the following iterative algorithm. For a given point $(x_0, y_0) \in E \times E$, compute an approximate solution (x_n, y_n) given by

$$x_{n+1} = (1 - a_n)x_n + a_n P_K(y_n - \rho_1 F(y_n, x_n)),$$

$$y_{n+1} = (1 - b_n)x_{n+1} + b_n P_K(x_{n+1} - \rho_2 G(x_{n+1}, y_n)),$$

where $a_n, b_n \in [0, 1]\ \ \forall n \geq 0$.

(3) If $a_n \equiv b_n \equiv 1$; $K(y, x) = K(x, y) \equiv K$ and $g \equiv I$, MTPIIA 5.1 reduces to the following iterative algorithm. For a given point $(x_0, y_0) \in E \times E$, compute an approximate solution (x_n, y_n) given by

$$x_{n+1} = P_K(y_n - \rho_1 F(y_n, x_n)),$$

$$y_{n+1} = P_K(x_{n+1} - \rho_2 G(x_{n+1}, y_n)).$$

Finally, we discuss the convergence criteria for MTPIIA 5.1.

Theorem 5.1. Let the mappings E, F, G, g be same as in Theorem 4.1 and let conditions (4.1) and (4.2) of Theorem 4.1 hold. Then approximate solution (x_n, y_n) generated by MTPIIA 5.1, converges strongly to the unique solution (x, y) of SGQVIP(3.1).

Proof. It follows from Theorem 4.1 that SGQVIP (3.1) has the unique solution $(x, y) \in E \times E$. Hence, by Lemma 3.1, we have

$$x = (1 - a_n)x + a_n[x - g(x) + P_{K(y,x)}(g(y) - \rho_1 F(y, x))], \tag{5.3}$$

$$y = (1 - b_n)y + b_n[y - g(y) + P_{K(x,y)}(g(x) - \rho_2 G(x, y))]. \tag{5.4}$$

From (4.1),(5.1) and (5.3), we have

$$\|x_{n+1} - x\| \leq (1 - a_n)\|x_n - x\| + a_n\|x_n - x - (g(x_n) - g(x))\|$$
$$+ a_n\|P_{K(y_n,x_n)}(g(y_n) - \rho_1 F(y_n, x_n)) - P_{K(y,x)}(g(y) - \rho_1 F(y, x))\|$$

$$\begin{aligned} &\leq\ (1-a_n)\|x_n-x\| + a_n\|x_n-x-(g(x_n)-g(x))\| \\ &\quad +a_n\|P_{K(y,x)}(g(y_n)-\rho_1 F(y_n,x_n)) - P_{K(y,x)}(g(y)-\rho_1 F(y,x))\| \\ &\quad +a_n\|P_{K(y_n,x_n)}(g(y_n)-\rho_1 F(y_n,x_n)) - P_{K(y,x)}(g(y_n)-\rho_1 F(y_n,x_n))\| \\ &\leq\ (1-a_n)\|x_n-x\| + a_n\|x_n-x-(g(x_n)-g(x))\| \\ &\quad +a_n\|g(y_n)-g(y)-\rho_1(F(y_n,x_n)-F(y,x))\| + a_n\lambda\|y_n-y\|. \end{aligned} \tag{5.5}$$

Since g is σ-strongly accretive and μ-Lipschitz continuous, using Lemma 2.1, we have

$$\|x_n-x-(g(x_n)-g(x))\|^2\ \leq\ (1-2\sigma+64c\mu^2)\|x_n-x\|^2. \tag{5.6}$$

Also, F is α_1-strongly accretive with respect to g in the first argument and β_1-Lipschitz continuous in the first argument, we have

$$\|g(y_n)-g(y)-\rho_1(F(y_n,x_n)-F(y,x))\|^2\ \leq\ (\mu^2-2\rho_1\alpha_1+64c\rho_1^2\beta_1^2)\|y_n-y\|^2. \tag{5.7}$$

From (5.5),(5.6) and (5.7), it follows that

$$\begin{aligned} &\|x_{n+1}-x\| \\ &\leq\ (1-a_n)\|x_n-x\|+a_n\sqrt{(1-2\sigma+64c\mu^2)}\|x_n-x\| \\ &+a_n\sqrt{(\mu^2-2\rho_1\alpha_1+64c_1\rho_1^2\beta_1^2)}\|y_n-y\|+a_n\lambda\|y_n-y\| \\ &= (1-a_n[1-\sqrt{(1-2\sigma+64c\mu^2)}])\|x_n-x\|+a_n\left(\sqrt{(\mu^2-2\rho_1\alpha_1+64c\rho_1^2\beta_1^2)}+\lambda\right)\|y_n-y\| \\ &=\ (1-a_n(1-\theta))\|x_n-x\|+a_n\theta_1\|y_n-y\|. \end{aligned} \tag{5.8}$$

Now, from (4.1), (5.2) and (5.4), we have

$$\begin{aligned} \|y_{n+1}-y\|\ \leq\ &(1-b_n)\|x_{n+1}-x\|+(1-b_n)\|y-x\|+b_n\|y_{n+1}-y-(g(y_{n+1})-g(y))\| \\ &+b_n\|g(x_{n+1})-g(x)-\rho_2(G(x_{n+1},y_n)-G(x,y))\|+b_n\lambda\|x_{n+1}-x\| \end{aligned} \tag{5.9}$$

Since g is σ-strongly accretive and μ-Lipschitz continuous, using Lemma 2.1, we have

$$\|y_n-y-(g(y_n)-g(y))\|^2\ \leq\ (1-2\sigma+64c\mu^2)\|y_n-y\|^2. \tag{5.10}$$

Since G is α_2-strongly accretive with respect to g in the first argument and β_2-Lipschitz continuous in the first argument, we have

$$\|g(x_{n+1})-g(x)-\rho_2(G(x_{n+1},y_n)-G(x,y))\|^2 \leq\ (\mu^2-2\rho_2\alpha_2+64c\rho_2^2\beta_2^2)\|x_{n+1}-x\|^2. \tag{5.11}$$

From (5.6), (5.9), (5.10) and (5.11), it follows that

$$\begin{aligned}
&\|y_{n+1}-y\|\\
&\leq (1-b_n)\|x_{n+1}-x\|+(1-b_n)\|y-x\|+b_n\sqrt{(1-2\sigma+64c\mu^2)}\|y_{n+1}-y\|\\
&+b_n\sqrt{(\mu^2-2\rho_2\alpha_2+64c\rho_2^2\beta_2^2)}\|x_{n+1}-x\|+b_n\lambda\|x_{n+1}-x\|\\
&= \Big(1-b_n\Big(1-(\sqrt{(\mu^2-2\rho_2\alpha_2+64c\rho_2^2\beta_2^2)}+\lambda)\Big)\Big)\|x_{n+1}-x\|\\
&+b_n\sqrt{(1-2\sigma+64c\mu^2)}\|y_{n+1}-y\|+(1-b_n)\|y-x\|\\
&= (1-b_n(1-\theta_2))\|x_{n+1}-x\|+b_n\theta\|y_{n+1}-y\|+(1-b_n)\|y-x\|\\
\leq\ & \|x_{n+1}-x\|+\theta\|y_{n+1}-y\|+(1-b_n)\|y-x\|, \qquad (5.12)
\end{aligned}$$

where $\theta_2 := \sqrt{(\mu^2-2\rho_2\alpha_2+64c\rho_2^2\beta_2^2)}+\lambda < 1$; $b_n \in (0,1]$.

From (5.12), we have

$$\|y_{n+1}-y\| \leq \frac{1}{(1-\theta)}\Big(\|x_{n+1}-x\|+(1-b_n)\|y-x\|\Big). \qquad (5.13)$$

Combining (5.8) and (5.13), we have

$$\begin{aligned}
\|x_{n+1}-x\| &\leq (1-a_n(1-\theta))\|x_n-x\|+a_n\theta_1\Big\{\frac{1}{(1-\theta)}\Big(\|x_n-x\|+(1-b_{n-1})\|y-x\|\Big)\Big\}\\
&= \Big[1-a_n\Big(1-\Big[\theta+\frac{\theta_1}{1-\theta}\Big]\Big)\Big]\|x_n-x\|+a_n\frac{\theta_1(1-b_{n-1})}{1-\theta}\|y-x\|. \qquad (5.14)
\end{aligned}$$

By condition (4.2) it follows that $\Big(1-\Big(\theta+\frac{\theta_1}{1-\theta}\Big)\Big) \in (0,1]$, $\sum_{n=0}^{\infty} a_n\Big(1-\Big(\theta+\frac{\theta_1}{1-\theta}\Big)\Big) = \infty$, and $\frac{a_n\theta_1(1-b_{n-1})}{(1-\theta)} = o(a_n)$, and then by Lemma 3.2, it follows that $\lim_{n\to\infty}\|x_n-x\| = 0$. Further, the result $\lim_{n\to\infty}\|y_n-y\| = 0$ follows from (5.13) and $\lim_{n\to\infty} b_n = 1$. This completes the proof.

Remark 5.1. The proof of theorems presented in this paper for SGQVIP(3.1) under the assumption of relaxed cocoercivity on mappings F and G need further research effort.

Acknowledgement : The authors express their sincere thanks to the referee for his/her valuable comments and suggestions for improving the first version of this paper.

References

[1] I. Cioreneseu, Geometry of Banach spaces, duality mappings and nonlinear problems, Kluwer Academic Publishers, Dordrecht 1990.

[2] K. Goebel and S. Reich, Uniform convexity, hyperbolic geometry and nonexpansive mappings, Dekker, New York, 1984.

[3] Z. Huang and M.A. Noor, An explicit projection method for a system of nonlinear variational inequalities with different (γ, r)-cocoercive mappings, Appl. Math. Comput. **190** (2007) 356–361.

[4] Z. Huang and M.A. Noor, Some new unified iteration schemes with errors for nonexpansive mappings and variational inequalities, Appl. Math. Comput. **194** (2007) 135–142.

[5] M.A. Noor, General variational inequalities and nonexpansive mappings, J. Math. Anal. Appl. **331**(2) (2007) 810–822.

[6] M.A. Noor and Z. Huang, Three step methods for nonexpansive mappings and variational inequalities, Appl. Math. Comput. **187** (2007) 680–685.

[7] M.A. Noor and Z. Huang, An iterative scheme for a system of quasi-variational inequalities, J. Math. Inequal. **1**(1) (2007) 31–38.

[8] R.U. Verma, Generalized system for relaxed cocoercive variational inequalities and projection methods, J. Optim. Theory Appl. **121**(1) (2004) 203–210.

[9] X.L. Weng, Fixed point iteration for local strictly pseudocontractive mappings, Proc. Amer. Math. Soc. **113**(3) (1991) 727–731.

[10] Z. Xu and G.F. Roach, Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. **157** (1991) 189–210.

(Received 18 August 2009)

K.R. Kazmi, F.A. Khan
Department of Mathematics,
Aligarh Muslim University,
Aligarh 202002 India.
e-mail : `krkazmi@gmail.com (K.R. Kazmi)`

Mohd. Shahzad
Department of Applied Mathematics,
Aligarh Muslim University,
Aligarh 202002 India.