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Intuitionistic Fuzzy n-Ary Subgroups

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Abstract : In this paper, we introduce a notion of an intuitionistic fuzzy n-ary subgroup in an n-ary group (G, f) and have studied their related properties.

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1 Introduction

The theory of fuzzy set was first developed by Zadeh [15] and has been applied to many branches in mathematics. Later fuzzification of the concept "group" into "fuzzy subgroup" was made by Rosenfeld [14]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. The study of n-ary systems was initiated by Kasner [11] in 1904, but the important study on n-ary groups was done by Dörnte [4]. The theory of n-ary systems have many applications. For example, in the theory of automata [9] n-ary semigroup and n-ary groups are used. The n-ary groupoids are applied in the theory of quantum groups [13]. Also the ternary structures in physis are described by Kerner in [10]. The first fuzzification of n-ary system was introduced by Dudek [5]. Moreover Davvz et. al [3] have studied fuzzy n-ary groups as a generalization of Rosenfeld's fuzzy groups and have investigated their related properties.
The notion of intuitionistic fuzzy sets introduced by Atanassov [1,2] , is a generalization of the notion of fuzzy set. Dudek [7] has introduced the Atanassov idea's in n-ary systems. In this paper, we introduce the notion of intuitionistic fuzzy n-ary subgroup in n-ary group (G, f) and have investigated their related properties.

2 Preliminaries

A non-empty set G together with one n-ary operation $f : G^n \to G$, where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (G, f). According to the general convention used in the theory of n-ary groupoids the sequence of elements $x_i, x_{i+1}, ..., x_j$ is denoted by x_i^j. In the case, if $j < i$, it denotes the empty symbol. If $x_{i+1} = x_{i+2} = ... = x_{i+t} = x$, then instead of x_{i+1}^{i+t}, we write $\overset{(t)}{x}$. In this convention

$$f(x_1, ..., x_n) = f(x_1^n)$$

and

$$f(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (G, f) is called an (i, j)-*associative* if

$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\right)$$

hold for all $x_1, ..., x_{2n-1} \in G$. If this identity holds for all $1 \leq i \leq j \leq n$,then we say that the operation f is associative and (G, f) is called an *n-ary semigroup*. It is clear that an n-ary groupoid is associative if and only if it is $(1, j)$-associative for all $j = 2, .., n$. In the binary case (i.e. n=2) it is a usual semigroup.If for all $x_0, x_1, ..., x_n \in G$ and fixed $i \in \{1, ..., n\}$ there exists an element $z \in G$ such that

$$f\left(x_1^{i-1}, z, x_{i+1}^n\right) = x_0 \tag{1}$$

then we say that this equation is *i-solvable* or *solvable at the place i*.If the solution is unique, then we say that (1) is *uniquely i-solvable*. An n-ary groupoid (G, f) uniquely solvable for all $i = 1, ..., n$ is called an *n-ary quasigroup*. An associative n-ary quasigroup is called an *n-ary group* .
Fixing an n-ary operation f, where $n \geq 3$, the elements a_2^{n-2} we obtain the new binary operation $x \diamond y = f(x, a_2^{n-2}, y)$. If (G, f) is an n-ary group then $(G, \diamond)$ is a group. Choosing different elements a_2^{n-2} we obtain different groups. All these groups are isomorphic[8]. So, we can consider only the groups of the form

$$ret_a(G, f) = (G, \circ), \text{ where } x \circ y = f(x, \overset{(n-2)}{a}, y).$$

In this group $e = \overline{a}, x^{-1} = f(\overline{a}, \overset{(n-3)}{a}, \overline{x}, \overline{a})$.

In the theory of n-ary groups, the following Theorem plays an important role.

Theorem 2.1.[8] *For any n-ary group (G, f) there exist a group $(G, \circ)$,its automorphism φ and an element $b \in G$ such that*

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \phi^2(x_3) \circ ... \circ \phi^{n-1}(x_n) \circ b \tag{2}$$

holds for all $x_1^n \in G$.

In what follows, G is a non-empty set and (G,f) is an n-ary group unless otherwise specified.

An intuitionistic fuzzy set (briefly,IFS) A in a non-empty set G is an object having the form [1]

$$A = \{(x, \mu_A(x), \nu_A(x)|x \in G)\}$$

where the functions $\mu_A : G \to [0,1]$ and $\nu_A : G \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively, and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall \in G.$$

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x))|x \in G\}$ in G can be identified to an ordered pair (μ_A, ν_A) in $I^G \times I^G$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IFS $A = \{(x, \mu_A(x), \nu_A(x)|x \in G)\}$.

Definition 2.2.[3] Let (G,f) be an n-ary group. A fuzzy subset of G is called a *fuzzy subgroup* of (G,f) if the following axioms holds:
(FnS1)$(\forall x_1^n \in G), (\mu(f(x_1^n) \geq min\{\mu(x_1), ..., \mu(x_n)\})$,
(FnS2)$(\forall x \in G), (\mu(\overline{x}) \geq \mu(x))$.

Note that for $n = 3$ the second condition (FnS2) of definition 2.2 can be replaced by the condition
(FnS3)$(\forall x \in G), (\mu(\overline{x}) = \mu(x))$.
because in this case $n = 3$, we have $\overline{\overline{x}} = x$. These two conditions are equivalent for all n-ary groups in which for every $x \in G$ there exists a natural number k such that $\overline{x}^{(k)} = x$, where $\overline{x}^{(k)}$ denotes the elements skew to $\overline{x}^{(k-1)}$ and $\overline{x}^{(0)} = x$.But, as it was observed in [6], there are fuzzy n-ary subgroups in which $\mu(\overline{x}) > \mu(x)$ for all $x \in G$.

3 Intuitionistic fuzzy n-ary subgroups

Definition 3.1. An $IFS\,A = (\mu_A, \nu_A)$ in G is called an *intuitionistic fuzzy n-ary subgroup* of (G,f) if the following axioms holds:
(IFnS1) $(\forall x_1^n \in G), (\mu(f(x_1^n) \geq min\{\mu(x_1), ..., \mu(x_n)\})$,
(IFnS2) $(\forall x_1^n \in G), (\nu(f(x_1^n) \leq max\{\nu(x_1), ..., \nu(x_n)\})$,
(IFnS3) $(\forall x \in G), (\mu(\overline{x}) \geq \mu(x))$,
(IFnS4) $(\forall x \in G), (\nu(\overline{x}) \leq \nu(x))$.

Example 3.2. Consider $(\mathbb{Z}_4, f)$,where $f : \mathbb{Z}_4^4 \to \mathbb{Z}_4$ is defined by
$f(x_1, x_2, x_3, x_4) = max(x_1, x_2, x_3, x_4)$. Clearly $(\mathbb{Z}_4, f)$ is a 4-ary subgroup derived from additive group $\mathbb{Z}_4$. Define IFS $A = (\mu_A, \nu_A)$ in $(\mathbb{Z}_4, f)$ as follows:

$$\mu_A(x) = \begin{cases} 0.7 & if\ x = 0, \\ 0.2 & if\ x = 1, 2, 3. \end{cases} \qquad \nu_A(x) = \begin{cases} 0.2 & if\ x = 0, \\ 0.9 & if\ \ x = 1, 2, 3. \end{cases}$$

Then it is easy to verify that $IFS\ \ A = (\mu_A, \nu_A)$ is an intuitionistic 4-ary fuzzy subgroup of $(\mathbb{Z}_4, f)$.

Theorem 3.3. *If $\{A_i | i \in \Lambda\}$ is an arbitrary family of an intuitionistic fuzzy n-ary subgroup of (G, f) then $\bigcap A_i$ is an intuitionistic fuzzy n-ary subgroup of (G, f), where $\bigcap A_i = \{(x, \wedge\mu_{A_i}(x), \vee\nu_{A_i}(x)) | x \in G)\}$.*

Proof. The proof is trivial. □

Theorem 3.4. *If an $IFS\, A = (\mu_A, \nu_A)$ in G is an intuitionistic fuzzy n-ary subgroup of (G, f) then so is $\Box A$, where $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) | x \in G\}$.*

Proof. It is sufficient to show that $\overline{\mu}_A$ satisfies condition (IFnS2) and (IFnS4). Let $x_1^n \in G$. Then

$$\begin{aligned} \overline{\mu}_A(f(x_1^n)) &= 1 - \mu_A(f(x_1^n)) \\ &\leq 1 - min\{\mu_A(x_1), ..., \mu_A(x_n)\} \\ &= max\{\mu_A(x_1), ..., \mu_A(x_n)\}. \end{aligned}$$

and

$$\overline{\mu}_A(\overline{x}) = 1 - \mu_A(\overline{x}) \leq 1 - \mu(\overline{x}) = \overline{\mu}(x).$$

Hence $\Box A$ is an intuitionistic fuzzy n-ary subgroup of (G, f). □

Definition 3.5.[7] Let $A = (\mu_A, \nu_A)$ be an IFS in G and let $t \in [0, 1]$. Then the set

$$U(\mu_A, t) := \{x \in G | \mu_A(x) \geq t\}(\text{resp. } L(\nu_A, t) := \{x \in G | \nu_A(x) \leq t\})$$

is called μ_A-*level* t-*cut*(resp. ν_A-*level* t-*cut*) of G.

The following Theorem is a consequence of the Transfer Principle described in [12].

Theorem 3.6. *An $IFS\ \ A = (\mu_A, \nu_A)$ in G with the images $Im(\mu_A) = \{t_i : i \in I\}$ and $Im(\nu_A) = \{t_j : j \in I\}$, is an intuitionistic fuzzy n-ary subgroup of (G, f) if and only if the μ_A-level t-cut and ν_A-level t-cut of G are n-ary subgroup of (G, f) for every $t \in [0, 1]$ such that $t \in Im(\mu_A) \cap Im(\nu_A)$, which are called μ_A-level n-ary subgroup and ν_A-level n-ary subgroups respectively.*

Proof. Let $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G, f). If $x_1^n \in G$and $t \in [0, 1]$, then $\mu(x_i) \geq t$ for all $i = 1, 2, ..., n$. Thus

$$\mu_A(f(x_1^n) \geq min\{\mu_A(x_1), ..., \mu_A(x_n)\} \geq t,$$

which implies $f(x_1^n) \in U(\mu_A, t)$ and

$$\nu_A(f(x_1^n) \leq max\{\nu_A(x_1), ..., \nu_A(x_n)\} \leq t,$$

which implies $f(x_1^n) \in L(\nu_A, t)$. Moreover, for some $x \in U(\mu_A, t)$ and $x \in L(\nu_A, t)$, we have

$$\mu_A((\overline{x}) \geq \mu_A(x)) \geq t \text{ and } \nu_A((\overline{x}) \leq \nu_A(x)) \leq t,$$

which implies $\overline{x} \in U(\mu_A, t)$ and $\overline{x} \in L(\nu_A, t)$. Thus μ_A-level t-cut and ν_A-level t-cut are n-ary subgroup of $(G, .)$.

Conversely, assume that μ_A-level t-cut and ν_A-level t-cut are n-ary subgroup of $(G, .)$. Let us define

$$t_0 = min\{\mu_A(x_1), ..., \mu_A(x_n)\},$$

and

$$t_1 = max\{\nu_A(x_1), ..., \nu_A(x_n)\},$$

for some $x_1^n \in G$. Then obviously $x_1^n \in U(\mu_A, t_0)$ and $x_1^n \in L(\nu_A, t_1)$, consequently $f(x_1^n) \in U(\mu_A, t_0)$ and $f(x_1^n) \in L(\nu_A, t_1)$. Thus

$$\mu_A(f(x_1^n)) \geq t_0 = min\{\mu_A(x_1), ..., \mu_A(x_n)\}$$

and

$$\nu_A(f(x_1^n)) \leq t_1 = max\{\nu_A(x_1), ..., \nu_A(x_n)\}.$$

Now let $x \in U(\mu_A, t)$ and $x \in L(\nu_A, t)$. Then $\mu(x) = t_0 \geq t$ and $\nu(x) = t_1 \leq t$. Thus $x \in U(\mu_A, t_0)$ and $x \in L(\nu_A, t_1)$. Since, by the assumption, $\overline{x} \in U(\mu_A, t_0)$ and $\overline{x} \in L(\nu_A, t_1)$. Whence $\mu_A(\overline{x}) \geq t_0 = \mu_A(x)$ and $\nu_A(\overline{x}) \geq t_1 = \nu_A(x)$. This complete the proof. □

Using the above theorem,we can prove the following characterization of intuitionistic fuzzy n-ary subgroup.

Theorem 3.7. *An IFS $A = (\mu_A, \nu_A)$ in G, is an intuitionistic fuzzy n-ary subgroup of (G, f) if and only if the μ_A-level t-cut and ν_A-level t-cut of G are n-ary subgroup of (G, f) for all $i = 1, 2, ..., n$ and all $x_1^n \in G$, A satisfies the following conditions:*

(*i*) $\mu_A(f(x_1^n) \geq min\{\mu_A(x_1), ..., \mu_A(x_n)\}$,

(*ii*) $\nu_A(f(x_1^n) \leq max\{\nu_A(x_1), ..., \nu_A(x_n)\}$,

(iii) $\mu_A(x_i) \geq min\{\mu_A(x_1), ..., \mu_A(x_{i-1}), \mu_A(f(x_1^n)), \mu_A(x_{i-1}), ..., \mu_A(x_n)\}$,

(iv) $\nu_A(x_i) \leq max\{\nu_A(x_1), ..., \nu_A(x_{i-1}), \nu_A(f(x_1^n)), \nu_A(x_{i-1}), ..., \nu_A(x_n)\}$.

Proof. Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G, f) .Similarly as in the proof of Theorem 3.5, we can prove each non-empty level subset $U(\mu_A, t)$ and $L(\nu_A, t)$ are closed under the operation f, that is $x_1^n \in U(\mu_A, t)$ and $x_1^n \in L(\nu_A, t)$ implies $f(x_1^n) \in U(\mu_A, t)$ and $f(x_1^n) \in L(\nu_A, t)$.
Now let $x_0, x_1^{i-1}, x_{i+1}^n$, where $x_0 = f(x_1^{i-1}, z, x_{i+1}^n)$ for some $i = 1, 2, ..., n$ and $z \in G$ which implies $x_0 \in U(\mu_A, t)$ and $x_0 \in L(\nu_A, t)$.Then,according to (iii) and (iv), we have $\mu_A(z) \geq t$ and $\nu_A(z) \leq t$. So, the equation (1) has a solution $z \in \mu_A(t)$ and $z \in \nu_A(t)$. This mean that μ_A-level t-cut and ν_A-level t-cut are an n-ary subgroups.
Conversely, assume that μ-level t-cut and ν-level t-cut are an n-ary subgroups. Then it is easy to prove the conditions (i) and (ii). For $x_1^n \in G$, we define

$$t_0 = min\{\mu_A(x_1), ..., \mu_A(x_{i-1}), \mu_A(f(x_1^n)), \mu_A(x_{i-1}), ..., \mu_A(x_n)\}$$

and

$$t_1 = max\{\nu_A(x_1), ..., \nu_A(x_{i-1}), \nu_A(f(x_1^n)), \nu_A(x_{i-1}), ..., \nu_A(x_n)\}.$$

Then $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in U(\mu_A, t_0)$ and $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in L(\nu_A, t_1)$. Whence, according to the definition of n-ary group, we conclude $x_i \in U(\mu_A, t_0)$ and $x_i \in L(\nu_A, t_1)$. Thus $\mu(x_i) \geq t_0$ and $\nu(x_i) \leq t_1$. This proves the conditions (iii) and (iv). □

Definition 3.8. Let (G, f) and (G', f) be an n-ary groups. A mapping $\alpha : G \to G'$ is called an *n-ary homomorphism* if $\alpha(f(x_1^n)) = f(\alpha^n(x_1^n))$, where $\alpha^n(x_1^n) = (\alpha(x_1), ..., \alpha(x_n))$ for all $x_1^n \in G$.

For any IFS $A = (\mu_A, \nu_A)$ in G', we define the *preimage* of A under α, denoted by $\alpha^{-1}(A)$, is an IFS in G defined by

$$\alpha^{-1}(A) = (\mu_{\alpha^{-1}(A)}, \nu_{\alpha^{-1}(A)}),$$

where

$$\mu_{\alpha^{-1}(A)}(x) = \mu_A(\alpha(x)) \text{ and } \nu_{\alpha^{-1}(A)}(x) = \nu_A(\alpha(x)), \forall x \in G.$$

For any IFS $A = (\mu_A, \nu_A)$ in G, we define the *image* of A under α, denoted by $\alpha(A)$, is an IFS in G' defined by

$$\alpha(A) = (\alpha_{sup}(\mu_A), \alpha_{inf}(\nu_A)),$$

where

$$\alpha_{\sup}(\mu_A)(y) = \begin{cases} \sup\limits_{x \in \alpha^{-1}(y)} \mu_A(x), & if \;\; \alpha^{-1}(y) \neq \phi, \\ 0, & otherwise. \end{cases}$$

and

$$\alpha_{\inf}\left(\nu_A\right)(y)=\begin{cases}\inf\limits_{x\in\alpha^{-1}(y)} \nu_A(x), & if\ \ \alpha^{-1}(y)\neq\phi,\\ 0, & otherwise.\end{cases}$$

for all $x \in G$ and $y \in G'$.

Theorem 3.9. *Let α be a n-ary homomorphism mapping from G into G' with $\alpha(\overline{x}) = \alpha(x)$ for all $x \in G$ and $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G', f). Then $\alpha^{-1}(A)$ is an intuitionistic fuzzy n-ary subgroup of (G, f).*

Proof. Let $x_1^n \in G$, we have

$$\begin{aligned}\mu_{\alpha^{-1}(A)}(f(x_1^n)) &= \mu_A(\alpha(f(x_1^n)) = \mu_A(f(\alpha^n(x_1^n)))\\ &\geq min\{\mu_A(\alpha(x_1), ..., \mu_A(\alpha(x_n)\}\\ &= min\{\mu_{\alpha^{-1}(A)}(x_1), ..., \mu_{\alpha^{-1}(A)}(x_n)\}.\end{aligned}$$

$$\begin{aligned}\nu_{\alpha^{-1}(A)}(f(x_1^n)) &= \nu_A(\alpha(f(x_1^n)) = \mu_A(f(\alpha^n(x_1^n)))\\ &\leq max\{\nu_A(\alpha(x_1), ..., \nu_A(\alpha(x_n)\}\\ &= max\{\nu_{\alpha^{-1}(A)}(x_1), ..., \nu_{\alpha^{-1}(A)}(x_n)\}.\end{aligned}$$

$$\begin{aligned}\mu_{\alpha^{-1}(A)}(\overline{x}) &= \mu_A(\alpha(\overline{x})) \geq \mu_A(\alpha(x)) = \mu_{\alpha^{-1}(A)}(x)\\ \nu_{\alpha^{-1}(A)}(\overline{x}) &= \nu_A(\alpha(\overline{x})) \leq \mu_A(\alpha(x)) = \nu_{\alpha^{-1}(A)}(x).\end{aligned}$$

This completes the proof. □

If we strengthen the condition of α, then we can construct the converse of Theorem 3.9 as follows.

Theorem 3.10. *Let α be an n-ary homomorphism from G into G' and $\alpha^{-1}(A) = (\mu_{\alpha^{-1}(A)}, \nu_{\alpha^{-1}(A)})$ is an intuitionistic fuzzy n-ary subgroup of (G, f).Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G', f).*

Proof. For any $x_1 \in G'$,there exists $a_1 \in G$ such that $\alpha(a_1) = x_1$ and For any $f(x_1^n) \in (G', f)$, there exists $f(a_1^n) \in (G, f)$ such that $\alpha(f(a_1^n)) = f(x_1^n)$. Then

$$\begin{aligned}\mu_A(f(x_1^n)) &= \mu_A(\alpha(f(a_1^n)) = \mu_{\alpha^{-1}(A)}(f(a_1^n))\\ &\geq min\{\mu_{\alpha^{-1}(A)}(a_1), \mu_{\alpha^{-1}(A)}(a_2), ..., \mu_{\alpha^{-1}(A)}(a_n)\}\\ &= min\{\mu_A(\alpha(a_1), ..., \mu_A(\alpha(a_n)\}\\ &= min\{\mu_A(x_1), ..., \mu_A(x_n)\}.\end{aligned}$$

$$\begin{aligned}\nu_A(f(x_1^n)) &= \nu_A(\alpha(f(a_1^n)) = \nu_{\alpha^{-1}(A)}(f(a_1^n))\\ &\leq max\{\nu_{\alpha^{-1}(A)}(a_1), \nu_{\alpha^{-1}(A)}(a_2), ..., \nu_{\alpha^{-1}(A)}(a_n)\}\\ &= max\{\nu_A(\alpha(a_1), ..., \nu_A(\alpha(a_n)\}\\ &= max\{\nu_A(x_1), ..., \nu_A(x_n)\}.\end{aligned}$$

For any $\overline{x} \in G'$, there exists $\overline{a} \in G$ such that $\alpha(\overline{a}) = \overline{x}$, we have

$$\begin{aligned}\mu_A(\overline{x}) &= \mu_A(\alpha(\overline{a})) = \mu_{\alpha^{-1}(A)}(\overline{a}) \geq \mu_{\alpha^{-1}(A)}(a) = \mu_A(\alpha(a)) = \mu_A(x).\\ \nu_A(\overline{x}) &= \nu_A(\alpha(\overline{a})) = \nu_{\alpha^{-1}(A)}(\overline{a}) \leq \nu_{\alpha^{-1}(A)}(a) = \nu_A(\alpha(a)) = \nu_A(x).\end{aligned}$$

This completes the proof. □

Theorem 3.11. *Let α be a mapping from G into G'. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G, f), then $\alpha(A) = (x, \alpha_{sup}(\mu_A), \alpha_{inf}(\nu_A))$ is an intuitionistic fuzzy n-ary subgroup of (G', f).*

Proof. Let α be a mapping from G into G' and let $x_1^n \in G$, $y_1^n \in G'$. Noticing that

$$\begin{aligned}\{x_i(i = 1, 2, ..., n)|x_i \in \alpha^{-1}(f(y_1^n))\} \subseteq &\{f(x_1^n) \in G|x_1 \in \alpha^{-1}(y_1),\\ &x_2 \in \alpha^{-1}(y_2), ..., x_n \in \alpha^{-1}(y_n))\}.\end{aligned}$$

we have

$$\begin{aligned}\alpha_{sup}(\mu_A)(f(y_1^n)) &= sup\{\mu_A(x_1^n)|x_i \in \alpha^{-1}(f(y_1^n))\}\\ &\geq sup\{\mu_A(f(x_1^n)|x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2), ...,\\ &\quad x_n \in \alpha^{-1}(y_n))\}\\ &\geq sup\{min\{\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)\}|x_1 \in \alpha^{-1}(y_1),\\ &\quad x_2 \in \alpha^{-1}(y_2), ..., x_n \in \alpha^{-1}(y_n))\}\\ &= min\{sup\{\mu_A(x_1)|x_1 \in \alpha^{-1}(y_1)\},\\ &\quad sup\{\mu_A(x_2)|x_1 \in \alpha^{-1}(y_2)\}, ..., sup\{\mu_A(x_n)|x_1 \in \alpha^{-1}(y_n)\}\}\\ &= min\{\alpha_{sup}(\mu_A)(y_1), \alpha_{sup}(\mu_A)(y_2), ..., \alpha_{sup}(\mu_A)(y_n)\}.\end{aligned}$$

$$\begin{aligned}\alpha_{inf}(\nu_A)(f(y_1^n)) &= inf\{\nu_A(x_1^n)|x_i \in \alpha^{-1}(f(y_1^n))\}\\ &\leq inf\{\nu_A(f(x_1^n))|x_1 \in \alpha^{-1}(y_1), x_2 \in \alpha^{-1}(y_2), ...,\\ &\quad x_n \in \alpha^{-1}(y_n))\}\\ &\leq inf\{max\{\nu_A(x_1), \nu_A(x_2), ..., \nu_A(x_n)\}|x_1 \in \alpha^{-1}(y_1),\\ &\quad x_2 \in \alpha^{-1}(y_2), ..., x_n \in \alpha^{-1}(y_n))\}\\ &= max\{inf\{\nu_A(x_1)|x_1 \in \alpha^{-1}(y_1)\}, inf\{\nu_A(x_2)|x_1 \in \alpha^{-1}(y_2)\},\\ &\quad ..., inf\{\nu_A(x_n)|x_1 \in \alpha^{-1}(y_n)\}\}\\ &= max\{\alpha_{inf}(\nu_A)(y_1), \alpha_{inf}(\nu_A)(y_2), ..., \alpha_{inf}(\nu_A)(y_n)\}.\end{aligned}$$

$$\begin{aligned}\alpha_{sup}(\mu_A)(\overline{x}) &= sup\{\mu_A(\overline{x})|\overline{x}\in\alpha^{-1}(f(\overline{y}))\}\\ &\geq sup\{\mu_A(x)|x\in\alpha^{-1}(f(y))\}\\ &= \alpha_{sup}(\mu_A)(x).\end{aligned}$$

$$\begin{aligned}\alpha_{inf}(\nu_A)(\overline{x}) &= inf\{\nu_A(\overline{x})|\overline{x}\in\alpha^{-1}(f(\overline{y}))\}\\ &\leq inf\{\nu_A(x)|x\in\alpha^{-1}(f(y))\}\\ &= \alpha_{inf}(\nu_A)(x).\end{aligned}$$

This completes the proof. □

Corollary 3.12. *An IFS $A=(\mu_A,\nu_A)$ defined on group $(G,.)$ is an Intuitionistic fuzzy subgroup if and only if*
(1) $\mu_A(xy)\geq min\{\mu_A(x),\mu_A(y)\}$ and $\nu_A(xy)\leq max\{\nu_A(x),\nu_A(y)\}$,
(2) $\mu_A(x)\geq min\{\mu_A(y),\mu_A(xy)\}$ and $\nu_A(x)\leq max\{\nu_A(y),\nu_A(xy)\}$,
(3) $\mu_A(y)\geq min\{\mu_A(x),\mu_A(xy)\}$ and $\nu_A(y)\leq max\{\nu_A(x),\nu_A(xy)\}$.
holds for all $x,y\in G$.

Theorem 3.13. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy n-ary subgroup of (G,f). If there exists an element $a\in G$ such that $\mu_A(a)\geq\mu_A(x)$ and $\nu_A(a)\leq\nu_A(x)$ for every $x\in G$, then $A=(\mu_A,\nu_A)$ is an intuitionistic fuzzy n-ary subgroup of a group $ret_a(G,f)$.*

Proof. For all $x,y,a\in G$ we have

$$\begin{aligned}\mu_A(x\circ y) &= \mu_A(f(x,\overset{(n-2)}{a},y)\geq min\{\mu_A(x),\mu_A(a),\mu_A(y)\}=min\{\mu_A(x),\mu_A(y)\}.\\ \nu_A(x\circ y) &= \nu_A(f(x,\overset{(n-2)}{a},y)\leq max\{\nu_A(x),\nu_A(a),\nu_A(y)\}=max\{\nu_A(x),\nu_A(y)\}.\\ \mu_A(x^{-1}) &= \mu_A(f(\overline{a},\overset{(n-3)}{x}\ \overline{x},\overline{a}))\geq min\{\mu_A(x),\mu_A(\overline{x}),\mu_A(a),\mu_A(\overline{a})\}=\mu_A(x).\\ \nu_A(x^{-1}) &= \nu_A(f(\overline{a},\overset{(n-3)}{x}\ \overline{x},\overline{a}))\leq max\{\nu_A(x),\nu_A(\overline{x}),\nu_A(a),\nu_A(\overline{a})\}=\nu_A(x).\end{aligned}$$

which complete the proof. □

In Theorem 3.13, the assumptions that $\mu_A(a)\geq\mu_A(x)$ and $\nu_A(a)\leq\nu_A(x)$ cannot be omitted.

Examples 3.14. Consider $(\mathbb{Z}_4,f)$, where $f:\mathbb{Z}_4^3\to\mathbb{Z}_4$ is defined by $f(x_1,x_2,x_3)=max(x_1,x_2,x_3)$. Clearly, $(\mathbb{Z}_4,f)$ is a ternary subgroup derived from $\mathbb{Z}_4$. Define an IFS $A=(\mu_A,\nu_A)$ as follows:

$$\mu_A(x)=\begin{cases}1 & if\ x=0,\\ 0.2 & \quad x=1,2,3.\end{cases}\qquad \nu_A(x)=\begin{cases}0 & if\ x=0,\\ 0.9 & \quad x=1,2,3.\end{cases}$$

clearly $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ternary subgroup of $(\mathbb{Z}_4, f)$.For $ret_1(\mathbb{Z}_4, f)$,we have

$$\mu_A(0 \circ 0) = \mu_A((f(0,1,0)) = \mu_A(1) = 0.2 \quad \ngeq \quad min\{\mu_A(0), \mu_A(0)\} = 1.$$
$$\nu_A(0 \circ 0) = \nu_A((f(0,1,0)) = \nu_A(1) = 0.9 \quad \nleq \quad max\{\nu_A(0), \nu_A(0)\} = 0.$$

Hence the assumptions $\mu_A(a) \geq \mu_A(x)$ and $\nu_A(a) \leq \nu_A(x)$ cannot be omitted.

Theorem 3.15. *Let (G, f) be an n-ary group. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of a group $ret_a(G, f)$ and $\mu_A(a) \geq \mu_A(x)$, $\nu_A(a) \leq \nu_A(x)$ for all $a, x \in G$, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy n-ary subgroup of (G, f).*

Proof. According to Theorem 2.1, any n-ary group can be represented of the form (2), where $(G, \circ) = ret_a(G, f), \varphi(x) = f(\overline{a}, x, \overset{(n-2)}{x})$ and $b = f(\overline{a}, ..., \overline{a})$. Then we have

$$\mu_A(\varphi(x)) = \mu_A(f(\overline{a}, x, \overset{(n-2)}{x})) \geq min\{\mu_A(\mu_A(\overline{a}), \mu_A(x), \mu_A(a)\} = \mu_A(x).$$

$$\begin{aligned} \mu_A(\varphi^2(x)) &= \mu_A(f(\overline{a}, \varphi(x), \overset{(n-2)}{x})) \\ &\geq min\{\mu_A(\mu_A(\overline{a}), \mu_A(\varphi(x)), \mu_A(a)\} \\ &= \mu_A(\varphi(x)) \\ &\geq \mu_A(x). \end{aligned}$$

Consequently, $\mu_A(\varphi^k(x)) \geq \mu_A(x)$ for all $x \in G$ and $k \in \mathbb{N}$ and

$$\nu_A(\varphi(x)) = \nu_A(f(\overline{a}, x, \overset{(n-2)}{x})) \leq max\{\nu_A(\nu_A(\overline{a}), \nu_A(x), \nu_A(a)\} = \nu_A(x).$$

$$\begin{aligned} \nu_A(\varphi^2(x)) &= \nu_A(f(\overline{a}, \varphi(x), \overset{(n-2)}{x})) \\ &\leq max\{\nu_A(\nu_A(\overline{a}), \nu_A(\varphi(x)), \nu_A(a)\} \\ &= \nu_A(\varphi(x)) \leq \nu_A(x). \end{aligned}$$

Consequently, $\nu_A(\varphi^k(x)) \leq \nu_A(x)$ for all $x \in G$ and $k \in \mathbb{N}$. Similarly, for all $x \in G$ we have

$$\begin{aligned} \mu_A(b) &= \mu_A(f(\overline{a}, ..., \overline{a})) \geq \mu_A(\overline{a}) \geq \mu_A(x). \\ \nu_A(b) &= \nu_A(f(\overline{a}, ..., \overline{a})) \leq \nu_A(\overline{a}) \leq \nu_A(x). \end{aligned}$$

Thus

$$\begin{aligned} \mu_A(f(x_1^n)) &= \mu_A(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-2}(x_n) \circ b) \\ &\geq min\{\mu_A(x_1), \mu_A(\varphi(x_2)), \mu_A(\varphi^2(x_3)), ..., \mu_A(\varphi^{n-2}(x_n)), \mu_A(b)\} \\ &\geq min\{\mu_A(x_1), \mu_A((x_2)), \mu_A((x_3)), ..., \mu_A(x_n), \mu_A(b)\} \\ &\geq min\{\mu_A(x_1), \mu_A((x_2)), \mu_A((x_3)), ..., \mu_A(x_n)\}. \end{aligned}$$

$$\begin{aligned}\nu_A(f(x_1^n)) &= \nu_A(x_1\circ\varphi(x_2)\circ\varphi^2(x_3)\circ...\circ\varphi^{n-2}(x_n)\circ b)\\ &\le max\{\nu_A(x_1),\nu_A(\varphi(x_2)),\nu_A(\varphi^2(x_3)),...,\nu_A(\varphi^{n-2}(x_n)),\nu_A(b)\}\\ &\le max\{\nu_A(x_1),\nu_A((x_2)),\nu_A((x_3)),...,\nu_A(x_n),\nu_A(b)\}\\ &\le max\{\nu_A(x_1),\nu_A((x_2)),\nu_A((x_3)),...,\nu_A(x_n)\}.\end{aligned}$$

From (4) and (7)of [3], we have

$$\overline{x} = \left(\varphi(x)\circ\varphi^2(x)\circ...\circ\varphi^{n-2}(x)\circ b\right)^{-1}$$

Thus

$$\begin{aligned}\mu_A(\overline{x}) &= \mu_A\left(\left(\varphi(x)\circ\varphi^2(x)\circ...\circ\varphi^{n-2}(x)\circ b\right)^{-1}\right)\\ &\ge \mu_A\left(\varphi(x)\circ\varphi^2(x)\circ...\circ\varphi^{n-2}(x)\circ b\right)\\ &\ge min\{\mu_A(\varphi(x)),\mu_A(\varphi^2(x)),...,\mu_A(\varphi^{n-2}(x)),\mu_A(b)\}\\ &\ge min\{\mu_A(x),\mu_A(b)\} = \mu_A((x)).\end{aligned}$$

$$\begin{aligned}\nu_A(\overline{x}) &= \nu_A\left(\left(\varphi(x)\circ\varphi^2(x)\circ...\circ\varphi^{n-2}(x)\circ b\right)^{-1}\right)\\ &\le \nu_A\left(\varphi(x)\circ\varphi^2(x)\circ...\circ\varphi^{n-2}(x)\circ b\right)\\ &\le max\{\nu_A(\varphi(x)),\nu_A(\varphi^2(x)),...,\nu_A(\varphi^{n-2}(x)),\nu_A(b)\}\\ &\le max\{\nu_A(x),\nu_A(b)\} = \nu_A(x).\end{aligned}$$

This completes the proof. □

Corollary 3.16. *If (G,f) is a ternary group, then any intuitionistic fuzzy subgroup of $ret_a(G,f)$ is an intuitionistic fuzzy ternary subgroup of (G,f).*

Proof. Since $\overline{a}$ is a neutral element of a group $ret_a(G,f)$ then $\mu_A(\overline{a}) \ge \mu_A(x)$ and $\nu_A(\overline{a}) \le \nu_A(x)$ for all $x \in G$. Thus $\mu_A(\overline{a}) \ge \mu_A(a)$ and $\nu_A(\overline{a}) \le \nu_A(a)$. But in ternary group $\overline{\overline{a}} = a$ for any $a \in G$, whence $\mu(a) = \mu_A(\overline{\overline{a}}) \ge \mu_A(\overline{a}) \ge \mu_A(x)$ and $\nu(a) = \nu_A(\overline{\overline{a}}) \le \nu_A(\overline{a}) \le \nu_A(x)$. So, $\mu_A(a) = \mu_A(\overline{\overline{a}}) \ge \mu_A(x)$ and $\nu_A(a) = \nu_A(\overline{\overline{a}}) \le \nu_A(x)$ for all $x \in G$. This means that the assumptions of Theorem 3.15 are satisfied. □

Example 3.17. Consider the ternary group $(\mathbb{Z}_{12}, f)$, where $f : \mathbb{Z}_{12}^3 \to \mathbb{Z}_{12}$ is defined by $f(x_1,x_2,x_3) = max(x_1,x_2,x_3)$, derived from the additive group $\mathbb{Z}_{12}$. Let $A = (\mu_A,\nu_A)$ be an intuitionistic fuzzy subgroup of the group of $ret_1(G,f)$ induced by subgroups $S_1 = \{11\}, S_2 = \{5,11\}$ and $S_3 = \{1,3,5,7,9,11\}$. Define the IFS $A = (\mu_A,\nu_A)$ as follows:

$$\mu_A(x) = \begin{cases} 0.8 & if\ \ x = 11,\\ 0.6 & if\ \ x = 5,\\ 0.4 & if\ \ x = 1,3,7,9,\\ 0.2 & if\ \ x \notin S_3.\end{cases} \qquad \nu_A(x) = \begin{cases} 0.1 & if\ \ x = 11,\\ 0.3 & if\ \ x = 5,\\ 0.5 & if\ \ x = 1,3,7,9,\\ 0.9 & if\ \ x \notin S_3.\end{cases}$$

Then

$$\mu_A(\overline{5}) = \mu_A(7) = 0.4 \quad \not\geq \quad = 0.6 = \mu_A(5).$$
$$\nu_A(\overline{5}) = \mu_A(7) = 0.5 \quad \not\leq \quad = 0.3 = \nu_A(5).$$

Hence $A = (\mu_A, \nu_A)$ is not an intuitionistic fuzzy ternary subgroup of (G, f).

Observations. From the above Example 3.16 it follows that:
(1) There are intuitionistic fuzzy subgroups of $ret_a(G, f)$ which are not intuitionistic fuzzy n-ary subgroups of (G, f).
(2) In Theorem 3.15 the assumptions $\mu_A(a) \geq \mu_A(x)$ and $\nu_A(a) \leq \nu_A(x)$ can not be omitted.In the above example we have $\mu_A(1) = 0.4 \not\geq 0.6 = \mu_A(5)$ and $\nu_A(1) = 0.5 \not\leq 0.3 = \nu_A(5)$.
(3) The assumptions $\mu_A(a) \geq \mu_A(x)$ and $\nu_A(a) \leq \nu_A(x)$ cannot be replaced by the natural assumption $\mu_A(\overline{a}) \geq \mu_A(x)$ and $\nu_A(\overline{a}) \leq \nu_A(x)$.($\overline{a}$ is the identity of $ret_a(G, f)$). In the above example $\overline{1} = 11$, then $\mu_A(11) \geq \mu_A(x)$ and $\nu_A(11) \leq \nu_A(x)$ for all $x \in \mathbb{Z}_{12}$.

Theorem 3.18. *Let (G, f) be an n-ary group of b-derived from the group $(G, \circ)$. Any intuitionistic fuzzy n-ary subgroup $A = (\mu_A, \nu_A)$ of $(G, \circ)$ such that $\mu_A(b) \geq \mu_A(x)$ and $\nu_A(b) \leq \nu_A(x)$ for every $x \in G$ is an intuitionistic fuzzy n-ary subgroup of (G, f).*

Proof. The conditions (IFnS1) and (IFns2) are obvious. To prove (IFnS3) and (IFns4), we have n-ary group (G, f) b-derived from the group $(G, \circ)$, which implies

$$\overline{x} = (x^{n-2} \circ b)^{-1},$$

where x^{n-2} is the power of x in $(G, \circ)$[4] .
Thus, for all $x \in G$

$$\mu_A(\overline{x}) = \mu_A((x^{n-2} \circ b)^{-1}) \geq \mu_A(x^{n-2} \circ b) \geq min\{\mu_A(x^{n-2}), \mu_A(b)\} = \mu_A(x).$$
$$\nu_A(\overline{x}) = \nu_A((x^{n-2} \circ b)^{-1}) \leq \nu_A(x^{n-2} \circ b) \leq max\{\nu_A(x^{n-2}), \nu_A(b)\} = \nu_A(x).$$

This complete the proof. □

Corollary3.19. *Any intuitionistic fuzzy group of a group $(G, \circ)$ is a intuitionistic fuzzy n-ary subgroup of an n-ary group (G, f) derived from $(G, \circ)$.*

Proof. If n-ary group (G, f) is derived from the group $(G, \circ)$ then $b = e$. Thus $\mu_A(e) \geq \mu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$ for all $x \in G$. □

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