Thai Journal of Mathematics
Volume 8 (2010) Number 2 : 383–390

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A kind of product of submodules and some related results

M.J. Nikmehr[1], S. Heidari and R. Nikandish

Abstract : Let R be a commutative ring with identity, M an R-module and K_1, K_2 submodules of M. In this article, we define a kind of product between K_1 and K_2. In a special case of this product, we focus on M^2 as an $R(M)$-module and we show that, in many cases, the study of M as an R-module can be replaced by the study of M^2 as an $R(M)$-module.

Keywords : Commutative ring, Product of submodules, Idealization.
2000 Mathematics Subject Classification : 13C05 (2000 MSC)

1 Introduction

In this paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module; the idealization of M, $R(M)$, introduced by Nagata in [11], and many papers have been devoted to this concept. Idealization is useful for generalizing results from rings to modules and constructing examples of commutative rings with zero-divisors (see [2] and [9, Section 25]). Let K_1 and K_2 be submodules of M. In this article, we construct an algebraic object by K_1 and K_2 denoted by K_1K_2, called product of K_1 and K_2. We show that K_1K_2, with appropriate operations, has an $R(M)$-module structure. Our main aim of this paper is to study some of the most important properties of $R(M)$-module $MM = M^2$. For instance, in section 2, we give a necessary and sufficient condition under which M^2 is a projective $R(M)$-module. In section 3, we find primary and secondary decompositions for $R(M)$-module M^2. Now, we define the concepts that we will need. Recall that $R(M) = R(+)M$ with coordinate-wise addition and multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1),$$

[1]Corresponding author : nikmehr@kntu.ac.ir (M.J. Nikmehr)

is a commutative ring with identity, called the idealization of M. Note that R naturally embeds into $R(M)$ via $r \longrightarrow r(+)0$, if N is a submodule of M, then $0(+)N$ is an ideal of $R(M)$, $0(+)M$ is a nilpotent ideal of $R(M)$ of index 2, every ideal that contains $0(+)M$ has the form $I(+)M$ for some ideal I of R, and every ideal that is contained in $0(+)N$ has the form $0(+)K$ for some submodule K of N. The purpose of idealization is to put M inside a commutative ring A so that the structure of M as an R-module is essentially the same as that of M as an A-module, that is, an ideal of A. Since $R \cong R(M)/0(+)M$, $I \longrightarrow I(+)M$ gives a one-to-one correspondence between ideals of R and ideals of $R(M)$ that contains $0(+)M$. Thus the prime (maximal) ideals of $R(M)$ have the form $P(+)M$ where P is a prime (maximal) ideal of R. Some basic results about idealization can be found in [9] and [2]. Generalizing the case for ideals, an R-module M is defined to be a cancellation module if $IM = JM$ for ideals I and J of R implies $I = J$ (equivalently, $[IM : M] = I$ for all ideals I of R) see [3]. Examples of cancellation modules include invertible ideals, free modules and finitely generated faithful multiplication modules [4, Corollary to Theorem 9]. It is also defined that M is a weak cancellation module if $IM = JM$ implies $I + AnnM = J + AnnM$ (equivalently, $[IM : M] = I + AnnM$). An R-module M is cancellation if and only if it is a faithful weak cancellation module. A submodule N of M is said to be join principal if for all ideals A of R and all submodules K of M $[(AN + K) : N] = A + [K : N]$ (see [3]). Setting $K = 0$, N becomes weak cancellation. Thus join principal submodules are weak cancellation. The trace ideal of an R-module M is $Tr(M) = \Sigma_{f\in Hom(M,R)} f(M)$. If M is projective, then $M = Tr(M)M$, $Ann(M) = AnnTr(M)$ and $Tr(M)$ is a pure ideal of R [8, Proposition 3.30].

2 Product of Submodules

In this section first we introduce a new product between submodules of an R-module M.

Definition. Let M be an R-module and K_1, K_2 submodules of M. Define the product of K_1 and K_2 as follows:

$$K_1K_2 = \{(1, k_1 + k_2)|k_1 \in K_1, k_2 \in K_2\}.$$

One can check that K_1K_2 forms an $R(M)$-module under below operations:

$$(1, k_1 + k_2) + (1, k_1^{'} + k_2^{'}) = (1, k_1 + k_1^{'} + k_2 + k_2^{'}),$$

$$(r, m)(1, k_1 + k_2) = (1, rk_1 + rk_2),$$

where $k_1, k_1^{'} \in K_1$, $k_2, k_2^{'} \in K_2$, $r \in R$ and $m \in M$.
Our starting point is the following result.

Theorem 2.1. *Let M be an R-module. Then every submodule of M^2 is of the form N^2, in which N is a submodule of M.*

Proof. Let H be a submodule of M^2. Put $N = \{h \mid (1,h) \in H\}$. One can check that N is a submodule of M and $N^2 = H$. It is easily checked that if N is a submodule of M then N^2 is a submodule of M^2. $\square$

For an R-module M, following [7], we set

$$M(P) = \{x \in M \;\mid\; sx \in PM \;\; for \;\; some \;\; s \in R \setminus P\},$$

in which P is prime ideal of R. In [7], it is shown that $M(P) = M$ or $M(P)$ is a submodule of M, for every $P \in Spec(R)$. As usual, we will denote the Support of M by

$$Supp_R M = \{P \in Spec(R) \mid there\ exists\ 0 \neq x \in M\ s.t.\ Ann(x) \subseteq P\}.$$

Recall that an R-module M is called quasi multiplication if $M(P) = PM$, for all $P \in Supp_R M$. For a reference on quasi multiplication module see [7]. The next result will be used in the Theorem 2.2.

Lemma 2.1. *Let M be an R-module. Then*
(i) $Supp_{R(+)M} M^2 = \{P(+)M \mid P \in Supp_R M\}$.
(ii) $M^2(P(+)M) = \{(1,m) \in M^2 \mid m \in M(P)\}$, for every $P \in Spec(R)$.

Proof. (i) If $P \in Supp_R M$, then there exists a none-zero element $x \in M$ such that $Ann(x) \subseteq P$. Since $Ann(1,x) = \{(r,m) \mid r \in Ann(x)\} \subseteq P(+)M$, we have $P(+)M \in Supp_{R(M)} M^2$. Now, let $P(+)M \in Supp_{R(M)} M^2$. Then there exists $0 \neq (1,x) \in M^2$ such that $Ann(1,x) \subseteq P(+)M$ and hence $Ann(x) \subseteq P$. Therefore, $P \in Supp_R M$. It follows that

$$Supp_{R(+)M} M^2 = \{P(+)M \mid P \in Supp_R M\}.$$

(ii) Let $P \in Spec(R)$. Clearly, $\{(1,m) \in M^2 \mid m \in M(P)\} \subseteq M^2(P(+)M)$. So we have only to prove the converse. Let $(1,m) \in M^2(P(+)M)$. Then there exists $(s_1,m_1) \in R(M)\backslash P(+)M$ such that $(s_1,m_1)(1,m) \in (P(+)M)M^2$. It follows that $s_1 m \in PM$ and hence $m \in M(P)$, and so $M^2(P(+)M) \subseteq \{(1,m) \in M^2 \mid m \in M(P)\}$. $\square$

The next result shows that M is a quasi multiplication R-module if and only if M^2 is a quasi multiplication $R(M)$-module.

Theorem 2.2. *Let M be an R-module. Then M is a quasi multiplication R-module if and only if M^2 is a quasi multiplication $R(M)$-module.*

Proof. Suppose that M^2 is quasi multiplication and $P \in Supp_R M$. Then $M^2(P(+)M) = (P(+)M)M^2$. By Lemma 2.1 (ii), we have $M(P) = PM$. Hence M is quasi multiplication.
Conversely, let M be quasi multiplication and $P(+)M \in Supp_{R(M)} M^2$. By Lemma 2.1, we have $P \in Supp M$ and

$$M^2(P(+)M) = \{(1,m) \mid m \in M(P) = PM\} = (P(+)M)M^2.$$

Hence M^2 is a quasi multiplication $R(M)$-module. $\square$

The following question is interesting: Does M^2 as an $R(M)$-module have all properties of R-module M? It is easily checked that $\mathbb{Q}$ is a faithful $\mathbb{Z}$-module, but $Ann(\mathbb{Q}^2) = Ann(\mathbb{Q})(+)\mathbb{Q} = 0(+)\mathbb{Q}$. In fact, the answer of the above question is negative.

Theorem 2.3. *Let M be an R-module. Then*

$$Tr(M^2) = Tr(M)(+)\sum\nolimits_{g\in Hom(M,M)} g(M) = Tr(M)(+)M.$$

Proof. To see why this is true note first that if $f \in Hom_{R(+)M}(M^2, R(+)M)$, then there exist $g_1 \in Hom_R(M,R)$ and $g_2 \in Hom_R(M,M)$ such that $f = g_1(+)g_2$. Hence

$$\begin{aligned} Tr(M^2) &= \sum_{f\in Hom(M^2,R(M))} f(M^2) \\ &= \sum_{g_1\in Hom(M,R), g_2\in Hom(M,M)} g_1(M)(+)g_2(M) \\ &= \sum_{g_1\in Hom(M,R)} g_1(M)(+) \sum_{g_2\in Hom(M,M)} g_2(M) \\ &\subseteq Tr(M)(+)M. \end{aligned}$$

Conversely, let $g \in Hom(M,R)$. Define $f : M^2 \longrightarrow R(M)$ as follows: for each $(1, m_1 + m_2) \in M^2$, $f(1, m_1 + m_2) = g(m_1 + m_2)(+)id(m_1 + m_2)$. It is clear that f is well defined and $R(M)$-homomorphism. Hence

$$Tr(M)(+)M = \sum_{g\in Hom(M,R)} g(M)(+)M \subseteq \sum f(M^2) \subseteq Tr(M^2).$$

It follows that $Tr(M^2) = Tr(M)(+)M$. □

Lemma 2.2. *Let M be a projective R-module. Then $Tr(M)$ is a finitely generated ideal of R if and only if $Tr(M^2)$ is a finitely generated ideal of $R(M)$.*

Proof. Let $Tr(M)$ be finitely generated. By Theorem 2.3 and [8, Proposition 3.3]

$$Tr(M^2) = Tr(M)(+)M = Tr(M)(+)Tr(M)M.$$

Hence $Tr(M^2)$ is finitely generated if and only if $Tr(M)$ is finitely generated, by [1, Theorem 7(1)]. □

It is shown in [8, Lemma 3.23] that an R-module M is projective if and only if there exist families $\{m_i\}_{i\in I}$ in M and $\{f_i\}_{i\in I}$ in $M^* = Hom_R(M,R)$ such that every $m \in M$ is a finite sum $m = \Sigma m_i f_i(m)$ where $f_i(m) = 0$ almost for every $i \in I$. In the next theorem, we prove that M is a projective R-module if and only if M^2 is a projective $R(M)$-module.

Theorem 2.4. *Let M be an R-module. Then M is projective if and only if M^2 is projective.*

Proof. Let M be a projective R-module and $(1, m) \in M^2$. Then there exist families $\{m_i\}_{i \in I}$ in M and $\{f_i\}_{i \in I}$ in $M^* = Hom_R(M, R)$ such that $m = \Sigma m_i f_i(m)$. Thus

$$(1, m) = (1, \Sigma m_i f_i(m)) = \Sigma(1, m_i f_i(m)) = \Sigma(f_i(m), 0)(1, m_i).$$

Put $g_i = f_i(+)0$ and $t_i = (1, m_i)$. Hence $(1, m) = \Sigma t_i g_i(1, m)$, in which $g_i \in Hom_{R(M)}(M^2, R(M))$. Therefore, M^2 is a projective. The proof of the converse is similar. □

One may ask the following question. If M is a weak cancellation R-module, can we deduce that M^2 is a weak cancellation $R(M)$-module? The following corollary gives an affirmative answer in the case projective modules. But first note that by [13, Theorem 4.1], any projective module is a weak cancellation if and only if its trace is finitely generated ideal.

Corollary 2.5. *Let M be a projective R-module. Then M is a weak cancellation R-module if and only if M^2 is a weak cancellation $R(M)$-module.*

Proof. Let M be a weak cancellation projective R-module. By [13, Theorem 4.1], $Tr(M)$ is finitely generated. Hence $Tr(M^2)$ is finitely generated, by Lemma 2.2. Theorem 2.4 and [13, Theorem 4.1] follow that M^2 is a weak cancellation module. The proof of the other side is similar. □

Corollary 2.6. *Let M be a projective R-module. Then M is a cancellation R-module if and only if M^2 is a cancellation $R(M)$-module.*

Proof. Let M be a cancellation projective R-module. By [13, Theorem 4.2], $Tr(M) = R$. Hence $Tr(M^2) = Tr(M)(+)M = R(M)$, by Theorem 2.3. Thus M^2 is a cancellation module. The proof of the converse is similar. □

By Corollary 2.6 and [13, Example 1.3], if F is a free R-module, then F^2 is a cancellation $R(M)$-module.
It is shown in [10, Theorem 7.6], M is flat if and only if for every pair of finite subsets $\{x_1, ..., x_n\}$ and $\{a_1, ..., a_n\}$ of M and R, respectively, such that $\Sigma_{i=1}^n a_i x_i = 0$ there exist elements $z_1, ..., z_k \in M$ and $b_{ij} \in R$ ($i = 1, ..., n$ and $j = 1, ..., k$) such that $\Sigma_{i=1}^n b_{ij} a_i = 0$ $(j = 1, ..., k)$ and $x_i = \Sigma_{j=1}^k b_{ij} z_j$. Now, we show that M is flat if and only if M^2 is flat.

Theorem 2.7. *Let M be an R-module. Then M is a flat R-module if and only if M^2 is a flat $R(M)$-module.*

Proof. Let M be a flat R-module and $\Sigma_{i=1}^n (a_i, m_i)(1, x_i) = 0$, where $\{(a_i, m_i)\}_{i=1}^n$ and $\{(1, x_i)\}_{i=1}^n$ are arbitrary subsets of $R(M)$ and M^2. So $\Sigma_{i=1}^n a_i x_i = 0$. Since M is flat, there exist elements $z_1, ..., z_k \in M$ and $b_{ij} \in R$ ($i = 1, ..., n$ and $j = 1, ..., k$) such that $\Sigma_{i=1}^n b_{ij} a_i = 0$ $(j = 1, ..., k)$ and $x_i = \Sigma_{j=1}^k b_{ij} z_j$. Thus $\Sigma_{i=1}^n (b_{ij}, 0)(a_i, 0) = 0$ and $(1, x_i) = \Sigma_{j=1}^k (b_{ij}, 0)(1, z_j)$. Therefore, M^2 is a flat R-module. The proof of the converse is similar. □

3 Product of Submodules and Decompositions

In this section we show that if N has a primary (secondary) decomposition then N^2 has a primary (resp. secondary) decomposition. We recall from [10], that a submodule Q of M is said to be a primary submodule precisely when $M/Q \neq 0$ and for each $a \in Zdv_R(\frac{M}{Q})$ there exists $n \in \mathbb{N}$ such that $a^n(\frac{M}{Q}) = 0$. Now, if Q is a primary submodule of M, then $P := rad(Ann_R \frac{M}{Q})$ is a prime ideal of R. In this case we say that Q is a P-primary submodule of M, or Q is P-primary in M. Let N be a proper submodule of M. A primary decomposition of N in M is an expression for N as an intersection of finitely many primary submodules of M. We say that N is a decomposable submodule of M precisely when it has a primary decomposition in M.
First we need the following.

Theorem 3.1. *Let $Q_1, ..., Q_n$ be submodules of an R-module M. Then $(Q_1+Q_2+\cdots+Q_n)^2 = Q_1^2+Q_2^2+\cdots+Q_n^2$ and $(Q_1 \cap Q_2 \cdots \cap Q_n)^2 = Q_1^2 \cap Q_2^2 \cap \cdots \cap Q_n^2$.*

Proof. The proof is trivial. □

Theorem 3.2. *Let N be a submodule of an R-module M. Then N has a primary decomposition if and only if N^2 has a primary decomposition.*

Proof. Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a primary decomposition for N and $rad(Ann_R \frac{M}{Q_i}) = P_i$, for every i with $1 \leq i \leq n$. By Theorem 3.1, $N^2 = Q_1^2 \cap Q_2^2 \cap \cdots \cap Q_n^2$. To see why this is a primary decomposition for N^2, note first that $\frac{M^2}{Q_i^2} \neq 0$, for every i with $1 \leq i \leq n$. If $(r,m) \in Zdv(\frac{M^2}{Q_i^2})$, then $r \in Zdv(\frac{M}{Q_i})$ and hence there exists $n \in \mathbb{N}$ such that $r^n(\frac{M}{Q_i}) = 0$. Hence $(r,m)^n(\frac{M^2}{Q_i^2}) = (r^n, nr^{n-1}m)(\frac{M^2}{Q_i^2}) = 0$. It remains to show that $rad(Ann_{R(M)} \frac{M^2}{Q_i^2}) = P_i(+)M$, for every i with $1 \leq i \leq n$. Let $(t,m) \in rad(Ann_{R(M)} \frac{M^2}{Q_i^2})$. Then there exists $n \in \mathbb{N}$ such that $t^n M \subseteq Q_i$. Thus $t \in rad(Ann_R \frac{M}{Q_i}) = P_i$. It turns out that $rad(Ann_{R(M)} \frac{M^2}{Q_i^2}) \subseteq P_i(+)M$. One can easily check that $P_i(+)M \subseteq rad(Ann_{R(M)} \frac{M^2}{Q_i^2})$. It follows that $P_i(+)M = rad(Ann_{R(M)} \frac{M^2}{Q_i^2})$ and so Q_i^2 is a primary submodule of M^2, for every i with $1 \leq i \leq n$. The proof of the converse is similar. □

I.G. Macdonald in [5] has developed the theory of attached prime ideals and secondary representations of a module, which is, in a certain sense, dual to the theory of associated prime ideals and primary decompositions. Let us recall from [10], the definition of secondary module. Recall that an R-module M is said to be secondary if $M \neq 0$ and for each $a \in R$ the endomorphism $\varphi_a : M \to M$ defined by $\varphi_a(m) = am$ (for $m \in M$) is either surjective or nilpotent. If M is secondary, then $P = rad(AnnM)$ is a prime ideal and M is said to be P-secondary. A secondary representation of an R-module M is an expression of M as a finite sum of secondary submodules

$$M = N_1 + N_2 + \cdots + N_n.$$

Theorem 3.3. *Let N be a submodule of an R-module M. Then N has a secondary representation if and only if N^2 has a secondary representation.*

Proof. Let $N = Q_1 + Q_2 + \cdots + Q_n$ be a secondary representation of N with $rad(AnnQ_i) = P_i$, for every i with $1 \leq i \leq n$. By Theorem 3.1, $N^2 = Q_1^2 + Q_2^2 + \cdots + Q_n^2$. To see why this is a secondary representation of the N^2 note first that for each $(r, m) \in R(M)$ the endomorphism $\phi_{(r,m)} : Q_i^2 \to Q_i^2$ defined by $\phi_{(r,m)}((1, q)) = (r, m)(1, q) = (1, rq)$ induce endomorphism $\varphi_r : Q_i \to Q_i$ defined by $\varphi_r(q) = rq$. φ_r is either surjective or nilpotent. It follows that $\phi_{(r,m)}$ is either surjective or nilpotent. It remains to show that $rad(AnnQ_i^2) = P_i(+)M$, for every i with $1 \leq i \leq n$. Let $(t, m) \in rad(AnnQ_i^2)$. Then there exists $n \in \mathbb{N}$ such that $t^n Q_i = 0$. Hence $t \in rad(AnnQ_i) = P_i$. It turns out that $rad(AnnQ_i^2) \subseteq P_i(+)M$. One can easily check that $P_i(+)M \subseteq rad(AnnQ_i^2)$. Thus $P_i(+)M = rad(AnnQ_i^2)$ and so Q_i^2 is a secondary submodule of M^2, for every i with $1 \leq i \leq n$. The proof of the converse is similar. □

Example 3.1. *Let R be an integral domain and K be the quotient field of R. Then K^2 is a $0(+)M$ secondary $R(K)$-module. In particular, $\mathbb{Q}^2$ is a $0(+)\mathbb{Q}$ secondary $\mathbb{Z}(+)\mathbb{Q}$-module.*

Example 3.2. *If P is a maximal ideal of R, then $\frac{R^2}{(P^n)^2}$ is a $P(+)R$-secondary $R(R)$-module, for every $n \in \mathbb{N}$.*

Example 3.3. *Let R be a local ring with the unique maximal ideal P. If every element of P is nilpotent, then R^2 is a $P(+)R$-secondary $R(R)$-module.*

Acknowledgement : The authors thank to the referee for his valuable suggestions.

References

[1] M. M. Ali, Idealization and Theorems of D. D. Anderson, Comm. Algebra, 34 (2006), 4479-4501.

[2] D. D. Anderson and M. Winders, Idealization of a module, J. Comm. Algebra, 1(1) (2009), 3-56.

[3] D. D. Anderson, Cancellation Modules and Related Modules, Lecture Notes in Pure and Applied Mathematics, 220. New York: Dekker, (2001) pp. 13-25.

[4] D. D. Anderson, Some remarks on multiplication ideals, Math. Japan, 4 (1980), 463-469.

[5] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, 1969.

[6] M. Behboodi and H. Koohi, Weakly prime submodules, Vietnam J. Math, 32(2) (2004), 185-195.

[7] K. Divaani-Aazar and M. A. Esmkhani, Associated prime submodules of finitely generated modules, Comm. Algebra, 33 (2005), 4259-4266.

[8] C. Faith, Algebra I: Rings, Modules and Categories. Springer-Verlag, (1981).

[9] J. A. Huckaba, Commutative Rings with Zero-Divisors, New York: Marcel Dekker, (1988).

[10] H. Matsumara, Commutative Ring Theory, Cambridge: Cambridge University Press, (1986).

[11] M. Nagata, Local Rings, Interscience, New York, (1962).

[12] A. G. Naoum and F. H. Al-Awan, Dedekind modules, Comm. Algebra, 24 (1996), 397-412.

[13] A. G. Naoum and A.S. Mijbas, Weak cancellation modules, Kyungpook Math. J, 37 (1997), 73-82.

(Received 1 November 2009)

M. J. Nikmehr
Department of Mathematics,
Faculty of Science,
K. N. Toosi University of Technology, IRAN.
e-mail : nikmehr@kntu.ac.ir

S. Heidari
Department of Mathematics,
Faculty of Science,
K. N. Toosi University of Technology, IRAN.
e-mail : sajad_math82@yahoo.com

R. Nikandish
Department of Mathematics,
Faculty of Science,
K. N. Toosi University of Technology, IRAN.
e-mail : r_nikandish@sina.kntu.ac.ir